λ-Terms, Logic, Determiners and Quantifiers

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Abstract

In this paper, a theory T_H based on combining type freeness with logic is introduced and is then used to build a theory of properties which is applied to determiners and quantifiers.

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1 THE THEORY T_H

It is well known that mixing type freeness and logic leads to contradictions. For example, by taking the following syntax of terms:

$t := x|\lambda x.t|t_1 t_2|\neg t$

and applying the term $\lambda x.\neg xx$ to itself one gets a contradiction (known as Russell's paradox). Church was aware of the problem when he started the λ-calculus which he intended to be a theory of *functions* and *logic*. But his first theory of the λ-calculus was type free and so was inconsistent. The paradox could be described as follows: take a to be $\lambda x.(xx \to \bot)$. Then from Modus Ponens (MP), the Deduction Theorem (DT), and β-conversion, we could derive Curry's paradox:

1.	$aa = aa \to \bot$	by β -conversion
2.	$aa \vdash aa$	
3.	$aa \vdash \bot$	by MP + 1+2
4.	$\vdash aa \to \bot$	by DT +3
5.	aa	from 1
6.	$\vdash \bot$	by MP +4 +5

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The presence of these foundational difficulties led to the creation of two routes of research. The first route placed a big emphasis on logic and deduction systems, but avoided the difficulty by restricting the language used to first or higher order without allowing any self-reference or polymorphism. The second route placed the emphasis on the expressiveness of the language and the richness of functional application and self reference, but at the expense of including logic in the language except if restrictions are made (such as using non-classical logics). Church for example, followed Russell and introduced the simply typed λ-calculus. However, it became obvious that the theory was unattractive as one will have numbers at each level, no polymorphic functions and so on. Church and others then decided to enrich the syntax and the language but to avoid or restrict logic.

Mixing logic with the type free λ-calculus is not straigthforward (see [Scott 75] and [Aczel 80]). Furthermore there has been various attempts at so doing. All these attepmts have many points in common. The theory we put forward below is influenced by all these approaches and will be the constructive version of that of [Flagg, Myhill 84], as we shall be explaining.

Assume a denumerably infinite set of variables and use $x, y, z, x_1, y_1, \ldots$ to range over them. Use $t, t', t'', t_1, t_2, \ldots$ to range over terms. The syntax of terms written in BNF notation is as follows

$t := x|\lambda x.t|t_1t_2|t_1 = t_2|t_1 \wedge t_2|t_1 \vee t_2|t_1 \to t_2|Ht|\forall t|\exists t.$

All these terms are obvious except for Ht. H is essentially what enables the avoidance of the paradox and when Ht is derivable, t must be a proposition.

We use $\equiv$ for syntactic identity and define $\bot =_{df} 0 = 1$ where $0 \equiv \lambda fx.x$ and $1 \equiv \lambda fx.fx$. We define '$\neg$' and '$\Leftrightarrow$' out of the previous ones as follows:

$$\neg t =_{df} t \to \bot$$
$$t_1 \Leftrightarrow t_2 =_{df} (t_1 \to t_2) \wedge (t_2 \to t_1).$$

Bound/free variables and substitution are defined as usual; in $t_1[x := t_2]$ the bound variables of t_1 are changed to avoid collision. Moreover we write $t[x_1 := t_1, \ldots, x_n := t_n]$ for the simultaneous substitution of t_i for x_i, for $1 \leq i \leq n$, in t.

1.1 Type freeness

We consider the following axioms and rules

(α) $\quad \lambda x.t = \lambda y.t[x := y]$ where y is not free in t

(β) $\quad (\lambda x.t)t' = t[x := t']$

(γ) $$\frac{t_1 = t_2 \qquad t'_1 = t'_2}{t_1t'_1 = t_2t'_2}$$

(δ) $$\frac{t = t' \qquad t = t''}{t' = t''}$$

(ϵ) $$\frac{tx = t'x}{t = t'}$$ where x is not free in t, t' or any open assumptions .

From the axioms so far, we can deduce the following

Lemma 1.1

(i) $=$ *is reflexive, i.e.* $t = t$

(ii) $=$ *is symmetric, i.e.* $\dfrac{t = t'}{t' = t}$

(iii) $=$ *is transitive, i.e.* $\dfrac{t = t' \qquad t' = t''}{t = t''}$

(iv) $(\xi)\dfrac{t = t'}{\lambda x.t = \lambda x.t'}$

(v) $(\eta)(\lambda y.uy) = u$ *for* y *not free in* u.

Proof: *Easy.* □

Note, however, that from (ϵ) above we have been able to deduce both (ξ) and (η), but from (ξ) alone we cannot deduce (ϵ) as we will also need (η) in the derivation.

1.2 Logic

(α)-(ϵ) are just axioms and rules of the λ-calculus with extensionality; we still need a logic and we therefore add the following

$(\vee I)$ $\dfrac{t \qquad Ht'}{t \vee t'} \qquad \dfrac{t' \qquad Ht}{t \vee t'}$

$(\vee E)$ $\dfrac{t \vee t' \qquad \begin{matrix}\{t\}\\ \vdots \\ s\end{matrix} \qquad \begin{matrix}\{t'\}\\ \vdots \\ s\end{matrix}}{s}$

$(H\vee)$ $\dfrac{Ht \qquad Ht'}{H(t \vee t')}$

$(\wedge I)$ $\dfrac{t \qquad t'}{(t \wedge t')}$

$(\wedge E)$ $\dfrac{t \wedge t'}{t} \qquad \dfrac{t \wedge t'}{t'}$

$(H\wedge)$ $\dfrac{Ht \qquad Ht'}{H(t \wedge t')}$

$(\to I)$ $\dfrac{Ht \qquad \begin{matrix}\{t\}\\ \vdots \\ t'\end{matrix}}{t \to t'}$

$$(\to E) \qquad \frac{t \qquad t \to t'}{t'}$$

$$(H \to) \qquad \frac{\begin{array}{cc} & \{t\} \\ & \vdots \\ Ht & Ht' \end{array}}{H(t \to t')}$$

$$(\bot) \qquad \frac{\bot}{t}$$

$$(\exists I) \qquad \frac{tt'}{\exists t}$$

$$(\exists E) \qquad \frac{\begin{array}{cc} & \{tx\} \\ & \vdots \\ \exists t & s \end{array}}{s} \quad \text{provided } x \text{ is not free in } t, s \text{ or any open assumption}$$

$$(H\exists) \qquad \frac{H(tx)}{H(\exists t)} \quad x \text{ not free in } t \text{ or any open assumption}$$

$$(\forall I) \qquad \frac{tx}{\forall t} \quad x \text{ not free in } t \text{ or in any open assumption}$$

$$(\forall E) \qquad \frac{\forall t}{tt'}$$

$$(H\forall) \qquad \frac{H(tx)}{H(\forall t)} \quad x \text{ not free in } t \text{ or any open assumption}$$

$$(H) \qquad \frac{t}{Ht}$$

$$(H =) \qquad H(t = t')$$

$$(T_{sub}) \qquad \frac{t = t' \qquad t}{t'}$$

Now we stop to compare this theory with other ones based on combining type free λ-calculus with logic. In [Aczel 81] a formal language of Frege structures is presented but negation there is a primitive operator. Flagg and Myhill (in [Flagg, Myhill 84]) offered the classical version of the above theory. [Smith 84] offers a theory of Frege structures with the aim of interpreting Martin-Löf type theory and [Mönnich 83] provided a theory similar to the one found in [Aczel 81]. Finally [Beeson 87] offered an axiomatic theory of Frege structures.

