Generalized β-reduction and explicit substitutions [*]

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Abstract

Extending the λ-calculus with either explicit substitution or generalised reduction has been the subject of extensive research recently which still has many open problems. Due to this reason, the properties of a calculus combining both generalised reduction and explicit substitutions have never been studied. This paper presents such a calculus λsg and shows that it is a desirable extension of the λ-calculus. In particular, we show that λsg preserves strong normalisation, is sound and it simulates classical β-reduction. Furthermore, we study the simply typed λ-calculus extended with both generalised reduction and explicit substitution and show that well-typed terms are strongly normalising and that other properties such as subtyping and subject reduction hold.

1 Introduction

1.1 The λ-calculus with generalised reduction

In $((\lambda_x.\lambda_y.N)P)Q$, the function starting with λ_x and the argument P result in the redex $(\lambda_x.\lambda_y.N)P$ which when contracted will turn the function starting with λ_y and Q into a redex. This fact has been exploited by many researchers and reduction has been extended so that the future redex based on the matching λ_y and Q is given the same priority as the other redex. Attempts at generalising reduction can be summarized by three axioms:

$$(\theta) \quad ((\lambda_x.N)P)Q \to (\lambda_x.NQ)P, \qquad (\gamma) \quad (\lambda_x.\lambda_y.N)P \to \lambda_y.(\lambda_x.N)P,$$
$$(\gamma_C) \quad ((\lambda_x.\lambda_y.N)P)Q \to (\lambda_y.(\lambda_x.N)P)Q.$$

These (related) rules attempt to make more redexes visible. γ_C for example, makes sure that λ_y and Q form a redex even before the redex based on λ_x and P is contracted. Due to compatibility, γ implies γ_C. Moreover, $((\lambda_x.\lambda_y.N)P)Q \to_\theta (\lambda_x.(\lambda_y.N)Q)P$ and hence both θ and γ_C put λ adjacently next to its matching argument. θ moves the argument next to its matching λ whereas γ_C moves the λ next to its matching argument. θ can be equally applied to explicitly and implicitly typed systems. The transfer of γ or γ_C to explicitly typed systems is not straightforward however, since in these systems, the type of y may be affected by the reducible pair λ_x, P. For example, it is fine to write $((\lambda_{x:*}.\lambda_{y:x}.y)z)u \to_\theta (\lambda_{x:*}.(\lambda_{y:x}.y)u)z$ but not to write $((\lambda_{x:*}.\lambda_{y:x}.y)z)u \to_{\gamma_C} (\lambda_{y:x}.(\lambda_{x:*}.y)z)u$. For this reason, we study θ-like rules in this paper. Now, we discuss where generalised reduction has been used (cf. [25]).

[31] introduces the notion of a *premier redex* which is similar to the redex based on λ_y and Q above (which we call *generalised redex*). [32] uses θ and γ (and calls the combination

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σ) to show that the perpetual reduction strategy finds the longest reduction path when the term is Strongly Normalising (SN). [35] also introduces reductions similar to those of [32]. Furthermore, [23] uses θ (and other reductions) to show that typability in ML is equivalent to acyclic semi-unification. [34] uses a reduction which has some common themes with θ. [29] and [11] use θ whereas [26] uses γ to reduce the problem of β-strong normalisation to the problem of weak normalisation for related reductions. [24] uses θ and γ to reduce typability in the rank-2 restriction of the 2nd order λ-calculus to the problem of acyclic semi-unification. [2] uses θ (called "let-C") as a part of an analysis of how to implement sharing in a real language interpreter in a way that directly corresponds to a formal calculus. [19] uses a more extended version of θ where Q and N are not only separated by the redex $(\lambda_x.N)P$ but by many redexes (ordinary and generalised). [19] shows that generalised reduction makes more redexes visible allowing flexibility in reducing a term. [6] shows that with generalised reduction one may indeed avoid size explosion without the cost of a longer reduction path and that λ-calculus can be elegantly extended with definitions which result in shorter type derivations. Generalised reduction is strongly normalising (cf. [6]) for all systems of the cube (cf. [3]) and preserves strong normalisation of classical reduction (cf. [16]).

1.2   The λ-calculus with explicit substitution

Functional programming and in particular partial evaluation may benefit from explicit substitution. For example, given $xx[x := y]$, we may not be interested in having yy as the result of $xx[x := y]$ but rather only $yx[x := y]$. In other words, we only substitute one occurrence of x by y and continue the substitution later. This issue of being able to follow substitution and decide how much to do and how much to postpone, has become a major one in functional language implementation (cf. [30]). Another wish is to execute substitutions only when necessary. For this purpose one may decide to postpone substitutions as long as possible ("lazy evaluations"). This can yield profits, since substitution is an inefficient, maybe even exploding, process by the many repetitions it causes. This is the ground for the so-called graph reduction (cf. [30]). Most theorem provers (Nuprl [7], Coq [12], HOL [13]) use explicit substitutions in their implementation in order to replace locally (rather than globally) some abbreviated term. This avoids explosion when it is necessary that a variable be replaced by a huge term only in specific places so that a certain theorem can be proved.

Most literature on the λ-calculus considers substitution as an implicit operation. It means that the computations to perform substitution are usually described with operators which do not belong to the language of the λ-calculus. The last fifteen years however, have seen an interest in formalising substitution explicitly; various calculi including new operators to denote substitution have been proposed. Amongst these calculi we mention $C\lambda\xi\phi$ (cf. [10]); the calculi of categorical combinators (cf. [8]); $\lambda\sigma$, $\lambda\sigma_{\Uparrow}$, $\lambda\sigma_{SP}$ (cf. [1], [9], [33]) referred to as the $\lambda\sigma$-family; λv (cf. [4]), a descendant of the $\lambda\sigma$-family; $\varphi\sigma BLT$ (cf. [18]), $\lambda\mathtt{exp}$ (cf. [5]), λs (cf. [20]), λs_e (cf. [22]) and $\lambda\zeta$ (cf. [28]). All these calculi (except $\lambda\mathtt{exp}$) are described in a de Bruijn setting where natural numbers play the role of the classical variables.

In [20], we extended the λ-calculus with explicit substitutions by turning de Bruijn's meta-operators into object-operators offering a style of explicit substitution that differs from that of $\lambda\sigma$. The resulting calculus, λs, remains as close as possible to the λ-calculus from an intuitive point of view. The main interest in introducing the λs-calculus (cf. [20]) was to provide a calculus of explicit substitutions which would both preserve strong normalisation and have a confluent extension on open terms (cf. [22]). There are calculi of explicit substitutions which

are confluent on open terms: the $\lambda\sigma_{\Uparrow}$-calculus (cf. [15] and [9]), but the non-preservation of strong normalisation for $\lambda\sigma_{\Uparrow}$, as well as for the rest of the $\lambda\sigma$-family and for the categorical combinators, has recently been proved (cf. [27]). There are also calculi which satisfy the preservation property: the λv-calculus (cf. [4]), but this calculus is not confluent on open terms. Moreover, in order to get a confluent extension, the introduction of a composition operator for substitutions seemed unavoidable, but precisely this operator was the cause of the non-preservation of strong normalisation as shown in [27]. The $\lambda\zeta$-calculus (cf. [28]) solved the problem by introducing two new applications that allow the passage of substitutions within classical applications only if these applications have a head variable. This is done to cut the branch of the critical pair which is responsible for the non-confluence of λv on open terms. Hence, $\lambda\zeta$ preserves strong normalisation and is itself confluent on open terms. Unfortunately, $\lambda\zeta$ is not able to simulate one step of clasical β-reduction as shown in [28], it simulates only a "big step" β-reduction. This lack of the simulation property is an uncommon feature among calculi of explicit substitutions. On the other hand, the λs-calculus has been extended to λs_e which is confluent on open terms (cf. [22]) and simulates one step β-reduction but the preservation of strong normalisation is still an open problem.

1.3   The λ-calculus with generalised reduction and explicit substitution

All the research mentioned above is a living proof for the importance and usefulness of generalised reduction and explicit substitutions. Moreover, a system where reduction is generalised and substitution is explicit, gives a more flexible way of evaluating programs where step-wise substitution and the ability of reducing more redexes, may help in interleaving redexes in a way that might play a great role in lazy evaluation and parallel reduction.

Before such a combination can be used as a powerful basis for programming, we need to check that this combination is sound and safe exactly like we checked that each of explicit substitutions and generalised reductions are sound and safe to use. We need to check that extending the λ-calculus with both concepts results in theories that are confluent, preserve termination, and simulate β-reduction. This is what this paper does.

