On Π-conversion in the λ-cube and the combination with abbreviations

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Abstract

Typed λ-calculus uses two abstraction symbols (λ and Π) which are usually treated in different ways: $\lambda_{x:*}.x$ has as type the abstraction $\Pi_{x:*}.*$ yet $\Pi_{x:*}.*$ has type $\square$ rather than an abstraction; moreover, $(\lambda_{x:A}.B)C$ is allowed and β-reduction evaluates it, but $(\Pi_{x:A}.B)C$ is rarely allowed. Furthermore, there is a general consensus that λ and Π are different abstraction operators. While we agree with this general consensus, we find it nonetheless important to allow Π- to *act* as an abstraction operator. Moreover, experience with AUTOMATH and the recent revivals of Π-reduction as in [KN 95b, PM 97], illustrate the elegance of giving Π-redexes some similar status to λ-redexes. Alas however, Π-reduction in the λ-cube faces serious problems as shown in [KN 95b, PM 97]: it is not safe as regards subject reduction, it does not satisfy type correctness, it loses the property that the type of an expression is well-formed and it fails to make any expression that contains a Π-redex well-formed.

In this paper, we propose a solution to all those problems. The solution is to use a concept that is heavily present in most implementations of programming languages and theorem provers: abbreviations (viz. by means of a definition) or let expressions. We will show that the λ-cube extended with Π-conversion and abbreviations satisfies all the desirable properties of the cube and does not face any of the serious problems of Π-reduction. We believe that this extension of the λ-cube is very useful: it gives a full formal study of two concepts (Π-reduction and abbreviations) that are useful for theorem proving and programming languages.

1 Introduction

Π-reduction and using names to abbreviate large expressions, are useful for automating mathematics, for theorem proving and for programming languages. Evidence of this is their presence in the various implementations of mathematics, theorem proving and programming languages. In what follows, we explicit the advantages and/or problems of these two concepts and we explain why combining them is even more useful.

1.1 On Π-reduction

Type theory has almost always been studied without Π-conversion (which is the analogue of β-conversion on product type level). That is, $\to_\beta$: $(\lambda_{x:A}.b)C \to_\beta b[x := C]$ is always assumed but not $\to_\Pi$: $(\Pi_{x:A}.B)C \to_\Pi B[x := C]$. The exceptions to this are: some AUTOMATH-languages in [NGV 94], the λ-cube extended with Π-reduction in [KN 95b] and the intermediate language in compilers for source languages as in [PM 97]. We claim that $\to_\Pi$ is desirable for the following reasons:

A. Π is, in a sense, a kind of λ. In higher order type theory, arrow-types of the form $A \to B$ are replaced by dependent products $\Pi_{x:A}.B$, where x may be free in B, and thus B *depends on* x. This means that abstraction can be over types: $\Pi_{x:A}.B$ as well as over terms: $\lambda_{x:A}.b$. But, once we allow abstraction over types, it would be nice to discuss the reduction rules which govern these types. In fact, Π is indeed a kind of λ as regards the abstraction over a variable and hence is eligible for an application.

B. Compatibility. Here are two important rules in the λ-cube:

$$\text{(abstraction rule)} \qquad \frac{\Gamma.\langle x : A\rangle \vdash b : B \qquad \Gamma \vdash \Pi_{x:A}.B : S}{\Gamma \vdash \lambda_{x:A}.b : \Pi_{x:A}.B}$$

$$\text{(application rule)} \qquad \frac{\Gamma \vdash F : \Pi_{x:A}.B \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]}$$

The (abstraction rule) may be regarded as the compatibility property for the typing with respect to abstraction. That is: $b : B \Rightarrow \lambda_{x:A}.b : \Pi_{x:A}.B$.

The compatibility property for the typing with respect to application is lost however. In fact, from the (application rule), one does not have: $F : \Pi_{x:A}.B \Rightarrow Fa : (\Pi_{x:A}.B)a$,

but instead $F : \Pi_{x:A}.B \Rightarrow Fa : B[x := a]$.

To get compatibility for the typing with respect to application, one needs to add $\to_\Pi$ and to change the (application rule) to:

$$\text{(new application rule)} \qquad \frac{\Gamma \vdash F : \Pi_{x:A}.B \qquad \Gamma \vdash a : A}{\Gamma \vdash Fa : (\Pi_{x:A}.B)a}$$

C. The AUTOMATH experience. One might argue that *implicit* Π-reduction (as is the case of the ordinary λ-cube with the (application rule) above) is closer to intuition in the most usual applications. However, experiences with the AUTOMATH-languages ([NGV 94]), containing *explicit* Π-reduction, demonstrated that there exists no formal or informal objection against the use of this explicit Π-reduction in natural applications of type systems.