Models of this theory exist and can be constructed using Aczel's techniques in [Aczel 80] or Scott's method in [Scott 75]. The models of the λ-calculus cannot deal with logic added on top of the λ-calculus, since once logic is added, consistency might be threatened. Models of type free λ-calculus with logic were not obvious until they were initiated by Scott in [Scott 75] where simply the idea was to start from any model of the λ-calculus and build logic on top by inductively constructing two collections one of the true propositions, the other of the false ones and by taking the limit of these two collections. Frege structures in [Aczel 80] are essentially based on the same idea, except that logic is built by inductively constructing the collections of the possible propositions and the possible truths, and by taking the limit of these two collections.

As it sounds, the process is quite simple, yet it depends on having a clear idea of the structure and on proving some theorems which will ensure the existence of the various logical connectives in the model considered.

2 THE METATHEORY OF T_H

We write $\vdash t$ if t is a theorem of T_H and $\Gamma \vdash t$ if t is deducible from the set of hypothesis in Γ. The following are provable in T_H

($T0$) $\quad H\bot$

($T1$) $$\frac{t = t' \qquad t''[x := t]}{t''[x := t']}$$

($T2$) $$\frac{t_i = t'_i \text{ for } i = 0, \ldots, n \qquad t_0[x_1 := t_1, \ldots, x_n := t_n]}{t'_0[x_1 := t'_1, \ldots, x_n := t'_n]}$$

($T3$) $$\frac{t = t' \qquad Ht}{Ht'}$$

($T4$) $\quad$ If $\vdash t \wedge t'$ then $\vdash t' \wedge t$

($T5$) $\quad \{Ht, Ht'\} \vdash (t \vee t' \rightarrow t' \vee t)$

($T6$) $\quad$ If $\vdash t$ then $\vdash \neg\neg t$

($T7$) $\quad \{Ht\} \vdash t \Leftrightarrow t$

($T8$) $\quad$ If $\vdash t \Leftrightarrow t'$ and $\vdash t' \Leftrightarrow t''$ then $\{Ht, Ht'\} \vdash t \Leftrightarrow t''$

($T9$) $\quad$ If $\vdash t \Leftrightarrow t'$ then $\vdash t' \Leftrightarrow t$

($T10$) $\quad$ If $\vdash \forall t$ then $\vdash \neg\exists(\lambda x.\neg(tx))$

($T11$) $\quad \{H(tx)\} \vdash \forall t \rightarrow \neg\exists(\lambda x.\neg(tx))$

($T12$) $\quad$ If $\vdash t$ then $\vdash \forall(\lambda x.t)$

($T13$) $\quad$ If $\Gamma \vdash t$ then $\Gamma \vdash \forall(\lambda x.t)$

($T14$) $\quad$ If $\Gamma \vdash \forall(\lambda x.t)$ then $\Gamma \vdash t$

($T15$) $\quad \vdash \exists(\lambda x.(x = 0))$

($T16$) $\quad \vdash \exists(\lambda x.\neg(x = 0))$

(T17) If $t \in \Gamma$ then $\Gamma \vdash t$

(T18) If $\Gamma \vdash t$ and $\Gamma \vdash t \rightarrow t'$ then $\Gamma \vdash t'$

(T19) If $\Gamma \cup \{t\} \vdash t'$ then $\Gamma \cup \{Ht\} \vdash t \rightarrow t'$

(T20) $\{Ht\} \vdash t \rightarrow (\neg t \rightarrow t')$

(T21) $\{Ht\} \vdash t \rightarrow \neg\neg t$

(T22) $\{Ht\} \vdash \neg t \Leftrightarrow \neg\neg\neg t$

(T23) $\{Ht, Ht'\} \vdash \neg(t \vee t') \Leftrightarrow \neg t \wedge \neg t'$

(T24) If $\{Ht, Ht'\} \vdash t \Leftrightarrow t'$ then $\{Ht, Ht'\} \vdash \neg t \Leftrightarrow \neg t'$

(T25) If $\Gamma \vdash \bot$ then $\Gamma \vdash t$

(T26) If $\Gamma \vdash t$ then $\Gamma \vdash \exists(\lambda x.t)$

(T27) If $\Gamma \vdash t$ then $\Gamma \cup \Lambda \vdash t$

(T28) If $\Gamma \vdash \exists t$ and $\Lambda \cup \{ty\} \vdash t'$ then $\Gamma \cup \Lambda \vdash t'$ for y not free in Λ

(T29) If $\Gamma \vdash t$ and $\Lambda \vdash t'$ then $\Gamma \cup \Lambda \vdash t \wedge t'$

(T30) If $\Gamma \vdash t \vee t', \Lambda \cup \{t\} \vdash t_1$ and $\Xi \cup \{t'\} \vdash t_1$ then $\Gamma \cup \Xi \vdash t_1$

(T31) $\{Ha\} \vdash \neg(a \wedge \neg a)$

(T32) $\{Ht, Ht'\} \vdash t \rightarrow (t' \rightarrow t)$

(T33) $\{Ht, Ht'\} \vdash (t \rightarrow t') \rightarrow ((t \rightarrow \neg t') \rightarrow \neg t)$

(T34) $\{Ht, Ht'\} \vdash t \rightarrow t \vee t'$

(T35) $\{Ht, Ht'\} \vdash t \rightarrow t' \vee t$

(T36) $\{Ht, Ht'\} \vdash t \rightarrow (t' \rightarrow t \wedge t')$

(T37) $\{Ht, Ht', Ht'', t \Leftrightarrow t'\} \vdash t''[x := t] \Leftrightarrow t\text{"}[x := t']$

(T38) If $\Gamma \cup \{t\} \vdash t'$ and $\Gamma \vdash t$ then $\Gamma \vdash t'$

Proof:[1]

(T0) is an instance of $(H =)$.

(T1) is deducible by induction on the way terms are constructed, using (T_{sub}) among other things .

(T2) is deducible by induction on the way terms are constructed.

(T3) is deducible from (γ) and (T_{sub}).

(T4) – (T38) are easy exercises.

[1](T19) is known as the deduction Theorem; please note the insertion of $\{Ht\}$. This is important as without it we would get Curry's paradox.

3 A THEORY OF PROPERTIES

We introduce in our language T_H the operator Δ, understanding ΔP to mean that P is a property. Δ is defined as follows:

$$\Delta P =_{df} \forall x H(Px).$$

That is, something is a property iff whenever it applies to an object, the result is a proposition; e.g. $\lambda x.\neg(x = x)$.