Generalised reduction $g\beta$, has never been introduced in a de Bruijn setting. Explicit substitution, has almost always been presented in a de Bruijn setting. For this reason, we combine $g\beta$-reduction and explicit substitution in a de Bruijn setting giving the first calculus of generalised reduction à la de Bruijn. As we need to describe generalised redexes in an elegant way, we use a notation suitable for this purpose *the item notation* (cf. [17]).

In Section 2 we introduce the calculus of generalised reduction, the λg-calculus, in item notation with de Bruijn indices and prove its confluence.

In Section 3 we recall the λs-calculus and extend it with $g\beta$-reduction, into the λsg-calculus. We show that λsg is sound with respect to λg, simulates $g\beta$ and is confluent.

In Section 4 we prove that the λsg-calculus preserves λs-strong normalisation (i.e. a is λs-SN $\Rightarrow a$ is λsg-SN) and conclude that a is λ-SN $\Leftrightarrow a$ is λs-SN $\Leftrightarrow a$ is λg-SN $\Leftrightarrow a$ is λsg-SN.

In Section 5 the simply typed versions of the λs- and λsg-calculi are presented and subject reduction, subtyping, and strong normalisation of well typed terms are proved.

2   The λg-calculus

We assume familiarity with de Bruijn notation. For instance, $\lambda x.\lambda y.xy$ is written as $\lambda\lambda(21)$ and $\lambda x.\lambda y.(x(\lambda z.zx))y$ as $\lambda(\lambda(2(\lambda(13))1))$. To translate free variables, we assume a fixed

ordered list of binders (written from left to right) $\cdots, \lambda z, \lambda y, \lambda x$ and prefix it to the term to be translated. Hence, $\lambda x.yz$ translates as $\lambda 34$ whereas $\lambda x.zy$ translates as $\lambda 43$. Since generalized β-reduction is easily described in item notation, we adopt the item syntax (cf. [19, 17]) and write $a\,b$ as $(b\,\delta)a$ and $\lambda.a$ as $(\lambda)a$.

Definition 1 *The* set of terms Λ, *is defined as follows:* $\Lambda ::= \mathbb{N} \mid (\Lambda\,\delta)\Lambda \mid (\lambda)\Lambda$
We let $a, b, \ldots$ *range over* Λ *and* $m, n, \ldots$ *over* $\mathbb{N}$ *(positive natural numbers). Throughout, $a = b$ means that a and b are syntactically identical. A reduction $\to$ is compatible on Λ when for all $a, b, c \in \Lambda$, $a \to b$ implies $(a\,\delta)c \to (b\,\delta)c$, $(c\,\delta)a \to (c\,\delta)b$ and $(\lambda)a \to (\lambda)b$.*

$(\lambda x \lambda y.zxy)(\lambda x.yx) \to_\beta \lambda u.z(\lambda x.yx)u$ translates to $(\lambda\lambda 521)(\lambda 31) \to_\beta \lambda 4(\lambda 41)1$. Note that we did not simply replace 2 in $\lambda 521$ by $\lambda 31$. Instead, we decreased 5 as one λ disappeared, and incremented the free variables of $\lambda 31$ as they occur within the scope of one more λ. For incrementing the free variables we need updating functions U_k^i, where k tests for free variables and $i - 1$ is the value by which a variable, if free, must be incremented:

Definition 2 *The* updating functions $U_k^i : \Lambda \to \Lambda$ *for $k \geq 0$ and $i \geq 1$ are defined inductively:*

$$U_k^i((a\,\delta)b) = (U_k^i(a)\,\delta)U_k^i(b)$$
$$U_k^i((\lambda)a) = (\lambda)(U_{k+1}^i(a))$$

$$U_k^i(\mathtt{n}) = \begin{cases} \mathtt{n} + \mathtt{i} - 1 & \text{if } n > k \\ \mathtt{n} & \text{if } n \leq k. \end{cases}$$

Now we define meta-substitution. The last equality substitutes the intended variable (when $n = j$) by the updated term. If n is not the intended variable, it is decreased by 1 if it is free (case $n > j$) as one λ has disappeared and if it is bound (case $n < j$) it remains unaltered.

Definition 3 *The* meta-substitutions at level j, *for $j \geq 1$, of a term $b \in \Lambda$ in a term $a \in \Lambda$, denoted $a \{\!\{ \mathtt{j} \leftarrow b \}\!\}$, is defined inductively on a as follows:*

$$((a_1\delta)a_2)\{\!\{ \mathtt{j} \leftarrow b \}\!\} = ((a_1\{\!\{ \mathtt{j} \leftarrow b \}\!\})\delta)(a_2\{\!\{ \mathtt{j} \leftarrow b \}\!\})$$
$$((\lambda)c)\{\!\{ \mathtt{j} \leftarrow b \}\!\} = (\lambda)(c\{\!\{ \mathtt{j} + 1 \leftarrow b \}\!\})$$

$$\mathtt{n}\{\!\{ \mathtt{j} \leftarrow b \}\!\} = \begin{cases} \mathtt{n} - 1 & \text{if } n > j \\ U_0^j(b) & \text{if } n = j \\ \mathtt{n} & \text{if } n < j. \end{cases}$$

The following lemma establishes the properties of meta-substitution and updating (cf. [20]):

Lemma 1 *Let $a, b, c \in \Lambda$.*
   1. For $k < n < k + i$ we have: $U_k^{i-1}(a) = U_k^i(a)\{\!\{ \mathtt{n} \leftarrow b \}\!\}$.
   2. For $l \leq k < l + j$ we have: $U_k^i(U_l^j(a)) = U_l^{j+i-1}(a)$.
   3. For $k + i \leq n$ we have: $U_k^i(a)\{\!\{ \mathtt{n} \leftarrow b \}\!\} = U_k^i(a\{\!\{ \mathtt{n} - \mathtt{i} + 1 \leftarrow b \}\!\})$.
   4. For $i \leq n$ we have: $a\{\!\{ \mathtt{i} \leftarrow b \}\!\}\{\!\{ \mathtt{n} \leftarrow c \}\!\} = a\{\!\{ \mathtt{n} + 1 \leftarrow c \}\!\}\{\!\{ \mathtt{i} \leftarrow b\{\!\{ \mathtt{n} - \mathtt{i} + 1 \leftarrow c \}\!\} \}\!\}$.
   5. For $l + j \leq k + 1$ we have: $U_k^i(U_l^j(a)) = U_l^j(U_{k+1-j}^i(a))$.
   6. For $n \leq k + 1$ we have: $U_k^i(a\{\!\{ \mathtt{n} \leftarrow b \}\!\}) = U_{k+1}^i(a)\{\!\{ \mathtt{n} \leftarrow U_{k-n+1}^i(b) \}\!\}$.

In order to introduce generalized β-reduction we need some defintions.

Definition 4 Items, segments *and* well-balanced segments (w.b.) *are defined respectively by:*
   $\mathcal{I} ::= (\Lambda\,\delta) \mid (\lambda)$ $\mathcal{S} ::= \phi \mid \mathcal{I}\,\mathcal{S}$ $\mathcal{W} ::= \phi \mid (\Lambda\delta)\mathcal{W}(\lambda) \mid \mathcal{W}\,\mathcal{W}$
where ϕ is the empty segment. Hence, a segment is a sequence of items. $(a\,\delta)$ and (λ) are called δ- and λ-item respectively. We let $I, J, \ldots$ range over $\mathcal{I}$; $S, S', \ldots$ over $\mathcal{S}$ and $W, U, \ldots$ over $\mathcal{W}$. For a segment S, $\lg S$, is given by: $\lg \phi = 0$, $\lg(I\,S) = 1 + \lg S$. The number of main λ-items in S, $N(S)$, is given by: $N(\phi) = 0$, $N((a\,\delta)S) = N(S)$ and $N((\lambda)S) = 1 + N(S)$.

Definition 5 *The λ-calculus (à la de Bruijn) is the reduction system $(\Lambda, \to_\beta)$, where $\to_\beta$ is the least compatible reduction on Λ generated by the β-rule: $(a\delta)(\lambda)b \to a\{\!\{1 \leftarrow b\}\!\}$.*

Definition 6 Generalized β, noted $\to_{g\beta}$, is the least compatible reduction on Λ generated by:

$$(g\beta\text{-rule}) \qquad (a\delta)W(\lambda)b \to W(b\{\!\{1 \leftarrow U_0^{N(W)+1}(a)\}\!\}) \qquad \text{where } W \text{ is well-balanced.}$$

The λg-calculus is the reduction system $(\Lambda, \to_{g\beta})$.

Remark 1 *The β-rule is an instance of the $g\beta$-rule.(Take $W = \phi$ and check $U_0^1(a) = a$.)*

We shall define updating and meta-substitution for segments and prove some useful properties.