D. Preference types, higher degrees, conversion. In [KN 95b], Π-reduction was shown to have various advantages which include calculating the preference type $\tau(\Gamma, A)$ of a term A in a context Γ, the ability of incorporating different degrees (rather than just the two, λ and Π, as in the λ-cube), the splitting of $\Gamma \vdash A : B$ into $\Gamma \vdash A$ (A is typable in Γ) and $\tau(\Gamma, A) = B$ (B is convertible to the preference type of A), and the getting rid of the following rule of the λ-cube:

$$\text{(conversion rule)} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : S \qquad B = B'}{\Gamma \vdash A : B'}$$

E. Programming Languages. In programming language studies, a thriving area is that of the use of richly-typed intermediate languages in sophisticated compilers for higher-order,

typed source languages ([PJ 96, SA 95, TA 96]). The latest language [PM 97] proposes to reduce the number of data types and the volume of code required in the compiler by getting rid of all duplications. To do this, [PM 97] uses the whole λ-cube extended with Π-reduction and gives the following explanation:

- This isolates reduction into one set of rules $=_\beta$.
- With the old application rule, matters get very complicated when one adds further expressions (such as let and case).
- In a compiler, Π-reduction allows to separate the type finder from the evaluator since $\vdash$ no longer mentions substitution. One first extracts the type and only then evaluates it.

All the above are reasons why it is interesting to study Π-conversion in the λ-cube. Alas however, Π-conversion added to the λ-cube is not a straightforward adding of $(\Pi_{x:A}.B)C \to_\Pi B[x := C]$ and of the (new application rule) (see [KN 95b]). Changing in the λ-cube the (application rule) to the (new application rule) results in the following problems:

1. **The correctness of types no longer holds.** With Π-reduction, one can have $\Gamma \vdash A : B$ without $B \equiv \Box$ or $\exists S\,.\,\Gamma \vdash B : S$. For example, $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$ yet $(\Pi_{y:z}.z)x \not\equiv \Box$ and $\forall S\,.\,\langle z : *\rangle.\langle x : z\rangle \not\vdash (\Pi_{y:z}.z)x : S$. The problem arises because of the new terms that contain Π-redexes (which did not exist in the λ-cube) and because [KN 95b] showed that in the λ-cube extended with Π-reduction:

   $$(\dagger) \qquad \forall \Gamma, A, B, C, x, S\,.\,\Gamma \not\vdash (\Pi_{x:A}.B)C : S$$

2. **The system is no longer safe**. More precisely, subject reduction (SR) fails. That is, with Π-reduction and the (new application rule), $\Gamma \vdash A : B$ and $A \twoheadrightarrow A'$ may not imply $\Gamma \vdash A' : B$. For example, $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$ and $(\lambda_{y:z}.y)x \to x$, but one can't show $\langle z : *\rangle.\langle x : z\rangle \vdash x : (\Pi_{y:z}.z)x$. To show this last formula, one needs to use the above (conversion rule) and for this, one needs that $\exists S.\langle z : *\rangle.\langle x : z\rangle \vdash (\Pi_{y:z}.z)x : S$. But $(\dagger)$ in 1 above make this impossible.

3. **The type of an expression may not be well-formed.** This is related to type correctness above. We say that A is well-formed if $A \equiv \Box$ or $\exists \Gamma, B\,.\,\Gamma \vdash A : B$. Now take $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$. The type of $(\lambda_{y:z}.y)x$, namely $(\Pi_{y:z}.z)x$ is not well-formed because for every $\Gamma, B, \Gamma \not\vdash (\Pi_{y:z}.z)x : B$ as [KN 95b] showed that:

   $(\dagger\dagger)$ If $\Gamma \vdash A : B$ then neither Γ nor A contain Π-redexes and if B contained a Π-redex, then B is itself that Π-redex.

4. **Π-redexes are not well-formed.** From $(\dagger\dagger)$ above, no expression that contains a Π-redex is well-formed.

Despite these shortcomings of Π-reduction, [PM 97] claims that its advantages are persuasive. In this paper we will repair the shortcomings of Π-reduction. It is amazing that the way to repair the problem is itself a very useful way in type theory: the use of abbreviations.

1.2 On abbreviations

In many type theories and lambda calculi, there is no formal possibility to use abbreviations, i.e. to introduce names for large expressions which can be used several times in a program or a proof. This possibility is essential for practical use, and indeed implementations of Pure Type Systems such as Coq [Dow 91], Lego [LP 92] and Nuprl [Con 86] do provide this possibility. Moreover, most implementations of programming languages (Haskell, ML, CAML, etc.) use names for large expressions via a familiar programming language concept: *let expressions*.