Having defined properties in T_H let us now look at their closure conditions to see whether they 'behave properly'. We can construct properties in the following way:

1. $P \cup P' = \lambda x.(Px \vee P'x)$
2. $P \cap P' = \lambda x.(Px \wedge P'x)$
3. $P^c = \lambda x.\neg Px$
4. $P \to P' = \lambda x.[\forall y(Py \to P'(xy))]$
5. $\Theta = \lambda x.(x = x)$
6. $\bigtriangledown = \lambda x.\neg(x = x)$

(1) - (3) give us boolean combinations of properties, using join, meet and complement. (4) gives us function space, and (5), (6) give us the universal and the empty property, respectively. Now before moving on to proving some important results about the collection of properties, we note that while we understand ΔP to be <u>P is a property</u>, some people understand it to be <u>P is a class</u>. Both interpretations work in parallel and to illustrate this point we introduce $\in$ by

$$a \in P =_{df} Pa,$$

and we understand it as saying *a belongs to the class P*. We can hence easily prove the following

(i) $$\frac{P = P' \qquad \Delta P}{\Delta P'}$$

$ii)$ $$\frac{\Delta P}{H(t \in P)}$$

$iii)$ $$\frac{H(tx)}{\Delta(\lambda x.tx)}$$ where no assumption depends on x.

Now we can prove the following lemma

Lemma 3.1 *The following are provable*

(i) $\vdash \Delta\Theta$

(ii) $\vdash \Delta\bigtriangledown$

(*iii*) $\{\Delta P, \Delta P'\} \vdash \Delta(P \cup P')$

(*iv*) $\{\Delta P, \Delta P'\} \vdash \Delta(P \cap P')$

(*v*) $\{\Delta P, \Delta P'\} \vdash \Delta P^c$

(*vi*) $\{\Delta P, \Delta P'\} \vdash \Delta(P \rightarrow P')$

Proof:
We shall only prove (vi), as the others are similar.
We have to show that $\forall x H(\forall z(Pz \rightarrow P'(xz)))$.
If $\Delta P'$ *then* $\forall x H(P'x)$,
hence $H(P'(xz))$; *but* $H(Pz)$ *as* ΔP.
Therefore $H(Pz \rightarrow P'(xz))$.
Hence $H(\forall z(Pz \rightarrow P'(xz)))$ *and so* $\forall x H(\forall z(Pz \rightarrow P'(xz)))$; *hence* $\Delta(P \rightarrow P')$. □

Θ stands for the universal property, $\bigtriangledown$ stands for the empty property, and, of course, if P, P' are properties, then so are their disjunction and conjunction. Also, the complement of any property is a property. This lemma implies that our domain of properties satisfies some important closure conditions; note especially that if P and P' are properties then $P \rightarrow P'$ is also a property. It is well known that this would not hold if the notion of property was more comprehensive. For instance, in [Turner 87] and [Feferman 79], if P, P' are properties or classes then $P \rightarrow P'$ is not necessarily a property or a class because according to their approach, there were more propositions than there is according to the approach put forward here.

Lemma 3.2 *The following are provable*

(*i*) $a \in P \cap P' = ((a \in P) \wedge (a \in P'))$

(*ii*) $a \in P \cup P' = ((a \in P) \vee (a \in P'))$

Proof: *Obvious.* □

Operators such as $\cup, \cap$ and ^{c} are just ways of building new properties (or classes) out of old ones. We have not yet defined any relations <u>between</u> properties (those relations may not be properties). Here we take the first step and define the following between properties:

$$P \subseteq P' = (\forall x)(Px \rightarrow P'x)$$

We understand $P \subseteq P'$ to be P is a subproperty of P'.

We also define the following operation on properties, which we have not included with the previous ones because of its distinctive status — a status which will become clear below.

$$\Pi P = \lambda x.(\forall y(Py \rightarrow yx))$$

ΠP is the intersection of all properties that are themselves P. It is obvious that we should not deduce from $\Delta P'$ and $P \subseteq P'$ that ΔP; but if ΔP, do we then have $\Delta(\Pi P)$? Well, we need to add another condition, namely, $\forall y(Py \rightarrow \Delta y)$. With this new condition, things fit;

Lemma 3.3 *If* ΔP *and* $\forall y(Py \to \Delta y)$ *then* $\Delta(\Pi P)$.
Proof:

$(\Pi P)x = \forall y(Py \to yx)$; *and we can show by* $(H \to)$,
$H(Py \to yx)$ *if we can show both that*
(i) $H(Py)$ *is deducible, and that*
(ii) $H(yx)$ *is deducible from assumption that* Py.
(i) follows from ΔP *and (ii) follows from* Py *and* $\forall y(Py \to \Delta y)$.
Hence $\Delta(\Pi P)$. □

Now we start by listing some characteristics of our domain of properties. We have already seen two of these characteristics in Lemma 3.2. With the following lemma we reveal more of our domain of properties,

Lemma 3.4

(i)	$(\lambda x.\Phi)t \wedge (\lambda x.\Psi)t = (\lambda x.(\Phi \wedge \Psi))t$
(ii)	$(\lambda x.\neg\Phi)t = \neg(\lambda x.\Phi)t$
(iii)	$(\lambda x.\Phi)t \vee (\lambda x.\Psi)t = (\lambda x.(\Phi \vee \Psi))t$
(iv)	$(P^c)t = \neg Pt$
(v)	$(P \cap P')t = Pt \wedge P't$
(vi)	$(P \cup P')t = Pt \vee P't$
(vii)	$((P^c)^c)t = \neg\neg Pt$
(viii)	$\{\Delta P, \Delta P'\} \vdash (P \cup P')^c t \Leftrightarrow (P^c)t \wedge (P'^c)t$
(ix)	$\{\Delta P, \Delta P'\} \vdash (P^c \cup P'^c)t = (P^c)t \vee (P'^c)t$
(x)	$\{\Delta P, \Delta P'\} \vdash (P^c \cap P'^c)t \Leftrightarrow (P \cup P')^c t$
(xi)	$\{Pt\} \vdash ((P^c)^c)t$
(xii)	*If* Ht *then* $\forall y(\lambda x.t)t' \to (\lambda x.\forall yt)t'$
(xiii)	*If* Ht *then* $\exists y(\lambda x.t)t' \to (\lambda x.\exists yt)t'$

Proof:[2] *We only prove (x) as (i)-(vii) are similar cases of β-conversion, (viii) comes from (x) and (v), (ix) is a particular case of (vi) and (xi) comes from (vii) and the fact that from a we deduce $\neg\neg a$. Also, (xii) and (xiii) are easy to prove.*

$$\begin{aligned}(P^c \cap P'^c)t &= (\lambda x.(P^c)x \wedge (P'^c)x)t \\ &= (P^c)t \wedge (P'^c)t \\ &= \neg Pt \wedge \neg P't \\ &\Leftrightarrow \neg(Pt \vee P't),\ \textit{from } (T23), \Delta P \textit{ and } \Delta P'.\end{aligned}$$

$$\begin{aligned}\textit{and } ((P \cup P')^c)t &= (\lambda x.\neg(P \cup P')x)t \\ &= \neg(P \cup P')t \\ &= \neg(Pt \vee P't)\end{aligned}$$