Definition 7 *Let $S \in \mathcal{S}$, $a, b \in \Lambda$, $k \geq 0$ and $n, i \geq 1$. We define $U_k^i(S)$ and $S\{\!\{n \leftarrow a\}\!\}$ by:*

$$U_k^i(\phi) = \phi \qquad\qquad \phi\{\!\{n \leftarrow a\}\!\} = \phi$$
$$U_k^i((b\,\delta)S) = (U_k^i(b)\,\delta)U_k^i(S) \quad ((b\,\delta)S)\{\!\{n \leftarrow a\}\!\} = (b\{\!\{n \leftarrow a\}\!\}\,\delta)(S\{\!\{n \leftarrow a\}\!\})$$
$$U_k^i((\lambda)S) = (\lambda)(U_{k+1}^i(S)) \qquad ((\lambda)S)\{\!\{n \leftarrow a\}\!\} = (\lambda)(S\{\!\{n + 1 \leftarrow a\}\!\})$$

Lemma 2 *Let S, T be segments and $a, b \in \Lambda$. The following hold:*
  1. $U_k^i(S\,T) = U_k^i(S)U_{k+N(S)}^i(T)$ *and* $U_k^i(S\,a) = U_k^i(S)U_{k+N(S)}^i(a)$
  2. $\lg(S) = \lg(U_k^i(S))$, $N(S) = N(U_k^i(S))$ *and if S is w.b. then $U_k^i(S)$ is w.b.*
  3. $(S\,\xi)\{\!\{n \leftarrow a\}\!\} = S\{\!\{n \leftarrow a\}\!\}\,\xi\{\!\{n + N(S) \leftarrow a\}\!\}$ *for ξ a segment or a term*
  4. $\lg(S) = \lg(S\{\!\{n \leftarrow a\}\!\})$, $N(S) = N(S\{\!\{n \leftarrow a\}\!\})$, *if S is w.b. then $S\{\!\{n \leftarrow a\}\!\}$ is w.b.*

Proof: 1. and 3. Induction on S. 2. and 4. Induction on S using 1. and 3. respectively. $\square$

Lemma 3 *Let $a, b \in \Lambda$. If $a \twoheadrightarrow_{g\beta} b$ then $a =_\beta b$.*

Proof: We prove $a \to_{g\beta} b \Rightarrow a =_\beta b$ by induction on a. We just prove by induction on $\lg W$, the case $(c\delta)W(\lambda)d \to_{g\beta} W(d\{\!\{1 \leftarrow U_0^{N(W)+1}(c)\}\!\})$. Remark that $W = (e\delta)W_1(\lambda)W_2$, where W_1 and W_2 are well balanced. Let $w_1 = N(W_1)$ and $w_2 = N(W_2)$.

$$(c\delta)W(\lambda)d = (c\delta)(e\delta)W_1(\lambda)W_2(\lambda)d \stackrel{IH}{=}_\beta (c\delta)W_1((W_2(\lambda)d)\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})$$
$$\stackrel{L\,2.3}{=} (c\delta)W_1(W_2\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})(\lambda)(d\{\!\{2 + \mathtt{w}_2 \leftarrow U_0^{w_1+1}(e)\}\!\})$$
$$\stackrel{IH\,\&\,L\,2.4}{=}_\beta W_1(W_2\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})(d\{\!\{2 + \mathtt{w}_2 \leftarrow U_0^{w_1+1}(e)\}\!\}\{\!\{1 \leftarrow U_0^{w_1+w_2+1}(c)\}\!\})$$
$$\stackrel{L\,1.1}{=} W_1(W_2\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})$$
$$(d\{\!\{2 + \mathtt{w}_2 \leftarrow U_0^{w_1+1}(e)\}\!\}\{\!\{1 \leftarrow U_0^{w_1+w_2+2}(c)\}\!\}\{\!\{1 + \mathtt{w}_2 \leftarrow U_0^{w_1+1}(e)\}\!\}\})$$
$$\stackrel{L\,1.4}{=} W_1(W_2\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})d\{\!\{1 \leftarrow U_0^{w_1+w_2+2}(c)\}\!\}\{\!\{1 + \mathtt{w}_2 \leftarrow U_0^{w_1+1}(e)\}\!\}$$
$$\stackrel{L\,2.3\,\&\,L\,2.4}{=} W_1(W_2(d\{\!\{1 \leftarrow U_0^{w_1+w_2+2}(c)\}\!\})\{\!\{1 \leftarrow U_0^{w_1+1}(e)\}\!\})$$
$$\stackrel{IH}{=}_\beta (e\delta)W_1(\lambda)W_2(d\{\!\{1 \leftarrow U_0^{w_1+w_2+2}(c)\}\!\}) = W(d\{\!\{1 \leftarrow U_0^{N(W)+1}(c)\}\!\}) \qquad \square$$

Theorem 1 (Confluence of λg) *The λg-calculus is confluent.*

Proof: This proof is the de Bruijn version of the proof given in [19].
Let $a \twoheadrightarrow_{g\beta} b$ and $a \twoheadrightarrow_{g\beta} c$. By lemma 3, $a =_\beta b$ and $a =_\beta c$, hence $b =_\beta c$. By confluence of β, $\exists d \in \Lambda$ where $b \twoheadrightarrow_\beta d$ and $c \twoheadrightarrow_\beta d$. By Remark 1, $b \twoheadrightarrow_{g\beta} d$ and $c \twoheadrightarrow_{g\beta} d$. $\square$
   Finally, the following ensures the good passage of $g\beta$-reduction through $\{\!\{ \leftarrow \}\!\}$ and U_k^i:

Lemma 4 *Let $a, b, c, d \in \Lambda$. The following hold:*
  1. *If $c \to_{g\beta} d$ then $U_k^i(c) \to_{g\beta} U_k^i(d)$.*
  2. *If $c \to_{g\beta} d$ then $a\{\!\{n \leftarrow c\}\!\} \twoheadrightarrow_{g\beta} a\{\!\{n \leftarrow d\}\!\}$.*
  3. *If $a \to_{g\beta} b$ then $a\{\!\{n \leftarrow c\}\!\} \to_{g\beta} b\{\!\{n \leftarrow c\}\!\}$.*

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Proof: 1. By induction on c. We just check case $c = (c_1\delta)W(\lambda)c_3$, W well balanced, and $d = W(c_3\{\!\{1 \leftarrow U_0^{N(W)+1}(c_1)\}\!\}):$

$$U_k^i(c) = U_k^i((c_1\delta)W(\lambda)c_3) \stackrel{L\,2.1}{=} (U_k^i(c_1)\delta)U_k^i(W)(\lambda)U_{k+N(W)+1}^i(c_3) \stackrel{L\,2.2}{\rightarrow}_{g\beta}$$

$$U_k^i(W)(U_{k+N(W)+1}^i(c_3)\{\!\{1 \leftarrow U_0^{N(W)+1}(U_k^i(c_1))\}\!\}) \stackrel{L\,1.5}{=}$$

$$U_k^i(W)(U_{k+N(W)+1}^i(c_3)\{\!\{1 \leftarrow U_{k+N(W)}^i(U_0^{N(W)+1}(c_1))\}\!\}) \stackrel{L\,1.6}{=}$$

$$U_k^i(W)U_{k+N(W)}^i(c_3\{\!\{1 \leftarrow U_0^{N(W)+1}(c_1)\}\!\}) \stackrel{L\,2.1}{=} U_k^i(W(c_3\{\!\{1 \leftarrow U_0^{N(W)+1}(c_1)\}\!\})) = U_k^i(d).$$