Example 1.1 $\texttt{Let}\ id : A \to A\ \texttt{be}\ (\lambda_{x:A}.x)\ \texttt{in}\ (\lambda_{y:A\to A}.id)id$ abbreviates the complex expression $(\lambda_{x:A}.x)$ as id in a more complex expression in which id occurs two times.

The intended meaning of "$\texttt{let}\ x : A\ \texttt{be}\ a\ \texttt{in}\ b$" is that a can be substituted for x in the expression b. In a sense, the expression $\texttt{let}\ x : A\ \texttt{be}\ a\ \texttt{in}\ b$ is similar to $(\lambda_{x:A}.b)a$. It is not intended however to necessarily replace all the occurrences of x in b by a. Nor is it intended that such a let expression is a part of our term. Rather, the let expression will live in the environment (or context) in which we evaluate or reason about the expression.

One of the advantages of the expression $\texttt{let}\ x : A\ \texttt{be}\ a\ \texttt{in}\ b$ over the redex $(\lambda_{x:A}.b)a$ is that it is convenient to have the freedom of substituting only some of the occurrences of an expression in a given formula. Another advantage is efficiency; one evaluates a in $\texttt{let}\ x : A\ \texttt{be}\ a\ \texttt{in}\ b$ only once, even in lazy languages. A further advantage is that using x to be a in b can be used to type b efficiently, since the type A of a has to be calculated only once. Furthermore, practical experiences with type systems show that let expressions are absolutely indispensable for any realistic application. Without let expressions, terms soon become forbiddingly complicated. By using let expressions one can avoid such an explosion in complexity. This is, by the way, a very natural thing to do: the apparatus of mathematics, for instance, is unimaginable without a form of let expressions (viz. definitions).

There exist already two formal studies of let expressions in the λ-cube [BKN 96, SP 93] where those let expressions are called *definitions*. In this paper we differ from both accounts and use the simplest way of renaming large expressions and describe such renaming as *abbreviations*. We differ from [SP 93] in that we do not introduce new terms (let terms) into our syntax and do not extend β-reduction to deal with those new terms. We differ from [BKN 96] in that we do not use nested definitions, which are needed for generalised reduction in [BKN 96] and not for Π-reduction.

We write $\langle x : A\rangle a$ to describe that x of type A, abbreviates a. We include abbreviations in contexts such that if an abbreviation occurs in a context then it can be used anywhere in the term we are reasoning about in that context.

In this paper, we will use abbreviations to repair all the problems of Π-reduction mentioned in Subsection 1.1. During this process, we shall show that the λ-cube extended with abbreviations satisfies all its original properties.

1.3 On combining abbreviations and Π-reductions

We shall show in this paper that the λ-cube extended with both abbreviations and Π-reduction ($\to_\Pi$ and the new application rule), preserves all its original properties (including safety, correctness of types and well-formedness of the type of an expression) and allows a non-limited occurrence of Π-redexes in the well-formed terms. This means that abbreviations (being important on their own) have repaired the problems of Π-reduction in the λ-cube. Let us here explain why the shortcomings of Π-reduction disappear with abbreviations:

Looking at the four problems of Π-reduction in Subsection 1.1, one sees that one needs to be able to type Π-redexes. This is not possible if the λ-cube was simply extended with the (new application rule) and $\rightarrow_\Pi$ as [KN 95b] showed. There are several ways one can follow:

A. Infinite levels of abstractions One may abandon the two levels (λ and Π) as used in the current type systems and reformulate all type theory using different levels of λ's, where say, λ^0 is what we call λ, λ^1 is the Π, etc. Then, one will be able to give every $\lambda^n_{x:A}.B$ the type $\lambda^{n+1}_{x:A}.C$ where C is a type of B, and the current type theory will be reproduced at each level. This would be rather interesting to investigate, but we feel it a drastic change from current type theory and we are not sure what complications or contradictions will arise from different levels of λ's. We leave it as a point for future research.

B. Abbreviations One may introduce (λ- and Π-) redexes as a separate (compound) term, which can be typed using abbreviations. In the type, the abbreviation is unfolded. The idea is simple: extend contexts with abbreviations and add the following rule:

$$\text{(abb rule)} \qquad \frac{\Gamma.\langle x : A\rangle B \vdash C : D}{\Gamma \vdash (\pi_{x:A}.C)B : D[x := B]} \text{ where } \pi \in \{\lambda, \Pi\}$$

This rule says that if $C : D$ can be typed using the abbreviation that x of type A is B, then $(\pi_{x:A}.C)B : D[x := B]$ can be typed without this abbreviation. This simple extension solves all the problems of Π-reduction mentioned in Subsection 1.1. Here is how:

1. **Type correctness.** We demonstrate this with the example of problem 1 of Subsection 1.1. Recall that we have $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$ and want that, $\exists S \,.\, \langle z : *\rangle.\langle x : z\rangle \vdash (\Pi_{y:z}.z)x : S$. Here is how the latter formula now holds:

$$\begin{array}{ll} \langle z : *\rangle.\langle x : z\rangle \vdash z : * & \text{(start and weakening)} \\ \langle z : *\rangle.\langle x : z\rangle.\langle y : z\rangle x \vdash z : * & \text{(weakening)} \\ \langle z : *\rangle.\langle x : z\rangle \vdash (\Pi_{y:z}.z)x : *[y := x] \equiv * & \text{(abb rule)} \end{array}$$

2. **Safety or subject reduction.** We demonstrate this with the example of problem 2 of Subsection 1.1. Recall that we have $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$ and $(\lambda_{y:z}.y)x \rightarrow x$ and we need to show that $\langle z : *\rangle.\langle x : z\rangle \vdash x : (\Pi_{y:z}.z)x$. Here is how the latter formula now holds:

$$\begin{array}{lll} a. & \langle z : *\rangle.\langle x : z\rangle \vdash x : z & \text{(start and weakening)} \\ b. & \langle z : *\rangle.\langle x : z\rangle \vdash (\Pi_{y:z}.z)x : * & \text{(from 1 above)} \\ & \langle z : *\rangle.\langle x : z\rangle \vdash x : (\Pi_{y:z}.z)x & \text{(conversion, } a, b, \text{ and } z = (\Pi_{y:z}.z)x) \end{array}$$

3. **Well-formedness of the types of expressions.** We demonstrate this with the example of problem 3 of Subsection 1.1. Recall that we have $\langle z : *\rangle.\langle x : z\rangle \vdash (\lambda_{y:z}.y)x : (\Pi_{y:z}.z)x$ and we want to show that $(\Pi_{y:z}.z)x$ is typable (note that $(\Pi_{y:z}.z)x \not\equiv \Box$). By 1 above, we have that $\langle z : *\rangle.\langle x : z\rangle \vdash (\Pi_{y:z}.z)x : *$.

4. **Unlimited Π-redexes.** Π-redexes can now occur in contexts, terms, and types and all expected ones are indeed well-formed.

Remark 4.2, Lemma 4.12 and Theorem 4.13 will show that indeed all the problems of Π-reduction are solved. Intuitively, the reason is that abbreviations keep information in the context about the defined values of variables.

We divide the paper as follows:

1. In Section 2, we set up the machinery for both abbreviations and Π-reduction.

2. In Section 3, we introduce the original relation of the λ-cube $\vdash_\beta$ and the extended relation $\vdash_{\beta\Pi}$ as in [KN 95b]. We list the properties of both $\vdash_\beta$ and $\vdash_{\beta\Pi}$.

3. In Section 4, we introduce $\vdash_{re}$ which is $\vdash_r$ (for $r = \beta$ or $\beta\Pi$) extended with abbreviations. We show that all the properties of the λ-cube remain valid for $\vdash_{re}$. This establishes that extending the λ-cube with abbreviations or with both abbreviations and Π-reduction results in a well-behaved system. Due to the elegance of our presentation, we prove almost all the results for $\vdash_{re}$ rather than separately prove them for $\vdash_{\beta e}$ and $\vdash_{\beta\Pi e}$.

2 The Formal Machinery

The systems of the λ-cube (see [Ba 92]), are based on a set of *pseudo-expressions* or *terms* $\mathcal{T}$ defined by the following abstract syntax:

$$\mathcal{T} = * \mid \Box \mid V \mid \mathcal{T}\mathcal{T} \mid \pi_{V:\mathcal{T}}.\mathcal{T}$$

where $\pi \in \{\lambda, \Pi\}$, V is an infinite collection of variables over which $\alpha, \beta, x, y, z, \ldots$ range, $*$ and $\Box$ are sorts over which $S, S_1, S_2, \ldots$ range. We let $A, B, a, b \ldots$ range over $\mathcal{T}$.

Bound and free variables and substitution are defined as usual where the binding power of Π is similar to that of λ. We write $BV(A)$ and $FV(A)$ to represent the bound and free variables of A respectively. We write $A[x := B]$ to denote the term where all the free occurrences of x in A have been replaced by B. Furthermore, we take terms to be equivalent up to variable renaming and $\equiv$ for syntactic equality. For example, we take $\lambda_{x:A}.x \equiv \lambda_{y:A}.y$. We assume moreover, the Barendregt variable convention which is formally stated as follows:

Convention 2.1 *(BC: Barendregt's Convention)*
Names of bound variables will always be chosen such that they differ from the free ones in a term. Moreover, different abstraction operators have different variables as subscript. Hence, we will not have $(\pi_{x:A}.x)x$, but $(\pi_{y:A}.y)x$ instead.