Hence $(P^c \cap P'^c)t \Leftrightarrow ((P \cup P')^c)t$. □

Now we discuss what would happen to the lemmas above if we change the functional application of the λ-calculus by a more intentional application, call it *pred*. That is, from $Px = Qy$, we can deduce nothing about the relationship between P and Q and x and y. pred on the other hand, will satisfy that if $pred(P, a) = pred(Q, b)$ then $P = Q$ and $a = b$. So let us introduce pred such that

(P_1) $\quad \dfrac{pred(P,x)}{Px} \qquad \dfrac{Px}{pred(P,x)} \qquad \dfrac{H(pred(P,x))}{H(Px)} \qquad \dfrac{H(Px)}{H(pred(P,x))}$

(P_2) $\quad \forall x(pred(P,x) = pred(Q,x)) \rightarrow P = Q$

(P_3) $\quad pred(P,a) = pred(Q,b) \rightarrow (P = Q \wedge a = b)$

Lemma 3.5 *If* ΔP *then* $\forall x(Px \Leftrightarrow pred(P,x))$.
Proof*: obvious.* □

Now it is interesting to see what would happen to the closure of our properties if we understand the predication relation to be given in terms of pred and not functional application.

We start from our definition of Δ above. We see that $\overline{\Delta^* P} =_{df} \forall x H(pred(P,x))$ does not give anything new. Suppose, however, that we introduce a relation $\in^*$ such that $a \in^* P =_{df} pred(P,a)$, then in Lemmas 3.1 and 3.3, nothing new results, since if ΔP then $pred(P,x) \equiv Px$ for any x. In Lemmas 3.2 and 3.4, let us replace any occurrences of $\in$ by $\in^*$, $=$ by $\equiv$ and functional application by pred. We combine the things that work for pred in one lemma, Lemma 3.6, and we add the condition that ΔP and $\Delta P'$

[2]But not necessarily:

$(viii')$ $\quad \{\Delta P, \Delta P'\} \vdash ((P \cap P')^c)t = (P^c)t \vee (P'^c)t$

(x') $\quad (P^c \cup P'^c)t = ((P \cap P')^c)t$

(xi') $\quad ((P^c)^c t \vdash (P)t$

Lemma 3.6
If $\Delta P, \Delta P'$ then the following holds,

1. $pred(P,t) \wedge pred(P',t) \Leftrightarrow pred(P \cap P', t)$
2. $pred(P^c \cup P'^c, t) \Leftrightarrow pred(P^c, t) \wedge pred(P'c, t)$
3. $pred(P^c \cap P'^c, t) \Leftrightarrow pred((P \cup P')^c, t)$.[3]
4. $pred(P,t) \rightarrow pred((P^c)^c, t)$[4]
5. $pred(P^c, t) \Leftrightarrow \neg pred(P,t)$

Proof:

1. *If $\Delta P, \Delta P'$ then $\Delta(P \cap P')$.*
   Therefore $H(pred(P \cap P', t)), H(pred(P,t))$ and $H(pred(P',t))$.

   $$\begin{aligned} \text{But } pred(P \cap P', t) &= pred(\lambda x.Px \wedge P'x, t) \\ &\Leftrightarrow Pt \wedge P't, \text{ as } \Delta(P \cap P'). \end{aligned}$$

   Since $pred(P,t) \Leftrightarrow Pt$ and $pred(P',t) \Leftrightarrow P't$
   then $pred(P,t) \wedge pred(P',t) \Leftrightarrow Pt \wedge P't$.
   Hence 1 is a theorem.

2. $\Delta P \Longrightarrow \Delta P^c \Longrightarrow pred(P^c, t) \Leftrightarrow P^c t$.
   $\Delta P' \Longrightarrow \Delta P'^c \Longrightarrow pred(P'^c, t) \Leftrightarrow P'^c t$.
   ΔP^c *and* $\Delta P'^c \Longrightarrow \Delta P^c \cup P'^c$
   $\Longrightarrow pred(Pc \cup P'c, t) \Leftrightarrow (P^c \cup P'^c)t$.
   But by Lemma 3.4 (vi), $(P^c \cup P'^c)t = (P^c)t \vee (P'^c)t$,
   hence $pred(P^c \cup P'^c, t) \Leftrightarrow (P^c)t \vee (P'^c)t \Leftrightarrow pred(P^c, t) \vee pred(P'^c, t)$.

3. $\Delta P \Longrightarrow \Delta P^c$
   $\Delta P' \Longrightarrow \Delta P'^c$
   ΔP^c *and* $\Delta P'^c \Longrightarrow \Delta(P^c \cap P'^c) \Longrightarrow$
   $pred(P^c \cap P'^c, t) \Leftrightarrow (P^c \cap P'^c)t$.
   ΔP *and* $\Delta P' \Longrightarrow \Delta(P \cup P'^c) \Longrightarrow \Delta((P \cup P')^c) \Longrightarrow$
   $pred(P^c \cup P'^c, t) \Leftrightarrow (P \cup P')^c t$.
   But by Lemma 3.4, (V), $(P^c \cap P'^c)t = (P^c)t \wedge (P'^c)t$
   and by Lemma 3.4, (Viii), $(P \cup P')^c t \Leftrightarrow (P^c)t \wedge (P'^c)t$.
   Hence $(P^c \cap P'^c)t \Leftrightarrow ((P \cup P')^c)t$
   and so $pred(P^c \cap P'^c, t) \Leftrightarrow pred((P \cup P')^c, t)$.

[3] Not necessarily $pred(P^c \cup P'^c, t) \equiv pred((P \cap P')^c, t)$, as we have: $\{Ht, Ht'\} \vdash \neg(t \vee t') \equiv \neg t \wedge \neg t'$ but not: $\{Ht, Ht'\} \vdash \neg(t \wedge t') \equiv \neg t \vee \neg t'$.

[4] But not necessarily $pred((p^c)^c, t) \rightarrow pred(P,t)$; this will only be the case if DP where DP will be defined below.

4. $\Delta P \Longrightarrow H(pred(P,t))$

$\Delta P \Longrightarrow \Delta P^c \Longrightarrow \Delta(P^c)^c$.

But by Lemma 3.4, (Vii), $((P^c)^c t = \neg\neg Pt$

$$\frac{H(pred(P,t)) \qquad \begin{array}{c}\{pred(P,t)\}\\ Pt\\ \neg\neg Pt\\ ((P^c)^c)t\\ pred((P^c)^c,t)\end{array}}{pred(P,t) \rightarrow pred((P^c)^c,t)}$$

5. $pred(P^c,t) \Leftrightarrow (P^c)t$ *when* ΔP.

$(P^c)t = \neg(P)t$;

hence $pred(P^c,t) \Leftrightarrow \neg Pt$.

But $Pt \Leftrightarrow pred(P,t)$; *hence by (T24),* $\neg Pt \Leftrightarrow pred(P,t)$.

Therefore, $pred(P^c,t) \Leftrightarrow \neg pred(P,t)$.