2. By induction on a using 1. .

3. Induction on a. We prove case $a = (a_1\delta)W(\lambda)a_2$ and $b = W(a_2\{\!\{1 \leftarrow U_0^{N(W)+1}(a_1)\}\!\})$:

$$a\{\!\{\mathtt{i} \leftarrow c\}\!\} = ((a_1\delta)W(\lambda)a_2)\{\!\{\mathtt{i} \leftarrow c\}\!\} \stackrel{L\,2.3}{=}$$

$$(a_1\{\!\{\mathtt{i} \leftarrow c\}\!\}\delta)(W\{\!\{\mathtt{i} \leftarrow c\}\!\})(\lambda)(a_2\{\!\{\mathtt{i} + N(W) + 1 \leftarrow c\}\!\}) \stackrel{L\,2.4}{\rightarrow}_{g\beta}$$

$$W\{\!\{\mathtt{i} \leftarrow c\}\!\}(a_2\{\!\{\mathtt{i} + N(W) + 1 \leftarrow c\}\!\}\{\!\{1 \leftarrow U_0^{N(W)+1}(a_1\{\!\{\mathtt{i} \leftarrow c\}\!\})\}\!\}) \stackrel{L\,1.3}{=}$$

$$W\{\!\{\mathtt{i} \leftarrow c\}\!\}(a_2\{\!\{\mathtt{i} + N(W) + 1 \leftarrow c\}\!\}\{\!\{1 \leftarrow U_0^{N(W)+1}(a_1)\{\!\{\mathtt{i} + N(W) \leftarrow c\}\!\}\}\!\}) \stackrel{L\,1.4}{=}$$

$$W\{\!\{\mathtt{i} \leftarrow c\}\!\}(a_2\{\!\{1 \leftarrow U_0^{N(W)+1}(a_1)\}\!\}\{\!\{\mathtt{i} + N(W) \leftarrow c\}\!\}) \stackrel{L\,2.3}{=}$$

$$(W(a_2\{\!\{1 \leftarrow U_0^{N(W)+1}(a_1)\}\!\}))\{\!\{\mathtt{i} \leftarrow c\}\!\} = b\{\!\{\mathtt{i} \leftarrow c\}\!\} \qquad\qquad \Box$$

3   The λs- and λsg-calculi

The idea is to handle explicitly the meta-operators of definitions 2 and 3. Therefore, the syntax of the λs-calculus is obtained by adding to Λ two families of operators:

1. Explicit substitution operators $\{\sigma^j\}_{j\geq 1}$ where $(b\,\sigma^j)a$ stands for a where all free occurrences of the variable representing the index j are to be substituted by b.

2. Updating operators $\{\varphi_k^i\}_{k\geq 0\ i\geq 1}$ necessary for working with de Bruijn numbers.

Definition 8 *The* set of terms, *noted* Λs, *of the λs-calculus is given as follows:*

$$\Lambda s ::= \mathbb{N} \mid (\Lambda s\,\delta)\Lambda s \mid (\lambda)\Lambda s \mid (\Lambda s\,\sigma^j)\Lambda s \mid (\varphi_k^i)\Lambda s \qquad where \quad j, i \geq 1,\ k \geq 0.$$

We let a, b, c range over Λs. A term $(a\,\sigma^j)b$ is called a closure. *Furthermore, a term containing neither σ's nor φ's is called a* pure term. *Λ denotes the set of pure terms. $\delta\lambda$-segments are those whose main items are either δ- or λ-items, i.e. $\mathcal{DL} ::= \phi \mid (\Lambda s\,\delta)\mathcal{DL} \mid (\lambda)\mathcal{DL}$.*

A reduction $\rightarrow$ on Λs is compatible *if for all $a, b, c \in \Lambda s$, if $a \rightarrow b$ then $(a\,\delta)c \rightarrow (b\,\delta)c$, $(c\,\delta)a \rightarrow (c\,\delta)b$, $(\lambda)a \rightarrow (\lambda)b$, $(a\,\sigma^j)c \rightarrow (b\,\sigma^j)c$, $(c\,\sigma^j)a \rightarrow (c\,\sigma^j)b$ and $(\varphi_k^i)a \rightarrow (\varphi_k^i)b$.*

Definition 9 Items, segments *and* well-balanced segments *for Λs are defined as follows:*

$$\mathcal{I}s ::= (\Lambda s\,\delta)\mid(\lambda)\mid(\Lambda s\,\sigma^j)\mid(\varphi_k^i) \qquad \mathcal{S}s ::= \phi\mid\mathcal{I}s\,\mathcal{S}s \qquad \mathcal{W}s ::= \phi\mid(\Lambda s\,\delta)\mathcal{W}s(\lambda)\mid\mathcal{W}s\,\mathcal{W}s$$

We let I, J, ... range over $\mathcal{I}s$; S, S', ... over $\mathcal{S}s$ and W, U, ... over $\mathcal{W}s$. We call $(a\,\sigma^j)$ and (φ_k^i), σ- and φ-item respectively. $\lg(S)$ is trivially extended to $S \in \mathcal{S}s$ and $N(S)$ is extended by: $N((a\,\sigma^j)S) = N(S)$ and $N((\varphi_k^i)S) = N(S)$.

As λs-calculus should carry out updating and substitution explicitly, we include a set of rules which are the equations in definitions 2 and 3 oriented from left to right.

Definition 10 *The λs-calculus is the reduction system $(\Lambda s, \rightarrow_{\lambda s})$, where $\rightarrow_{\lambda s}$ is the least compatible reduction on Λs generated by the rules given in Figure 1. We use λs to denote this set of rules. The calculus of substitutions associated with the λs-calculus is the reduction system generated by the set of rules $s = \lambda s - \{\sigma\text{-generation}\}$ and we call it the s-calculus. The λsg-calculus is the calculus whose set of rules is $\lambda sg = \lambda s + \{g\sigma\text{-generation}\}$ where:*

$$
\begin{array}{lrcl}
(\sigma\text{-generation}) & (b\,\delta)(\lambda)a & \longrightarrow & (b\,\sigma^1)a \\
\sigma\text{-}\lambda\text{-transition} & (b\,\sigma^j)(\lambda)a & \longrightarrow & (\lambda)(b\,\sigma^{j+1})a \\
\sigma\text{-app-transition} & (b\,\sigma^j)(a_1\delta)a_2 & \longrightarrow & ((b\,\sigma^j)a_1\delta)\,(b\,\sigma^j)a_2 \\[2mm]
\sigma\text{-destruction} & (b\,\sigma^j)\mathtt{n} & \longrightarrow & \left\{ \begin{array}{ll} \mathtt{n-1} & if \ \ n>j \\ (\varphi_0^j)b & if \ \ n=j \\ \mathtt{n} & if \ \ n<j \end{array}\right. \\[5mm]
\varphi\text{-}\lambda\text{-transition} & (\varphi_k^i)(\lambda)a & \longrightarrow & (\lambda)(\varphi_{k+1}^i)a \\
\varphi\text{-app-transition} & (\varphi_k^i)(a_1\delta)a_2 & \longrightarrow & ((\varphi_k^i)a_1\delta)(\varphi_k^i)a_2 \\[2mm]
\varphi\text{-destruction} & (\varphi_k^i)\mathtt{n} & \longrightarrow & \left\{ \begin{array}{ll} \mathtt{n+i-1} & if \ \ n>k \\ \mathtt{n} & if \ \ n\le k \end{array}\right.
\end{array}
$$

Figure 1: The λs-calculus

$$g\sigma\text{-generation} \qquad (b\,\delta)W(\lambda)a \longrightarrow W((\varphi_0^{N(W)+1})b\,\sigma^1)a \qquad W \ \ well\ balanced,\ W\neq\phi$$

$\sigma\text{-generation}$ starts β-reduction by generating a substitution operator (σ^1). $\sigma\text{-app}$ and $\sigma\text{-}\lambda$ allow this operator to travel throughout the term until its arrival to the variables. If a variable should be affected by the substitution, $\sigma\text{-destruction}$ (case $j=n$) carries out the substitution by the updated term, thus introducing the updating operators. Finally the $\varphi\text{-rules}$ compute the updating. We state now the following theorem of the λs-calculus (cf. [22]).

Theorem 2 *The s-calculus is strongly normalising and confluent on Λs, hence s-normal forms are unique. The set of s-normal forms is exactly Λ. Furthermore, if $s(a)$ denotes the s-normal form of a, then for every $a,b \in \Lambda s$: $s((a\,\delta)b) = (s(a)\,\delta)s(b)$, $s((\lambda)a) = (\lambda)(s(a))$, $s((\varphi_k^i)a) = U_k^i(s(a))$ and $s((b\,\sigma^j)a) = s(a)\{\!\{\mathtt{j} \leftarrow s(b)\}\!\}$.*

Lemma 5 *Let $a,b \in \Lambda s$, $a \rightarrow_{\sigma-gen} b \Rightarrow s(a) \twoheadrightarrow_\beta s(b)$ and $a \rightarrow_{g\sigma-gen} b \Rightarrow s(a) \twoheadrightarrow_{g\beta} s(b)$.*

Proof: Induction on a using Lemma 4 and Thm. 2. For the second item note that if W is well balanced then $s(W\,a) = s(W)s(a)$, where the s-nf of a $\delta\lambda$-segment is given by: $s(\phi) = \phi$, $s((a\,\delta)S) = (s(a)\,\delta)s(S)$ and $s((\lambda)S) = (\lambda)s(S)$. $\qquad\square$

Corollary 1 *Let $a,b \in \Lambda s$, if $a \twoheadrightarrow_{\lambda sg} b$ then $s(a) \twoheadrightarrow_{g\beta} s(b)$.*

Corollary 2 (Soundness) *Let $a,b \in \Lambda$, if $a \twoheadrightarrow_{\lambda sg} b$ then $a \twoheadrightarrow_{g\beta} b$.*

This last corollary says that the λsg-calculus is correct with respect to the λg-calculus, i.e. λsg-derivations of pure terms ending with pure terms can also be derived in the λg-calculus.
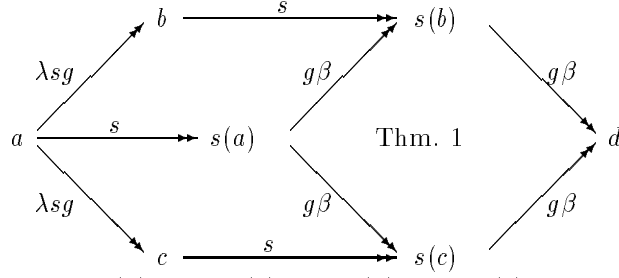
Moreover, the λsg-calculus is powerful enough to simulate $g\beta$-reduction.