Terms are related via a reduction relation $\to_r$. We assume the usual definition of the compatibility of a reduction relation, and define $\twoheadrightarrow_r$ to be its reflexive transitive closure and $=_r$ to be its equivalence closure. We take $r \in \{\beta, \beta\Pi\}$ throughout and use the relations: $\to_\beta$ and $\to_{\beta\Pi}$ generated by: $(\lambda_{x:A}.B)C \to_\beta B[x := C]$, and $(\Pi_{x:A}.B)C \to_{\beta\Pi} B[x := C]$ respectively.

In the following definition, declarations are familiar from the λ-cube. $\subseteq'$ says that changing a declaration into an abbreviation which preserves that declaration, does not decrease the information in the pseudocontext. We let $d, d_1, d_2, \ldots$ range over declarations and abbreviations and $\Gamma, \Delta, \Gamma', \Gamma_1, \Gamma_2, \ldots$ over pseudocontexts.

Definition 2.2 *(declarations, abbreviations, pseudocontexts, $\subseteq'$)*

1. *A declaration d is of the form $\langle x : A\rangle$. We define $\texttt{var}(d) \equiv x$ and $\texttt{type}(d) \equiv A$.*

2. *An abbreviation d is of the form $\langle x : A\rangle B$ and uses x of type A to abbreviate B. We define $\texttt{var}(d)$, $\texttt{type}(d)$ and $\texttt{ab}(d)$ to be x, A, and B respectively.*

3. *A pseudocontext Γ is a (possibly empty) concatenation of declarations and abbreviations $d_1.d_2.\cdots.d_n$ such that if $i \neq j$, then $\texttt{var}(d_i) \not\equiv \texttt{var}(d_j)$.*

4. *Define* $dom(\Gamma) = \{\mathtt{var}(d) \mid d \in \Gamma\}$, Γ-$\mathtt{decl} = \{d \in \Gamma \mid d$ *is a declaration* $\}$ *and* Γ-$\mathtt{abb} = \{d \in \Gamma \mid d$ *is an abbreviation* $\}$. *Note that* $dom(\Gamma) = \{\mathtt{var}(d) \mid d \in \Gamma\text{-}\mathtt{decl} \cup \Gamma\text{-}\mathtt{abb}\}$.

5. *Define* $\subseteq'$ *between pseudocontexts as the least reflexive transitive relation satisfying:*
   - $\Gamma.\Delta \subseteq' \Gamma.d.\Delta$ *for* d *a declaration or an abbreviation.*
   - $\Gamma.\langle x : A\rangle.\Delta \subseteq' \Gamma.\langle x : A\rangle B.\Delta$

In the rest of this section, we let $\vdash$ be a notion of derivability. The following is familiar:

Definition 2.3 *Let* Γ *be a pseudocontext.*

1. $A : B$ *is called a statement.* A *and* B *are its subject and predicate.*
2. $\Gamma \vdash_r A : B$ *is called a judgement, and* $\Gamma \vdash_r A : B : C$ *denotes* $\Gamma \vdash_r A : B \wedge \Gamma \vdash_r B : C$.
3. Γ *is called legal if* $\exists P, Q \in \mathcal{T}$ *such that* $\Gamma \vdash_r P : Q$.
4. $A \in \mathcal{T}$ *is called a* Γ*-term if* $\exists B \in \mathcal{T}[\Gamma \vdash_r A : B \vee \Gamma \vdash_r B : A]$.
5. $A \in \mathcal{T}$ *is called legal if* $\exists \Gamma[A$ *is a* Γ*-term*$]$.

The following is needed in the conversion rule where we replace $A =_r B$ by $\Gamma \vdash_r A =_{\mathtt{ab}} B$.

Definition 2.4 *(Abbreviational r-equality) For all pseudocontexts* Γ *we define the binary relation* $\Gamma \vdash_r \cdot =_{\mathtt{ab}} \cdot$ *to be the equivalence relation generated by*

- *if* $A =_r B$ *then* $\Gamma \vdash_r A =_{\mathtt{ab}} B$
- *if* $d \in \Gamma$-$\mathtt{abb}$ *and* $A, B \in \mathcal{T}$ *such that* B *arises from* A *by substituting one particular free occurrence of* $\mathtt{var}(d)$ *in* A *by* $\mathtt{ab}(d)$, *then* $\Gamma \vdash_r A =_{\mathtt{ab}} B$.

Remark 2.5 If no abbreviations are present in Γ then $\Gamma \vdash_r A =_{\mathtt{ab}} B$ is the same as $A =_r B$.

The following definition groups some preconditions of some typing rules. For example, instead of postulating for the start rule (in the case of a declaration) that $\Gamma \vdash \mathtt{type}(d) : S$ and $\mathtt{var}(d) \notin \Gamma$, we say $\Gamma \prec d$. This becomes particularly useful in the case of abbreviations.