□

If ΔP and $\Delta P'$ are not assumed then the version of Lemma 3.6 is as follows

Lemma 3.7 *The following holds in* T_H ,

(i) $\{pred(P \cap P',t)\} \vdash pred(P,t) \wedge pred(P',t)$

(ii) $\{pred(P,t) \wedge pred(P',t)\} \vdash pred(P \cap P',t)$

(iii) $\{pred(P^c,t)\} \vdash \neg pred(P,t)$

(iv) $\{\neg pred(P,t)\} \vdash pred(P^c,t)$.

Proof:

(i) *If we assume* $pred(P \cap P',t)$ *then* $H(pred(P \cap P',t))$,
hence $H((P \cap P')t))$ *and so* $H(Pt)$ *and* $H(P't)$.
This means that $H(pred(P,t))$ *and* $H(pred(P',t))$.
But $Pt \Leftrightarrow pred(P,t), P't \Leftrightarrow pred(P',t)$,
$(P \cap P')t \Leftrightarrow pred(P \cap P',t)$ *and* $(P \cap P')t = Pt \wedge P't$.
Hence $pred(P \cap P',t) \Leftrightarrow (pred(P,t) \wedge pred(P',t))$.
Therefore the assumption $pred(P \cap P',t)$
implies $pred(P,t) \wedge pred(P',t)$;
i.e.$pred(P \cap P',t) \vdash pred(P,t) \wedge pred(P',t)$.

Now (ii), (iii) and (iv) are easy. □

4 DETERMINATE PROPERTIES

Now, even if ΔP, we still do not have that $pred(P,c) \vee \neg pred(P,c)$; we therefore define a property to be determinate as follows

$$DP =_{df} \forall x(pred(P,x) \vee \neg pred(P,x))$$

E.g. $D\Theta$; this is because $\forall x(pred(\Theta,x) \vee \neg pred(\Theta,x))$ is true as it is equivalent (in terms of $\Leftrightarrow$) to $\forall x((x = x) \vee \neg(x = x))$. We know that $x = x$ is always true, therefore $(x = x) \vee \neg(x = x)$ is always true and so $\forall x((x = x) \vee \neg(x = x))$ is true. For $\bigtriangledown$, we know that $pred(\bigtriangledown,x) \Leftrightarrow \neg(x = x)$ and so for any $x, \neg pred(\bigtriangledown,x) \Leftrightarrow (x = x)$ which is true; therefore, $\forall x(pred(\bigtriangledown,x) \vee \neg pred(\bigtriangledown,x))$ is true and so $D\bigtriangledown$.
As an example of an undeterminate property, take: $P_a = \lambda x.(x = a)$; P_a is undeterminate for take $pred(P_a,x) \Leftrightarrow (x = a)$ and $pred(P_a,x) \Leftrightarrow \neg(x = a)$. Therefore, $pred(P_a,x) \vee \neg pred(P_a,x) \Leftrightarrow (x = a) \vee \neg(x = a)$ which we do not have a proof for and so we do not have that P_a is determinate.[5]

Lemma 4.1 *Let P be a property such that DP. Then for any t, $pred(P,t) \Leftrightarrow \neg\neg pred(P,t)$.*
Proof:
($\Longrightarrow$) We always have $pred(P,t) \rightarrow \neg\neg pred(P,t)$ for any property P.
($\Longleftarrow$)

1. $H(\neg\neg pred(P,t))$ *because* $H(pred(P,t))$.

2. $pred(P,t) V \neg pred(P,t)$ *because* DP.

3. $$\frac{H(\neg\neg pred(P,t) \qquad \begin{array}{c}\{\neg\neg pred(P,t)\}\\ \{pred(P,t)\}\\ pred(P,t) \vee \neg pred(P,t)\end{array} \qquad \begin{array}{c}\{\neg pred(P,t)\}\\ \bot\\ pred(P,t)\end{array} \qquad pred(P,t)}{\neg\neg pred(P,t) \rightarrow pred(P,t)}$$

□

The above lemma shows that the domain of determinate properties obeys classical logic; the following lemma shows that this domain is closed under $\cup, \cap$, and ^{c}.

Lemma 4.2 *If DP and DP' then $D(P \cup P'), DP^c, D(P \cap P')$.*
Proof: *Obvious.* □

The following lemma pushes negation inside $\underline{pred}$ in the definition of DP and shows that for any object we cannot predicate both a property and its complement to that object.

Lemma 4.3

(i) *For any P such that $\Delta P, DP \Leftrightarrow \forall x(pred(P,x) \vee pred(P^c,x))$.*
(ii) *$\forall x$, if Δx then $[\forall y[\neg[pred(x,y) \wedge pred(x^c,y)]]]$.*

[5] This is mainly because equality is not determinate in the λ-calculus; and we are using an intuitionistic theory.

Proof:

(i) *If* ΔP *then* $pred(P^c, x) \Leftrightarrow (P^c)x$ *and* $pred(P, x) \Leftrightarrow Px$;
but $(P^c)x = \neg Px$ *and* $Px \Leftrightarrow pred(P, x)$,
hence $pred(P^c, x) \Leftrightarrow \neg pred(P, x)$.
Therefore
$\forall x(pred(P, x) \vee \neg pred(P, x)) \Leftrightarrow \forall x(pred(P, x) \vee pred(P^c, x))$;
and so $DP \Leftrightarrow \forall x(pred(P, x) \vee pred(P^c, x))$.

(ii) *If* Δx *then* $pred(x^c, y) \Leftrightarrow \neg pred(x, y)$, *from above* .
But $\neg(pred(x, y) \wedge pred(x^c, y)) \Leftrightarrow \neg(pred(x, y) \wedge \neg pred(x, y))$
and we always have $\neg(pred(x, y) \wedge \neg pred(x, y))$.

$\square$

The next lemma is concerned with the conjunction of complements of properties

Lemma 4.4 *For any properties* P *and* P', *if* DP *and* DP' *then we can derive the following in* T_H

(i) $(P^c \cup P'c)t \Leftrightarrow ((P \cap P')^c)t$
(ii) $pred(P^c \cup P'^c, t) \Leftrightarrow pred((P \cap P')^c, t)$
(iii) $((P^c)^c)t \to Pt$
(iv) $pred((P^c)^c, t) \to pred(P, t)$

Proof:

(i) $(\Longrightarrow)$ *We have to show that*

$$\frac{Pt \vee \neg Pt \qquad P't \vee \neg P't \qquad \neg Pt \vee \neg' t}{\neg(Pt \wedge P't)}$$

In other words, we need to prove that

$$\frac{(a \vee \neg a) \qquad (b \vee \neg b) \qquad (\neg a \vee \neg b)}{\neg(a \wedge b)}$$

This is done as follows:

$$\cfrac{\cfrac{\begin{array}{cccc} (a \vee \neg a) & (b \vee \neg b) & & (\neg a \vee \neg b) \\ & \{a \wedge b\}(1) & & \\ a & b & \{\neg a\} & \{\neg b\} \\ \bot & \bot & (\vee E) & \text{and } \neg a \vee \neg b \end{array}}{\bot \text{ By discharging (1)}}}{\neg(a \wedge b)}$$

$(\Longleftarrow)$ *is similar* .