Lemma 6 (Simulation of $g\beta$-reduction) *Let $a,b \in \Lambda$, if $a \rightarrow_{g\beta} b$ then $a \twoheadrightarrow_{\lambda sg} b$.*

Proof: Induction on a using Lemma 4. $\qquad\square$

Theorem 3 (Confluence of λsg) *The λsg-calculus is confluent on Λs.*

Proof: We use the interpretation method (cf. [14, 9]). We interpret the λsg-calculus into the λg-calculus via s-normalisation:

The existence of the arrows $s(a) \twoheadrightarrow_{g\beta} s(b)$ and $s(a) \twoheadrightarrow_{g\beta} s(c)$ is guaranteed by Corollary 1. We can close the diagram thanks to the confluence of the λg-calculus and finally Lemma 6 ensures $s(b) \twoheadrightarrow_{\lambda sg} d$ and $s(b) \twoheadrightarrow_{\lambda sg} d$ proving thus the confluence for the λsg-calculus. $\quad\square$

4 The λsg-calculus preserves λs-SN

The technique used in this section to prove preservation of strong normalisation (PSN) is the same used in [4] to prove PSN for λv and in [20] to prove PSN for λs.

Notation 1 *We write $a \in \lambda$-SN resp. $a \in \lambda r$-SN when a is strongly normalising in the λ-calculus resp. in the λr-calculus for $r \in \{g, sg, s\}$. We write $a \underset{p}{\to} b$ to denote that p is the occurrence of the redex which is contracted. Therefore $a \underset{\epsilon}{\to} b$ means that the reduction takes place at the root. If no specification is made the reduction must be understood as a λsg-reduction. We denote by $\prec$ the prefix order between occurrences of a term. Hence if p, q are occurrences of the term a such that $p \prec q$, and we write a_p (resp. a_q) for the subterm of a at occurrence p (resp. q), then a_q is a subterm of a_p.*

For example, if $a = 2\sigma^3((\lambda 1)4)$, we have $a_1 = 2$, $a_2 = (\lambda 1)4$, $a_{21} = \lambda 1$, $a_{211} = 1$, $a_{22} = 4$. Since, for instance, $2 \prec 21$, a_{21} is a subterm of a_2.

The following three lemmas assert that all the σ's in the last term of a derivation beginning with a λ-term must have been created at some previous step by a (generalized) σ-generation and trace the history of these closures. The first lemma deals with one-step derivation where the redex is at the root; the second generalises the first; the third treats arbitrary derivations.

Lemma 7 *If $a \underset{\epsilon}{\to} C[(e\,\sigma^i)d]$ then one of the following must hold:*
1. $a = (e\,\delta)(\lambda)d$, $C = \square$ and $i = 1$.
2. $a = (e'\,\delta)W(\lambda)d$, $W \neq \phi$, $C = W\square$, $e = (\varphi_0^{N(W)+1})e'$ and $i = 1$.
3. $a = C'[(e\,\sigma^j)d']$ for some context C', some term d' and some natural j.

Proof: Since the reduction is at the root, we must check for every rule $a \to a'$ in λsg that if $(e\,\sigma^i)d$ occurs in a' then either 1. or 2. or 3. follows. We just check the interesting rules:

(σ-gen) : $a = (c\,\delta)(\lambda)b$ and $a' = (c\,\sigma^1)b$. If $(e\,\sigma^i)d$ matches $(c\,\sigma^1)d$ then 1. Else $(e\,\sigma^i)d$ must occur within b or c and hence 3. with $j = i$ and $d' = d$.

(gσ-gen) : $a = (c\,\delta)W(\lambda)b$, $W \neq \phi$ and $a' = W((\varphi_0^{N(W)+1})c\,\sigma^1)b$. If $(e\,\sigma^i)d$ matches $((\varphi_0^{N(W)+1})c\,\sigma^1)d$ then 2. Else $(e\,\sigma^i)d$ occurs in b, c or W, hence 3. with $j = i$, $d' = d$.

(σ-λ-trans) : $a = (c\,\sigma^h)(\lambda)b$ and $a' = (\lambda)(c\,\sigma^{h+1})b$. If $(e\,\sigma^i)d$ matches $(c\,\sigma^{h+1})b$ then 3. with $j = i - 1$, $d' = (\lambda)d$. Else $(e\,\sigma^i)d$ occurs in b or c, hence 3. with $j = i$, $d' = d$. $\qquad\square$

Lemma 8 *If $a \to C[(e\,\sigma^i)d]$ then one of the following must hold:*

1. $a = C[(e\,\delta)(\lambda)d]$ and $i = 1$.
2. $a = C'[(e'\,\delta)W(\lambda)d]$, $C = C'[W\,\square]$, $e = (\varphi_0^{N(W)+1})e'$ and $i = 1$.
3. $a = C'[(e'\,\sigma^i)d']$ where $e' = e$ or $e' \to e$.

Proof: Induction on a, using lemma 7 for the reductions at the root. $\qquad\square$

Lemma 9 *Let $a_1 \to \ldots \to a_n \to a_{n+1} = C[(e\,\sigma^i)d]$. There exists $e', d' \in \Lambda s$ such that $e' \twoheadrightarrow e$ and, either $a_1 = C'[(e'\,\sigma^j)d']$ or for some $k \le n$ and w.b. segment W, $a_k = C'[(e'\,\delta)W(\lambda)d']$ and $a_{k+1} = C'[W((\varphi_0^{N(W)+1})e'\,\sigma^1)d']$ or, if $W = \phi$, $a_{k+1} = C'[(e'\,\sigma^1)d']$.*

Proof: Induction on n and use the previous lemma. $\qquad\square$

We define now internal and external reductions. An internal reduction takes place somewhere at the left of a σ^i operator. An external reduction is a non-internal one. Our definition is inductive rather than starting from the notion of internal and external position as in [4].

Definition 11 *The reduction $\xrightarrow{\text{int}}_{\lambda sg}$ is defined by the following rules:*

$$\frac{a \longrightarrow_{\lambda sg} b}{(a\,\sigma^i)c \xrightarrow{\text{int}}_{\lambda sg} (b\,\sigma^i)c} \qquad \frac{a \xrightarrow{\text{int}}_{\lambda sg} b}{(a\,\delta)c \xrightarrow{\text{int}}_{\lambda sg} (b\,\delta)c} \qquad \frac{a \xrightarrow{\text{int}}_{\lambda sg} b}{(c\,\delta)a \xrightarrow{\text{int}}_{\lambda sg} (c\,\delta)b}$$

$$\frac{a \xrightarrow{\text{int}}_{\lambda sg} b}{(\lambda)a \xrightarrow{\text{int}}_{\lambda sg} (\lambda)b} \qquad \frac{a \xrightarrow{\text{int}}_{\lambda sg} b}{(c\,\sigma^i)a \xrightarrow{\text{int}}_{\lambda sg} (c\,\sigma^i)b} \qquad \frac{a \xrightarrow{\text{int}}_{\lambda sg} b}{(\varphi_k^i)a \xrightarrow{\text{int}}_{\lambda sg} (\varphi_k^i)b}$$

Therefore, $\xrightarrow{\text{int}}_{\lambda sg}$ is the least compatible relation closed under $\dfrac{a \longrightarrow_{\lambda sg} b}{(a\,\sigma^i)c \xrightarrow{\text{int}}_{\lambda sg} (b\,\sigma^i)c}$.