Definition 2.6 *For* $d \in \Gamma$-$\mathtt{abb} \cup \Gamma$-$\mathtt{decl}$, *we say* Γ *invites* d, *notation* $\Gamma \prec d$, *iff*

- $\Gamma.d$ *is a pseudocontext*
- $\Gamma \vdash_r \mathtt{type}(d) : S$ *for some sort* S.
- *if* d *is an abbreviation then* $\Gamma \vdash_r \mathtt{ab}(d) : \mathtt{type}(d)$

Finally, the following definition is again familiar from the λ-cube, but we extend it to deal with abbreviations. That is, $\Gamma \vdash_r (\pi_{x:A}.C)B$ (recall that $\pi \in \{\lambda, \Pi\}$) iff $\Gamma \vdash_r x : A$, $\Gamma \vdash_r B : A$, $\Gamma.\langle x : A\rangle \vdash C$ and $\Gamma \vdash_r x =_{\mathtt{ab}} B$.

Definition 2.7 *Let* Γ *be a pseudocontext. Let* $d, d_1, \ldots, d_n$ *be declarations and abbreviations. We define* $\Gamma \vdash_r d$ *and* $\Gamma \vdash_r d_1 \cdots d_n$ *as follows:*

- *If* d *is a declaration then* $\Gamma \vdash_r d$ *iff* $\Gamma \vdash_r \mathtt{var}(d) : \mathtt{type}(d)$. *Else, if* d *is an abbreviation then* $\Gamma \vdash_r d$ *iff* $\Gamma \vdash_r \mathtt{var}(d) : \mathtt{type}(d) \wedge \Gamma \vdash_r \mathtt{ab}(d) : \mathtt{type}(d) \wedge \Gamma \vdash_r \mathtt{var}(d) =_{\mathtt{ab}} \mathtt{ab}(d)$.
- $\Gamma \vdash_r d_1 \cdots d_n$ *iff* $\Gamma.d_1.\cdots.d_{i-1} \vdash_r d_i$ *for all* $1 \leq i \leq n$.

3 Extending the λ-cube with Π-reduction

In the λ-cube as presented in [Ba 92], the only declarations allowed are of the form $\langle x : A\rangle$. Hence there are no abbreviations in the contexts. Therefore, $\Gamma \prec d$ is of the form $\Gamma \prec \langle x : A\rangle$ and means that $\Gamma \vdash A : S$ for some S and that x is fresh in Γ, A. Moreover, recall that $\mathtt{var}(\langle x : A\rangle) \equiv x$ and $\mathtt{type}(\langle x : A\rangle) \equiv A$ and Π-reduction is not allowed.

Definition 3.1 *($\vdash_\beta$)*
The axioms and rules of the λ-cube are as follows (d **is a declaration,** *$=_{\mathtt{ab}}$* **is** *$=_\beta$):*

(axiom) $\langle\rangle \vdash_\beta * : \Box$

(start rule) $$\frac{\Gamma \prec d}{\Gamma.d \vdash_\beta \mathtt{var}(d) : \mathtt{type}(d)}$$

(weakening rule) $$\frac{\Gamma \prec d \qquad \Gamma \vdash_\beta D : E}{\Gamma.d \vdash_\beta D : E}$$

(formation rule) $$\frac{\Gamma \vdash_\beta A : S_1 \qquad \Gamma.\langle x : A\rangle \vdash_\beta B : S_2}{\Gamma \vdash_\beta \Pi_{x:A}.B : S_2} \quad \textit{if } (S_1, S_2) \textit{ is a rule}$$

(abstraction rule) $$\frac{\Gamma.\langle x : A\rangle \vdash_\beta b : B \qquad \Gamma \vdash_\beta \Pi_{x:A}.B : S}{\Gamma \vdash_\beta \lambda_{x:A}.b : \Pi_{x:A}.B}$$

(application rule) $$\frac{\Gamma \vdash_\beta F : \Pi_{x:A}.B \qquad \Gamma \vdash_\beta a : A}{\Gamma \vdash_\beta Fa : B[x := a]}$$

(conversion rule) $$\frac{\Gamma \vdash_\beta A : B \qquad \Gamma \vdash_\beta B' : S \qquad \Gamma \vdash_\beta B =_{\mathtt{ab}} B'}{\Gamma \vdash_\beta A : B'}$$

Each of the eight systems of the λ-cube is obtained by taking the (S_1, S_2) rules allowed from a subset of $\{(*,*), (*,\Box), (\Box,*), (\Box,\Box)\}$. These systems are given in the following table:

System	*Set of specific rules*			
$\lambda_\to$	$(*,*)$			
$\lambda 2$	$(*,*)$	$(\Box,*)$		
λP	$(*,*)$		$(*,\Box)$	
$\lambda P2$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$
$\lambda P\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$
$\lambda P\omega = \lambda C$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$