(ii) *Now it is enough to say that*
$pred(P^c \cup P'^c, t) \Leftrightarrow (P^c \cup P'^c)t$ *and*
$pred((P \cap P')^c, t) \Leftrightarrow ((P \cap P')^c)t$.

(iii) *We proved in Lemma 4.1 that if* P *is a property such that* DP *then* $pred(P, t) \Leftrightarrow \neg\neg pred(P, t)$. *We also proved in Lemma 3.6 that if* P *is a property then* $pred(Pc, t) \Leftrightarrow \neg pred(P, t)$. *Hence as* P^c *is a property,* $pred((P^c)^c, t) \Leftrightarrow \neg pred(P^c, t) \Leftrightarrow \neg\neg pred(P, t)$.
Therefore $pred((P^c)^c, t) \Leftrightarrow \neg\neg pred(P, t)$.
As $((P^c)^c)t \Leftrightarrow pred((P^c)^c, t)$ *and* $Pt \Leftrightarrow pred(P, t)$ *then*
$((P^c)^c)t \Leftrightarrow \neg\neg Pt$ *and so* $((P^c)^c)t \to Pt$.

(*iv*) *is a consequence from the proof of (iii) above.*

□

Now we define introduce $pr\ddot{e}d$ for those who are intersted in comparing the theory that is presented here with those theories presented elsewhere such as feferman's and Turner's. For this purpose we define

$$pr\ddot{e}d(P,x) =_{df} pred(P^c, x).$$

Lemma 4.5

(*i*) *If DP then we have $pred(P,x) \vee pr\ddot{e}d(P,x)$ for any x.*
(*ii*) *If HA then $pr\ddot{e}d(\lambda x.A, t) \Leftrightarrow \neg pred(\lambda x.A, t)$*
(*iii*) *For P a property, $\forall x \neg(pred(P,x) \wedge pr\ddot{e}d(P,x))$.*

Proof:

(*i*) $DP =_{df} \forall x pred(P,x) \vee \neg pred(P,x)$.
But $pred(P^c, x) \Leftrightarrow \neg pred(P,x)$ for any property P,
hence if DP then $pred(P,x) \vee pred(P^c,x)$ for any x.

(*ii*) $pr\ddot{e}d(\lambda x.A, t) = pred((\lambda x.A)^c, t)$.
If HA then $\Delta(\lambda x.A)$ and so
$pred((\lambda x.A)^c, t) \equiv pred(\lambda x.A, t)$.
Hence $pr\ddot{e}d((\lambda x.A), t) \Leftrightarrow \neg pred(\lambda x.A, t)$.

(*iii*) *If ΔP then $pred(P^c, x) \Leftrightarrow \neg pred(P,x)$;*
as we have $\forall x \neg(pred(P,x) \wedge \neg pred(P,x))$,
then $\forall x \neg(pred(P,x) \wedge pr\ddot{e}d(P,x))$.

□

One of the basic characteristics of the theory of property offered here is the full (even though weak) comprehension principle. This principle says that:

(CP) For f a propositional function, we have:
$(\lambda x.fx)t$ is true iff ft is true .

Here f is a propositional function iff for every x, we have $H(fx)$. This full comprehension principle would lead to inconsistency if the notion of property was strengthened. This is why the work of Turner, Feferman and others focussed on restricting the principle. The fullness of the principle however is very useful to have because, as we see from (CP), it relates the 'internal' logic to the 'external' one. To be more specific, let us understand by the term 'external' logic to be the logic which enables us to make the usual logical derivations such as the ability to deduce from $\Phi[t] \to \Psi[t]$ and $\Phi[t], \Psi[t]$. Let us moreover, understand by the term 'internal' logic to be the logic which enables us to make derivations inside the λ-operator, for example deducing from $pred(\lambda x.\Phi, t)$ and $pred(\lambda x.\Phi \to \Psi, t)$, that $pred(\lambda x.\Psi, t)$. In our theory, we have by (CP) that the laws of the 'internal' logic are a consequence of the laws of the 'external' one. That is, having (CP), one can make do with just the axioms of first order logic. On the other hand, because of the unavailability of (CP), [Turner 87] had to treat the 'internal' and 'external' logics separately. According to our theory, if we work with propositions then we could have the following;

$$pred(\lambda x.\Phi, t) \wedge pred(\lambda x.\Psi, t) \Leftrightarrow pred(\lambda x.\Phi \wedge \Psi, t) \text{ and:}$$

$$pred(\lambda x.\neg\Phi, t) \Leftrightarrow \neg\Phi(t).$$

Now if we want a more general version of the comprehension principle, we can introduce the following:

For any Ψ a propositional wff open in x,

$$(\exists P)(\forall t)(pred(P, t) \Leftrightarrow \Psi[t/x]);$$

the above principle is valid. Also we of course have extensionality:

$$(\forall x)(\forall y)((\forall z)(xz = yz) \rightarrow x = y)$$

5 DETERMINERS

One of Montague's main achievement in PTQ (see [Thoamson 74]) was to show how a logically adequate treatment of quantifier phrases could be systematically incorporated into a fragment of English. A further round of investigation into the characteristics of quantifiers and determiners was inaugurated by Barwise and Cooper's paper [Barwise, Cooper 81], which explored the way in which results in the area of generalised quantifiers could be applied to natural language. Since then, there has been a copious discussion of this topic - van Benthem provides a good summary of the main results (in [Benthem 83] and [Benthem 84]). Here, we inquire how natural language quantifiers and determiners might be incorporated into the framework of our theory of properties.

In a Montague treatment, a sentence like <u>Every boy runs</u> receives a translation of the following form:

(1) $(every'(boy'))(run')$.

Within the framework of [Barwise, Cooper 81], we say that <u>every'(boy')</u> is a quantifier - interpreted as a set of sets (or, intensionally, as a second order property of properties), and that <u>every'</u> is a determiner - interpreted as a function from sets to quantifiers. An alternative analysis, adopted by van Benthem, treats determiners as relations between sets. For example, (2) denotes an instance of the schema (3):

(2) $every'(boy', run')$

(3) $\delta(A, B)$

(3) provides a convenient notation for expressing interesting characteristics of determiners, Introducing δ in (3) above prepares us for the important concept of a <u>determiner relation</u>,

also known as the characteristic property of the determiner. A characteristic property of a determiner is that particular set theoretical relation which characterises this determiner set theoretically; e.g. for every', it is $\subseteq$ and for a' it is $\cap^1$. We shall see below what $\subseteq$ and $\cap^1$ are.

We start first by defining the two determiners every' and a' in our framework. Let

$$\underline{every'} =_{df} \lambda x.\lambda y.\forall z(xz \to yz)$$

$$\underline{a'} =_{df} \lambda x.\lambda y.\exists z(xz \wedge yz)$$

The meanings of every', a' are not classes but we can prove some important theorems about them. We need however to introduce the characteristic properties of these determiners. We have also to show that these characteristic properties (or for that matter the determiners themselves) behave properly; that is when we combine things together in the right way we get a proposition. This is shown to be the case in the following few definitions and lemmas. The characteristic property of every', namely $\subseteq$, has already been defined as follows:

If P_1, P_2 are properties ,

$$P_1 \subseteq P_2 =_{df} \forall x(P_1 x \to P_2 x)$$

Lemma 5.1 $\subseteq$ *is a transitive, reflexive relation on properties.*
Proof: *Obvious.* □

Another important property that one might desire is equisymmetry which is defined as follows: $\subseteq$ is equisymmetric iff

If $P_1 \subseteq P_2$ and $P_2 \subseteq P_1$ then $\forall x(P_1 x \equiv P_2 x)$.