Definition 12 *The reduction $\xrightarrow{\text{ext}}_{s}$ is defined by induction. The axioms are the rules of the s-calculus and the inference rules are the following:*

$$\frac{a \xrightarrow{\text{ext}}_{s} b}{(a\,\delta)c \xrightarrow{\text{ext}}_{s} (b\,\delta)c} \quad \frac{a \xrightarrow{\text{ext}}_{s} b}{(c\,\delta)a \xrightarrow{\text{ext}}_{s} (c\,\delta)b} \quad \frac{a \xrightarrow{\text{ext}}_{s} b}{(\lambda)a \xrightarrow{\text{ext}}_{s} (\lambda)b} \quad \frac{a \xrightarrow{\text{ext}}_{s} b}{(c\,\sigma^i)a \xrightarrow{\text{ext}}_{s} (c\,\sigma^i)b} \quad \frac{a \xrightarrow{\text{ext}}_{s} b}{(\varphi_k^i)a \xrightarrow{\text{ext}}_{s} (\varphi_k^i)b}$$

An external (generalized) σ-generation is defined by the rule $(g)\sigma$-generation taken as an axiom and the five inference rules stated above where $\xrightarrow{\text{ext}}_{s}$ is replaced by $\xrightarrow{\text{ext}}_{(g)\sigma-gen}$.

Remark 2 *By inspecting the inference rules one can check that $a \xrightarrow{\text{int}}_{\lambda sg} \mathbf{n}$ is impossible and:*

- *If $a \xrightarrow{\text{int}}_{\lambda sg} (\lambda)b$ then $a = (\lambda)c$ and $c \xrightarrow{\text{int}}_{\lambda sg} b$.*
- *If $a \xrightarrow{\text{int}}_{\lambda sg} (c\,\delta)b$ then $a = (e\,\delta)d$ and $((d \xrightarrow{\text{int}}_{\lambda sg} b$ and $e = c)$ or $(e \xrightarrow{\text{int}}_{\lambda sg} c$ and $d = b))$.*

Note that $\dfrac{a \xrightarrow{\text{ext}}_{s} b}{(a\,\sigma^i)c \xrightarrow{\text{ext}}_{s} (b\,\sigma^i)c}$ and $\dfrac{a \xrightarrow{\text{ext}}_{(g)\sigma-gen} b}{(a\,\sigma^i)c \xrightarrow{\text{ext}}_{(g)\sigma-gen} (b\,\sigma^i)c}$ are excluded from the definitions of external s-reduction and external (generalized) σ-generation, respectively. Thus, as expected, external reductions will not occur at the left of a σ^i operator. This enables us to write $\xrightarrow{+}_{\beta}$ instead of $\twoheadrightarrow_{\beta}$ in the following proposition (compare with Lemma 5).

Proposition 1 *Let $a, b \in \Lambda s$. $a \xrightarrow{\text{ext}}_{\sigma-gen} b \Rightarrow s(a) \xrightarrow{+}_{\beta} s(b)$ and $a \xrightarrow{\text{ext}}_{g\sigma-gen} b \Rightarrow s(a) \xrightarrow{+}_{g\beta} s(b)$.*

Proof: Induction on a, similar to the proof of Lemma 5. The point is that in the case $a = c\, \sigma^i d$, the reduction cannot take place within d because it is external, and this is the only case that forced us to consider the reflexive-transitive closure because of lemma 4.2. $\qquad \square$

The following lemma is needed in Lemma 11 and hence in the Preservation Theorem.

Lemma 10 (Commutation Lemma) *Let $a, b \in \Lambda s$ such that $s(a) \in \lambda$-SN and $s(a) = s(b)$. If $a \xrightarrow{\text{int}}_{\lambda sg} . \xrightarrow{\text{ext}}_s b$ then $a \xrightarrow{\text{ext}}_s^+ . \xrightarrow{\text{int}}_{\lambda sg} b$.*

Proof: By a careful induction on a while analysing the positions of the redexes. The proof is exactly the same as the proof of the Commutation Lemma in [20] $\qquad \square$

Lemma 11 *Let $a \in \lambda g$-SN $\cap \Lambda$. For every infinite λsg-derivation $a \to_{\lambda sg} b_1 \to_{\lambda sg} \cdots \to_{\lambda sg} b_n \to_{\lambda sg} \cdots$, there exists N such that for $i \geq N$ all the reductions $b_i \to_{\lambda sg} b_{i+1}$ are internal.*

Proof: An infinite λsg-derivation must contain infinite $(g)\sigma$-*generations*, since the s-calculus is SN, and the first rule must be a $(g)\sigma$-*generation* beacuse a is a pure term. Hence it looks like: $a = a_1 \to_{(g)\sigma-gen} a_1' \twoheadrightarrow_s a_2 \to_{(g)\sigma-gen} a_2' \twoheadrightarrow_s \cdots \twoheadrightarrow_s a_n \to_{(g)\sigma-gen} a_n' \twoheadrightarrow_s \cdots$. By Proposition 1, there must be only a finite number of *external $(g)\sigma$-generations* (otherwise we construct an infinite λg-derivation contradicting the hypothesis $a \in \lambda g$-SN). Therefore there exists P such that for $i \geq P$ we have $a_i \xrightarrow{\text{int}}_{(g)\sigma-gen} a_i'$. Furthermore, by Lemma 5, $s(a_i) \twoheadrightarrow_{(g)\beta} s(a_i')$ for all i, and therefore
$$a = s(a) = s(a_1) \twoheadrightarrow_{(g)\beta} s(a_1') = s(a_2) \twoheadrightarrow_{(g)\beta} s(a_2') = \cdots = s(a_n) \twoheadrightarrow_{(g)\beta} s(a_n') = \cdots$$
Since $a \in \lambda g$-SN, we conclude that $s(a_i) \in \lambda g$-SN for all i and therefore there exists $M \geq P$ such that for $i \geq M$ we have $s(a_i) = s(a_i')$. We claim that there exists $N \geq M$ such that for $i \geq N$ all the s-rewrites are also internal. Otherwise, there would be an infinity of external s-rewrites and at least one copy of each of these external rewrites can be brought, by the Commutation Lemma, in front of a_M, and so generate an infinite s-derivation beginning at a_M, which is a contradiction. This intuitive idea can be formally stated as:

Fact: *If there exists an infinite derivation $a_M \xrightarrow{\text{ext}}_s^n b \longtwoheadrightarrow^m c \xrightarrow{\text{ext}}_s d \longrightarrow \cdots$ where all the rewrites in $b \longtwoheadrightarrow c$ are either $\xrightarrow{\text{ext}}_s$ or $\xrightarrow{\text{int}}_{\lambda sg}$, then there exists an infinite derivation $a_M \xrightarrow{\text{ext}}_s^{n+1} b' \longtwoheadrightarrow d \longrightarrow \cdots$.*

Proof of Fact: By an easy induction on m, using the Commutation Lemma. We remark that M has been so chosen in order to satisfy the hypothesis of this lemma. $\qquad \square$

In order to prove the Preservation Theorem we need two definitions.

Definition 13 *An infinite λsg-derivation $a_1 \to \cdots \to a_n \to \cdots$ is* minimal *if for every step $a_i \xrightarrow{p}_{\lambda sg} a_{i+1}$, every other derivation beginning with $a_i \xrightarrow{q}_{\lambda sg} a_{i+1}'$ where $p \prec q$, is finite.*

The intuitive idea of a minimal derivation is that if one rewrites at least one of its steps within a subterm of the actual redex, then an infinite derivation is impossible.

Definition 14 *The syntax of* skeletons *and the* skeleton of a term *are defined as follows:*

$$\textbf{Skeletons} \quad K ::= \mathbb{N} \mid (K\, \delta) K \mid (\lambda) K \mid ([.]\, \sigma^j) K \mid (\varphi_k^i) K$$

$$Sk(\mathbf{n}) = \mathbf{n} \qquad Sk((a\, \delta) b) = (Sk(a)\, \delta) Sk(b) \qquad Sk((b\, \sigma^i) a) = ([.]\, \sigma^i) Sk(a)$$
$$Sk((\lambda) a) = (\lambda) Sk(a) \qquad\qquad Sk((\varphi_k^i) a) = (\varphi_k^i) Sk(a)$$

Remark 3 *Let a, $b \in \Lambda s$. If $a \xrightarrow{\texttt{int}}_{\lambda sg} b$ then $Sk(a) = Sk(b)$.*

Theorem 4 (Preservation of λs-SN) *For every $a \in \Lambda$, if a is strongly normalising in the λs-calculus then a is strongly normalising in the λsg-calculus.*