[KN 95b] extended this λ-cube by changing $\twoheadrightarrow_\beta$ to $\twoheadrightarrow_{\beta\Pi}$ and by changing $\vdash_\beta$ to $\vdash_{\beta\Pi}$ (note that $\Gamma \vdash_{\beta\Pi} B =_{\mathtt{ab}} B'$ is the same as $B =_{\beta\Pi} B'$, as there are no abbreviations):

Definition 3.2 *($\vdash_{\beta\Pi}$)*
$\vdash_{\beta\Pi}$ is $\vdash_\beta$ where β is replaced by $\beta\Pi$ throughout, and the application rule changes to:

(new application rule) $$\frac{\Gamma \vdash_{\beta\Pi} F : \Pi_{x:A}.B \qquad \Gamma \vdash_{\beta\Pi} a : A}{\Gamma \vdash_{\beta\Pi} Fa : (\Pi_{x:A}.B)a}$$

Now we list some properties of $\vdash_\beta$ and $\vdash_{\beta\Pi}$ without proofs (see [KN 95b]). These properties (except of course the loss of type correctness, of SR and the non well-formedness of some types and of Π-redexes) will be established for the λ-cube extended with either abbreviations alone, or with both abbreviations and Π-reduction in Section 4.

Theorem 3.3 *(The Church Rosser Theorem CR, for $\twoheadrightarrow_r$, $r = \beta$ or $\beta\Pi$)*
If $A \twoheadrightarrow_r B$ and $A \twoheadrightarrow_r C$ then there exists D such that $B \twoheadrightarrow_r D$ and $C \twoheadrightarrow_r D$ □

Lemma 3.4 *(Start Lemma for $\vdash_r$ for $r = \beta$ or $\beta\Pi$)*
Let Γ be a $\vdash_r$-legal context. Then $\Gamma \vdash_r * : \Box$ and $\forall d \in \Gamma[\Gamma \vdash_r d]$. □

Lemma 3.5 *(Correctness of types for $\vdash_\beta$,* not *for $\vdash_{\beta\Pi}$)*
For $r = \beta$, but not *for $r = \beta\Pi$ we have:*
If $\Gamma \vdash_r A : B$ then ($B \equiv \Box$ or $\Gamma \vdash_r B : S$ for some sort S).
(cf. the counterexample of Subsection 1.1, Problem 1.) □

Lemma 3.6 *(Subject Reduction SR, for $\vdash_\beta$,* not *for $\vdash_{\beta\Pi}$)*
For $r = \beta$, but not *for $r = \beta\Pi$ we have:*
If $\Gamma \vdash_r A : B$ and $A \twoheadrightarrow_\beta A'$ then $\Gamma \vdash_r A' : B$
(cf. Subsection 1.1, Problem 2.) □

However, a weak form of SR holds for $\vdash_{\beta\Pi}$. First we need the following definition which removes all Π-redexes of a $\vdash_{\beta\Pi}$-legal term:

Definition 3.7 *For A $\vdash_{\beta\Pi}$-legal, let $\hat{A}$ be $C[x := D]$ if $A \equiv (\Pi_{x:B}.C)D$ and A otherwise.*

Lemma 3.8 *(Weak Subject Reduction for $\vdash_{\beta\Pi}$ and $\twoheadrightarrow_{\beta\Pi}$)*
$\Gamma \vdash_{\beta\Pi} A : B \wedge A \twoheadrightarrow_{\beta\Pi} A' \Rightarrow \Gamma \vdash_{\beta\Pi} A' : \hat{B}$ □

Lemma 3.9 *(Well-formedness of types for $\vdash_\beta$,* not *for $\vdash_{\beta\Pi}$)*
For $r = \beta$, but not *for $r = \beta\Pi$ we have:*
If $\Gamma \vdash_r A : B$ then $B = \Box$ or $\exists C . \Gamma \vdash_r B : C$.
(cf. Subsection 1.1, Problem 3.) □

Lemma 3.10 *(Non-well-formedness of Π-redexes)*
For no Γ, A, B and C there is D such that $\Gamma \vdash_{\beta\Pi} (\Pi_{x:A}.B)C : D$.
(cf. Subsection 1.1, Problem 4.) □

Lemma 3.11 *(Uniqueness of Types for $\vdash_r$ and $\twoheadrightarrow_r$ for $r = \beta$ or $\beta\Pi$)*
$\Gamma \vdash_r A : B_1 \wedge \Gamma \vdash_r A : B_2 \Rightarrow B_1 =_r B_2$ □

Theorem 3.12 *(Strong Normalisation with respect to $\vdash_r$ and $\rightarrow_r$ for $r = \beta$ or $\beta\Pi$)*
If A is $\vdash_r$-legal then $SN_{\rightarrow_r}(A)$; i.e. A is strongly normalising with respect to $\rightarrow_r$. □

In the rest of the paper, we use the Π-cube to denote the λ-cube extended *with* Π*-reduction* and with the *new application rule.* We write π-cube for either the λ- or the Π-cube. Recall that $r \in \{\beta, \beta\Pi\}$ and $\pi \in \{\lambda, \Pi\}$.