As can be seen from the lemma below, equisymmetry holds for $\subseteq$.

Lemma 5.2 $\subseteq$ *is equisymmetric on properties.*
Proof: *Easy.* □

Of course one has to note here that only equivalence between P_1 and P_2 is obtained and not equality. I.e. we have equisymmetry rather than antisymmetry. This is due of course to the intensional notion that is embedded in the system.

Lemma 5.3 *If P_1 and P_2 are properties then*

(*i*) $every' P_1 P_2 = P_1 \subseteq P_2$ *and*
(*ii*) $H(every' P_1 P_2)$.

Proof: *Obvious.* □

We define $P_1 \cap^1 P_2 =_{df} \exists z(P_1 z \wedge P_2 z)$.

It is obvious that $P_1 \cap^1 P_2$ is a proposition when both P_1 and P_2 are properties.

Another concept that we introduce here is that of an empty property. We say that a property P is empty and write $\emptyset P$ iff $\forall z(\neg P z)$. E.g. $\bigtriangledown$ is an empty property.

Lemma 5.4 *If P, P_1 and P_2 are properties then the following holds:*

(i) If $\neg\emptyset P$ then $\neg\emptyset(P \cup P)$
(ii) If $\neg\emptyset(P_1 \cup P_2)$ then $\neg\emptyset(P_2 \cup P_1)$

Proof: *Easy.* □

Lemma 5.5 *If P_1 and P_2 are properties then*

(i) $a' P_1 P_2 = P_1 \cap^1 P_2$
(ii) $H(a' P_1 P_2)$

Proof: *Easy.* □

6 NON DETERMINATE RESULTS

Outside the collection of properties, we cannot draw useful conclusions about every' because we cannot decide the propositionhood of an arbitrary formula in which $\rightarrow$ is the main connective[6] This is not a disadvantage as we only want every' to have meaning when we are working with properties. Moreover, we cannot define the type of every' or of determiners inside our formal language. That is if we define Quant and Det[7] as follows

$$\text{Quant } t =_{df} \forall x(\Delta x \rightarrow H(tx))$$

$$\text{Det } t =_{df} \forall x(\Delta x \rightarrow \text{ Quant } (tx)).$$

then there is no way to prove that Det and Quant always return propositions when applied to terms, because

$$\forall x(\Delta x \rightarrow \text{ Quant } (tx)) \text{ and}$$

$$\forall x(\Delta x \rightarrow H(tx))$$

are not propositions for any t. In fact even if t is a property, we still do not have a guarantee that Det t and Quant t are propositions, due to the fact that Δx is not a proposition. This is not serious as there is no particular reason for wanting determiners and quantifiers to be determinate. Everything fits together properly, and we can prove many desirable features of our determiners, why insist on determinability? The following lemma proves inside the theory that combining a determiner and a property results in a quantifier.

[6]The reader is reminded again that $a \rightarrow b$ is a proposition in the case where a is a proposition and b is a proposition assumning a is true.

[7]Note that we could have defined it as: $\text{Det}(t) = \forall xy((\Delta x \wedge \Delta y) \rightarrow H((tx)y))$ which is closer to van Benthem's approach in [Benthem 83] and [Benthem 84].

Lemma 6.1 $\{\ Det\ Q, \Delta P\} \vdash\ \ Quant\ (QP)$.
Proof:

$$\{\Delta P\}$$
$$\{DetQ\}$$
$$\forall x(\Delta x \rightarrow Quant(Qx)) \qquad From\ Det\ Q\ By\ (\forall E)$$
$$\overline{\Delta P \rightarrow Quant(QP)}$$
$$\overline{\Delta P \qquad \Delta P \rightarrow Quant(QP) \qquad By\ (\rightarrow E)}$$
$$Quant(QP)$$

□

Having determiners such as every', a' is one thing; being able to deduce that every', a' are determiners is something else. I.e. can we prove that Det(every'), Det(a'), etc..? Take the formula for every':

$$\lambda x.\lambda y.\forall z[xz \rightarrow yz];$$

To show that Det(every') we have to show that

$$\forall x(\Delta x \rightarrow \forall y(\Delta y \rightarrow H(every'xy))).$$

But to be able to show the implication we need to have $H(\Delta x)$, and $H(\Delta y)$, which we cannot assume. For this we need an extension for implication as follows:

We always have that if $\{a\} \vdash b$ then $\{Ha\} \vdash a \rightarrow b$ (our version of the deduction theorem). We need that if $\{Ha\} \vdash b$ then $\vdash Ha \rightarrow b$. Can we assert this rule? That is:

$(*)$ If $\{Ha\} \vdash b$ then $\vdash Ha \rightarrow b$.

It may be claimed here that this rule leads to an inconsistency similar to Curry's paradox because if a is $\lambda x(H(xx) \rightarrow \bot)$, then a is a well-formed expression. However it is not the case that we will get Curry's paradox, for take the following chain of deductions:

$app(a,a) = H(aa) \rightarrow \bot$	by β -conversion
$aa \vdash H(aa) \rightarrow \bot$	from above
$aa \vdash H(aa)$	obvious
$aa \vdash \bot$	by MP
$H(aa) \vdash aa \rightarrow \bot$	by DT

But now applying (*) we get: $\vdash H(aa) \rightarrow (aa \rightarrow \bot)$
which is not contradictory.

Note that we should not always deduce from $\{a\} \vdash b$ that $\vdash a \rightarrow b$; because if we did then we get Curry's paradox. However, I am not sure whether the deduction from $\{Ha\} \vdash b$ to $\vdash Ha \rightarrow b$ is harmless and hence the following theorem that every',.a' and the' are determiners can only hold if we conjecture that (*) holds.

Lemma 6.2 *Det(every'), Det(a'), if (*) holds.*
Proof: *For every': We have to prove that* $\forall x(\Delta x \rightarrow y(\Delta y \rightarrow H(every'xy)))$.

$$\{\Delta x, \Delta y\} \vdash H(every'xy)).$$
$$\{\Delta x\} \vdash H(yz) \rightarrow H(every'xy) \text{ according to } (*).$$
From this we have: $\{\Delta x\} \vdash \forall z[H(yz) \rightarrow H(every'xy)]$
$$\{\Delta x\} \vdash [\forall z H(yz)] \rightarrow H(every'xy)$$
$$\{\Delta x\} \vdash \Delta y \rightarrow H(every'xy)$$
$$\{\Delta x\} \vdash \forall y(\Delta y \rightarrow H(every'xy)).$$
Repeating the same process, we get:
$$\vdash \Delta x \rightarrow \forall y(\Delta y \rightarrow H(every'xy))$$
$$\vdash \forall x(\Delta x \rightarrow \forall y(\Delta y \rightarrow H(every'xy))).$$

The proof of a' is similar to that of every'. □

7 Charactersistics of determiners and quantifiers

Here we are concerned with some characteristics of determiners that can be proven in our theory. We start with the first theorem that asserts that the result of applying a quantifier to a property results in a proposition.