Proof: Suppose $a \in \lambda s$-SN, $a \notin \lambda sg$-SN. Let us consider a minimal infinite λsg-derivation $\mathcal{D} : a \to a_1 \to \cdots \to a_n \to \cdots$. Lemma 11 gives N such that for $i \geq N$, $a_i \to a_{i+1}$ is internal. Hence, by Remark 3, $Sk(a_i) = Sk(a_{i+1})$ for $i \geq N$. As there are only a finite number of closures in $Sk(a_N)$ and as the reductions within these closures are independent, an infinite subderivation of $\mathcal{D}$ must take place within the same and unique closure in $Sk(a_N)$ and, evidently, this subderivation is also minimal. Let us call it $\mathcal{D}'$ and let C be the context such that $a_N = C[(d\,\sigma^i)c]$ and $(d\,\sigma^i)c$ is the closure where $\mathcal{D}'$ takes place. Therefore we have:

$$\mathcal{D}' : a_N = C[(d\,\sigma^i)c] \xrightarrow{\texttt{int}}_{\lambda sg} C[(d_1\,\sigma^i)c] \xrightarrow{\texttt{int}}_{\lambda sg} \cdots \xrightarrow{\texttt{int}}_{\lambda sg} C[(d_n\,\sigma^i)c] \xrightarrow{\texttt{int}}_{\lambda sg} \cdots$$

Since a is a pure term, Lemma 9 ensures the existence of $I \leq N$ such that either

$$a_I = C'[(d'\,\delta)(\lambda)c'] \to a_{I+1} = C'[(d'\sigma^1)c'] \ \text{ and } \ d' \twoheadrightarrow d \ \text{ or}$$

$$a_I = C'[(d'\,\delta)W(\lambda)c'] \to a_{I+1} = C'[W((\varphi_0^{N(W)+1})d'\sigma^1)c'] \ \text{ and } \ d' \twoheadrightarrow d.$$

Let us consider in the first and second cases respectively, the following infinite derivations:

$$\mathcal{D}'' : a \twoheadrightarrow a_I \twoheadrightarrow C'[(d\delta)(\lambda)c'] \to C'[(d_1\delta)(\lambda)c'] \to \cdots \to C'[(d_n\delta)(\lambda)c'] \to \cdots$$

$$\mathcal{D}''' : a \twoheadrightarrow a_I \twoheadrightarrow C'[(d\delta)W(\lambda)c'] \to C'[(d_1\delta)W(\lambda)c'] \to \cdots \to C'[(d_n\delta)W(\lambda)c'] \to \cdots$$

In $\mathcal{D}''$ and $\mathcal{D}'''$, the redex in a_I is within d' which is a proper subterm of $(d'\,\delta)(\lambda)c'$ (of $(d'\,\delta)W(\lambda)c'$ in the second case), whereas in $\mathcal{D}$ the redex in a_I is $(d'\,\delta)(\lambda)c'$ (in the second case $(d'\,\delta)W(\lambda)c'$) and this contradicts the minimality of $\mathcal{D}$. $\qquad\square$

Corollary 3 *For every $a \in \Lambda$, the following equivalences hold:*

$$a \in \lambda g\text{-}SN \Leftrightarrow a \in \lambda sg\text{-}SN \Leftrightarrow a \in \lambda\text{-}SN \Leftrightarrow a \in \lambda s\text{-}SN$$

Proof: By Remark 1 and Theorem 4, $a \in \lambda s$-SN $\Leftrightarrow a \in \lambda sg$-SN. Due to [16], $a \in \lambda$-SN $\Leftrightarrow a \in \lambda g$-SN. Due to [20], $a \in \lambda$-SN $\Leftrightarrow a \in \lambda s$-SN. Hence the corollary. $\qquad\square$

5 The typed λs- and λsg-calculi

We prove λsg-SN of well typed terms using the technique developped in [21] to prove λs-SN and suggested to us by P.-A. Melliès as a successful technique to prove λv-SN (personal communication).

We recall the syntax and typing rules for the simply typed λ-calculus in de Bruijn notation. The types are generated from a set of basic types T with the binary type operator $\to$. Environments are lists of types. Typed terms differ from the untyped ones only in the abstractions which are now marked with the type of the abstracted variable.

Definition 15 *The syntax for the simply typed λ-terms is given as follows:*

Types	$\mathcal{T}$	$::=$	$T \mid \mathcal{T} \to \mathcal{T}$
Environments	$\mathcal{E}$	$::=$	$nil \mid \mathcal{T}, \mathcal{E}$
Terms	Λ_t	$::=$	$\mathbf{n} \mid (\Lambda_t\,\delta)\Lambda_t \mid (\mathcal{T}\,\lambda)\Lambda_t$

We let A, B, ... range over $\mathcal{T}$; E, E_1, ... over $\mathcal{E}$ and a, b, ... over Λ_t.

The typing rules are given by the typing system **L1** *as follows:*

$$(\mathbf{L1} - var) \qquad A, E \vdash \mathtt{1} : A \qquad\qquad (\mathbf{L1} - \lambda) \qquad \frac{A, E \vdash b : B}{E \vdash (A\,\lambda)b : A \to B}$$

$$(\mathbf{L1} - varn) \quad \frac{E \vdash \mathtt{n} : B}{A, E \vdash \mathtt{n+1} : B} \qquad\qquad (\mathbf{L1} - app) \quad \frac{E \vdash b : A \to B \quad E \vdash a : A}{E \vdash (a\,\delta)b : B}$$

Before presenting the simply typed λs- and λsg-calculi we introduce the following notation concerning environments. If E is the environment $E_1, E_2, \ldots, E_n$, we shall use the notation $E_{\geq i}$ for the environment $E_i, E_{i+1}, \ldots, E_n$, analogously $E_{\leq i}$ stands for $E_1, \ldots, E_i$, etc.

Definition 16 *The syntax for the* simply typed λs-terms *is given as follows:*

$$\Lambda s_t \ ::= \ \mathbb{N} \mid (\Lambda s_t\,\delta)\Lambda s_t \mid (\mathcal{T}\,\lambda)\Lambda s_t \mid (\Lambda s_t\,\sigma^i)\Lambda s_t \mid (\varphi_k^i)\Lambda s_t \quad i \geq 1\,, \ k \geq 0.$$

Types and environments are as above. The typing rules of the system **Ls1** *are as follows: The rules* **Ls1**-var, **Ls1**-varn, **Ls1**-λ *and* **Ls1**-app *are exactly the same as* **L1**-var, **L1**-varn, **L1**-λ *and* **L1**-app, *respectively. The new rules are:*

$$(\mathbf{Ls1} - \sigma) \quad \frac{E_{\geq i} \vdash b : B \quad E_{<i}, B, E_{\geq i} \vdash a : A}{E \vdash (b\,\sigma^i)a : A} \qquad (\mathbf{Ls1} - \varphi) \quad \frac{E_{\leq k}, E_{\geq k+i} \vdash a : A}{E \vdash (\varphi_k^i)a : A}$$

The simply typed λs- *and* λsg-calculi *are defined by the same rules of the corresponding untyped versions, except that abstractions in the typed versions are marked with types.*

Definition 17 *We say that $a \in \Lambda s_t$ is a* well typed term *if there exists an environment E and a type A such that $E \vdash_{\mathbf{Ls1}} a : A$. We note Λs_{wt} the set of well typed terms.*

The aim of this section is to prove that every well typed λs-term a is λsg-SN (and hence λs-SN). To do so, we show $\Lambda s_{wt} \subseteq \Xi \subseteq \lambda sg$-SN, where

$$\Xi = \{a \in \Lambda s_t : \text{for every subterm } b \text{ of } a, \ s(b) \in \lambda g\text{-}SN\}.$$

To prove $\Lambda s_{wt} \subseteq \Xi$ (Proposition 2) we need to establish some useful results such as subject reduction, soundness of typing and typing of subterms:

Lemma 12 *Let S be a segment, A, B types and a, b, $c \in \Lambda s_t$. The following hold:*
1. *$E \vdash S((\varphi_0^i)a\,\delta)(c\,\delta)(B\,\lambda)b : A$ iff $E \vdash S(c\,\delta)(B\,\lambda)((\varphi_0^{i+1})a\,\delta)b : A$*
2. *$E \vdash S((\varphi_0^i)(\varphi_0^j)a\,\delta)b : A$ iff $E \vdash S((\varphi_0^{i+j-1})a\,\delta)b : A$*
3. *$E \vdash S(a\,\delta)(B\,\lambda)b : A$ iff $E \vdash S(a\,\sigma^1)b : A$*

Proof: All by induction on S. Here is the proof of two cases of the first item:

$S = \phi$: $E \vdash ((\varphi_0^i)a\,\delta)(c\,\delta)(B\,\lambda)b : A$ iff there exists C such that
$E \vdash (c\,\delta)(B\,\lambda)b : C \to A$ and $E \vdash (\varphi_0^i)a : C$ iff
$E \vdash (B\,\lambda)b : B \to (C \to A)$ and $E \vdash c : B$ and $E_{\geq i} \vdash a : C$ iff
$B, E \vdash b : C \to A$ and $E \vdash c : B$ and $(B, E)_{\geq i+1} \vdash a : C$ iff
$B, E \vdash b : C \to A$ and $E \vdash c : B$ and $B, E \vdash (\varphi_0^{i+1})a : C$ iff
$B, E \vdash ((\varphi_0^{i+1})a\,\delta)b : A$ and $E \vdash c : B$ iff
$E \vdash (B\,\lambda)((\varphi_0^{i+1})a\,\delta)b : B \to A$; and $E \vdash c : B$ iff
$E \vdash (c\,\delta)(B\,\lambda)((\varphi_0^{i+1})a\,\delta)b : A$

$S = (C\,\lambda)S'$: $E \vdash (C\,\lambda)S'((\varphi_0^i)a\,\delta)(c\,\delta)(B\,\lambda)b : A$ iff there exists D such that
$A = C \to D$ and $C, E \vdash S'((\varphi_0^i)a\,\delta)(c\,\delta)(B\,\lambda)b : D$ iff (IH)
$C, E \vdash S'(c\,\delta)(B\,\lambda)((\varphi_0^{i+1})a\,\delta)b : D$ iff $E \vdash (C\,\lambda)S'(c\,\delta)(B\,\lambda)((\varphi_0^{i+1})a\,\delta)b : A$. $\qquad \square$

Lemma 13 (Shuffle Lemma) *Let S be an arbitrary segment, W a well balanced segment and $a, b \in \Lambda s_t$, then $E \vdash S(a\,\delta)W\,b : A$ iff $E \vdash SW((\varphi_0^{N(W)+1})a\,\delta)\,b : A$.*

Proof: By induction on W. If $W = \phi$, it is immediate since $E' \vdash d : D$ iff $E' \vdash (\varphi_0^1)d : D$. Let us assume $W = (c\,\delta)U(B\,\lambda)V$, with U, V well balanced.

$E \vdash S(a\,\delta)(c\,\delta)U(B\,\lambda)V\,b : A$ iff (IH) $E \vdash S(a\,\delta)U((\varphi_0^{N(U)+1})c\,\delta)(B\,\lambda)V\,b : A$ iff (IH)

$E \vdash SU((\varphi_0^{N(U)+1})a\,\delta)((\varphi_0^{N(U)+1})c\,\delta)(B\,\lambda)V\,b : A$ iff (Lemma 12.1)

$E \vdash SU((\varphi_0^{N(U)+1})c\,\delta)(B\,\lambda)((\varphi_0^{N(U)+2})a\,\delta)V\,b : A$ iff (IH, twice)

$E \vdash S(c\,\delta)U(B\,\lambda)V((\varphi_0^{N(V)+1})(\varphi_0^{N(U)+2})a\,\delta)b : A$ iff (Lemma 12.2)

$E \vdash S(c\,\delta)U(B\,\lambda)V((\varphi_0^{N(V)+N(U)+2})a\,\delta)b : A$ $\qquad\qquad\qquad\qquad\qquad\square$

Lemma 14 (Subject reduction) *If $E \vdash_{\mathbf{Ls1}} a : A$ and $a \to_{\lambda sg} b$ then $E \vdash_{\mathbf{Ls1}} b : A$.*

Proof: By induction on a. If the reduction is not at the root, use IH. If it is, check that for each rule $a \to b$ we have $E \vdash_{\mathbf{Ls1}} a : A$ implies $E \vdash_{\mathbf{Ls1}} b : A$. Case σ-*generation*, use lemma 12.3. Case *generalized σ-generation*: If $E \vdash (a\,\delta)W(B\,\lambda)b : A$ then, by Lemma 13, $E \vdash W((\varphi_0^{N(W)+1})a\,\delta)(B\,\lambda)b : A$ and, by Lemma 12.3, $E \vdash W((\varphi_0^{N(W)+1})a\,\sigma^1)b : A$. $\qquad\square$

Corollary 4 *Let $E \vdash_{\mathbf{Ls1}} a : A$, if $a \twoheadrightarrow_{\lambda sg} b$ then $E \vdash_{\mathbf{Ls1}} b : A$.*

Lemma 15 (Typing of subterms) *If $a \in \Lambda s_{wt}$ and b is a subterm of a then $b \in \Lambda s_{wt}$.*

Proof: By induction on a. If b is not an immediate subterm of a, use the induction hypothesis. Otherwise, the last rule used to type a must contain a premise in which b is typed. $\qquad\square$

Lemma 16 (Soundness of typing) *If $a \in \Lambda_t$ and $E \vdash_{\mathbf{Ls1}} a : A$ then $E \vdash_{\mathbf{L1}} a : A$.*

Proof: Easy induction on a. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 2 $\Lambda s_{wt} \subseteq \Xi$.

Proof: Let $a \in \Lambda s_{wt}$ and let b a subterm of a. By Lemma 15, $b \in \Lambda s_{wt}$ and by Corollary 4, $s(b) \in \Lambda s_{wt}$. Since $s(b) \in \Lambda$ (Thm. 2), Lemma 16 yields that $s(b)$ is $\mathbf{L1}$-typable, and it is well known that classical typable λ-terms are strongly normalising in the λ-calculus. Hence, $s(b) \in \lambda$-SN and, by preservation (Corollary 3), $s(b) \in \lambda g$-SN. Therefore $a \in \Xi$. $\qquad\square$

We prove now $\Xi \subseteq \lambda sg$-SN.

Lemma 17 *If $a \in \Xi$ then for every infinite λs-derivation $a \to_{\lambda s} b_1 \to_{\lambda s} \cdots \to_{\lambda s} b_n \to_{\lambda s} \cdots$, there exists N such that for $i \geq N$ all the reductions $b_i \to_{\lambda s} b_{i+1}$ are internal.*

Proof: The proof is almost the same as the proof of lemma 11. $\qquad\qquad\qquad\square$

Proposition 3 *For every $a \in \Lambda s_t$, if $a \in \Xi$ then $a \in \lambda sg$-SN.*

Proof: Suppose there exists $a' \in \Xi$ and $a' \notin \lambda sg$-SN, then there must exist a term a of minimal size such that $a \in \Xi$ and $a \notin \lambda sg$-SN.

Let us consider a minimal infinite λsg-derivation $\mathcal{D} : a \to a_1 \to \cdots \to a_n \to \cdots$ and follow the proof of Theorem 4 to obtain:

$$\mathcal{D}' \; : \; a_N = C[(d\,\sigma^i)c] \xrightarrow{\text{int}}_{\lambda sg} C[(d_1\,\sigma^i)c] \xrightarrow{\text{int}}_{\lambda sg} \cdots \xrightarrow{\text{int}}_{\lambda sg} C[(d_n\,\sigma^i)c] \xrightarrow{\text{int}}_{\lambda sg} \cdots$$

Now three possibilities arise from lemma 9. Two of them have been considered in the proof of Theorem 4 and contradicted the minimality of $\mathcal{D}$. Let us consider the third one: $a = C'[(d'\,\sigma^i)c']$ where $d' \twoheadrightarrow d$. But now we have $d' \twoheadrightarrow d \to d_1 \to \cdots \to d_n \to \cdots$. Since d' is a subterm of a, $d' \in \Xi$, contradicting our choice of a with minimal size. $\qquad\square$

Therefore we conclude, using Propositions 2 and 3 and Corollary 3:

Theorem 5 *Every well typed λs-term is strongly normalising in the λsg-calculus.*

Corollary 5 *Every well typed λs-term is strongly normalising in the λs-calculus.*

6 Conclusion

In this paper, we started from the fact that generalised reduction and explicit substitution have been playing a vital role in useful extensions of the λ-calculus but have never been combined together. We commented that the combination might indeed join both benefits and hence a λ-calculus extended with both needs to be studied. We presented such a calculus and showed that it possesses most of the important properties that have been the center of research for each concept on its own. In particular, we showed that the resulting calculus is confluent, sound and simulates β-reduction. We showed moreover that it preserves strong normalisation of the unextended λ-calculus and of the λ-calculus extended with each of the two concepts independently. We studied furthermore, the simply typed version of our calculus of explicit substitution and generalised reduction and showed that it has again the important properties such as subject reduction, soundness of subtyping, typing of subterms and strong normalisation of well typed terms.

Now that a calculus combining both concepts have been shown to be theoretically correct, it would be interesting to extend our calculus λsg to one that is confluent on open terms as is the tradition with calculi of explicit substitution. It would be also interesting to study the polymorphically (rather than the simply) typed version of λsg. These are issues we are investigating at the moment.

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