Lemma 7.1 $\{Quant(Q), DP\} \vdash H(QP)$
Proof:

We have to prove that: $H(QP)$ *from assumptions:*
$\forall x(\Delta x \rightarrow H(Qx))$ *and* ΔP.
But $\{\forall x(\Delta x \rightarrow H(Qx)), DP\} \vdash \Delta P, \Delta P \rightarrow H(QP)$
and $\{\Delta P, \Delta P \rightarrow H(QP)\} \vdash H(Qx)$

□

We still cannot prove: $\{Quant(Q), \Delta P, \Delta P'\} \vdash (QP \wedge P \subseteq P') \rightarrow QP'$, but this is not worrying as it should not always hold. The following lemma is to show that the domain of quantifiers is closed under $\cup, \cap$ and ^{c}.

Lemma 7.2

(i) $\{Quant(a), Quant(b)\} \vdash Quant(a \cap b)$
(ii) $\{Quant(a), Quant(b)\} \vdash Quant(a \cup b)$
(iii) $\{Quant(a)\} \vdash Quant(a^c)$

Proof:[8] *We illustrate only (i):*

$$\frac{\forall x(\Delta x \rightarrow H(ax)) \qquad \forall x(\Delta x \rightarrow H(bx))}{\forall x(\Delta x \rightarrow H(ax) \wedge H(bx))}$$

$$\textit{Hence } \forall x(\Delta x \rightarrow H(ax \wedge bx))$$

□

Also the following lemma is concerned with the closure on the domain of determiners. Now closure is in term of $\cap_1, \cup_1, ^{c1}$.[9]

Lemma 7.3

$$\begin{array}{ll} (i) & \{Det(a), Det(b)\} \vdash Det(a \cap_1 b) \\ (ii) & \{Det(a), Det(b)\} \vdash Det(a \cup_1 b) \\ (iii) & \{Det(a)\} \vdash Det(a^{c1}) \end{array}$$

Where

$$\begin{array}{ll} a \cap_1 b & = \lambda x.(ax \cap bx) \\ a \cup_1 b & = \lambda x.(ax \cup bx) \\ a^{c1} & = \lambda x.(ax)^c \end{array}$$

Proof:[10] *We illustrate only (iii).*

$$\begin{array}{ll} Det(a) & = \forall x(\Delta x \rightarrow Quant(ax) \\ & \equiv \forall x(\Delta x \rightarrow Quant(ax^c) \textit{ By Lemma 7.2 }, (iii). \\ \textit{Hence} & Det(a^{c1}). \end{array}$$

□

We would be interested in proving something in general about these determiner relations. Let us consider monotonicity. We have two kinds of monotonicity: *upwards monotonicity* and *downwards monotonicity* (see [Benthem 83] and [Benthem 84]). These are defined as follows, where C is a property of sets:

$(upwards)$ If $A \subseteq A'$ and $C(A)$ then $C(A')$

[8] The following are examples of (i), (ii) and (iii) respectively:

- Every man and some women.
- Every man or some women.
- Not every man.

[9] We introduce the subscript '1' to make the distinction between the intersections considered.

[10] The following are examples of (i), (ii) and (iii) respectively:

- Many and many.
- Some or few.
- Not all, not every.

$(downwards)$ If $A' \subseteq A$ and $C(A)$ then $C(A')$

As an example of an upwards monotone determiner, we take a', a' is monotone in both arguments. E.g.a boy who sings walks entails a boy walks. Also a boy sings and dances entails a boy sings.
Hence to show that a' is upwards monotone in both arguments we need to show that

(i) $a'P_1P_2 \wedge P_1 \subseteq P_1' \rightarrow a'P_1'P_2$ and
(ii) $a'P_1P_2 \wedge P_2 \subseteq P_2' \rightarrow a'P_1P_2'$.

Both (i) and (ii) can be shown as follows:
For (i) :
$a'P_1P_2 \wedge P_1 \subseteq P_1' =$
$\exists z(P_1z \wedge P_2z) \wedge P_1 \subseteq P_1'$
$\rightarrow \exists z(P_1'z \wedge P_2z)$.
The proof of (ii) is similar.
every' can also be shown monotone in the right argument but not in the left.[11]

We now consider another property of determiner relations; that is *conservativity*. We say that a property of sets C is conservative if

$(CONS)$ $\delta(P_1, P_2) \equiv \delta(P_1, P_2 \cap P_1)$, where δ is the determiner relation of C.

As an example, a' and every' are conservative. E.g. a boy walks entails a boy is both a boy and he walks.
Also every man runs entails every man is both man and he runs. Now to show that a' and every' are conservative, we have to show that for any P_1 and P_2 properties,

$(a'CONS)$ $P_1 \cap^1 P_2 \equiv P_1 \cap^1 (P_2 \cap P_1)$.

$(every'CONS)$ $P_1 \subseteq P_2 \equiv P_1 \subseteq (P_2 \cap P_1)$.

This is shown by the following two lemmas:

Lemma 7.4 *If P_1 and P_2 are properties then $P_1 \cap^1 P_2 \equiv P_1 \cap^1 (P_2 \cap P_1)$.*
Proof: *The only thing worth mentioning here is that $(P_2 \cap P_1)z = P_2z \wedge P_1z$.* □

Lemma 7.5 *$P_1 \subseteq P_2 \equiv P_1 \subseteq (P_2 \cap P_1)$.*
Proof: *Trivial.* □

Of course we would like the conservativity condition to hold of any determiner we define and we would be happy if we could prove conservativity for determiner relations as a special type of their own. It is not obvious how to do so and we must be satisfied with proving properties about each determiner relation individually.

Now we can take the definition of properties of concepts which is given in [Benthem 83], page 459, to be:

"determiners only dependent upon the intersection of their arguments;

[11] For a clear discussion of such characteristics of determiners, the reader is referred to [Benthem 83].

that is if $C \cap D = A \cap B$ then $\delta(C, D) \equiv \delta(A, B)$".

Now we can prove that a' has such a property. This is because the determiner relation for a' is $\cap^1$ and we can prove that if $C \cap D = A \cap B$ then $C \cap^1 D \equiv A \cap^1 B$.

Before closing this section, we give the following lemma which shows that every' is a transitive relation

Lemma 7.6 $\{every' P_1 P_2, every' P_2, P_3\} \vdash every' P_1 P_3$.

Proof:

$every' P_1 P_2 \wedge every' P_2 P_3 \equiv$

$\forall z(P_1 z \rightarrow P_2 z) \wedge \forall z(P_2 z \rightarrow P_3 z)$.

Hence $\forall z(P_1 z \rightarrow P_3 z)$ *and so* $every' P_1 P_3$. □

Transitivity does not hold for a'.

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