4 Extending the π-cube with abbreviations

We shall extend the derivation rules of $\vdash_r$ so that we can use abbreviations in the context. The rules remain unchanged except for the following points:

- One rule, the (*abb rule*), is added.
- Not only declarations but also abbreviations are allowed in contexts.
- The use of $\Gamma \vdash B =_{\mathtt{ab}} B'$ in the conversion rule really has an effect, since $=_{\mathtt{ab}}$ is now a real extension of $=_r$ and Γ may contain abbreviations necessary to establish $B =_{\mathtt{ab}} B'$.

Note that the intended scope of $\langle x : A\rangle$ in $\Gamma.\langle x : A\rangle B.\Delta \vdash_r C : D$ is Δ, C and D. This is what should be expected since the scope in $\Gamma.\langle x : A\rangle.\Delta \vdash_r C : D$ is the same.

Definition 4.1 *(Axioms and rules of the λ-cube extended with abbreviations: d ranges over declarations and abbreviations. Recall that when $r = \beta$, π is only λ everywhere.)*

We extend the relation $\vdash_r$ to $\vdash_{re}$ by adding the following abbreviation rule:

$$\textit{(abb rule)} \qquad \frac{\Gamma.\langle x : A\rangle B \vdash_{re} C : D}{\Gamma \vdash_{re} (\pi_{x:A}.C)B : D[x := B]}$$

The (*abb rule*) says that if $C : D$ can be deduced using an abbreviation $d \equiv \langle x : A\rangle B$, then $(\pi_{x:A}.C)B$ will be of type D where d has been unfolded in D.

Remark 4.2 *(Well-Formedness of Π-redexes for $\vdash_{\beta\Pi e}$)* From the (abb rule) and the (new application rule), if $\Gamma \vdash_{\beta\Pi e} A : B$ then both A and B may contain Π-redexes.

Remark 4.3 When considering an abbreviation in a term to be equivalent to a redex, the (abb rule) is quite natural: for instance, deriving a type for $(\lambda_{x:\alpha}.x)y$ via abbreviating y to be x gives the same type as the derivation via abstraction followed by application (let $\Gamma \equiv \langle\alpha : *\rangle.\langle y : \alpha\rangle$):

$$\text{(abb rule)} \quad \frac{\Gamma.\langle x : \alpha\rangle y \vdash_{re} x : \alpha}{\Gamma \vdash_{re} (\lambda_{x:\alpha}.x)y : \alpha[x := y]}$$

$$\text{(appl)} \quad \frac{\text{(abstr)} \quad \dfrac{\Gamma.\langle x : \alpha\rangle \vdash_{re} x : \alpha \qquad \Gamma \vdash_{re} (\Pi_{x:\alpha}.\alpha) : *}{\Gamma \vdash_{re} \lambda_{x:\alpha}.x : \Pi_{x:\alpha}.\alpha} \quad \Gamma \vdash_{re} y : \alpha}{\Gamma \vdash_{re} (\lambda_{x:\alpha}.x)y : \alpha[x := y]}$$

Let us now give an example which shows why abbreviations are useful:

Example 4.4 $\langle\beta : *\rangle.\langle y : \beta\rangle \not\vdash_r (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y)\beta : \beta$. We need $y : \alpha$ to be able to type $(\lambda_{x:\alpha}.x)y$. Looking carefully however, we find that $(\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y)\beta$ is abbreviating β by α. So here is how the above derivation can be obtained using abbreviations (we present a short-cut and do not mention all the steps, nor the names of the rules):

$\langle\beta : *\rangle.\langle y : \beta\rangle.\langle\alpha : *\rangle\beta \vdash_{re} \lambda_{x:\alpha}.x : \Pi_{x:\alpha}.\alpha$
$\langle\beta : *\rangle.\langle y : \beta\rangle.\langle\alpha : *\rangle\beta \vdash_{re} y : \beta$
$\langle\beta : *\rangle.\langle y : \beta\rangle.\langle\alpha : *\rangle\beta \vdash_{re} \alpha =_{\mathtt{ab}} \beta$
$\langle\beta : *\rangle.\langle y : \beta\rangle.\langle\alpha : *\rangle\beta \vdash_{re} y : \alpha$
$\langle\beta : *\rangle.\langle y : \beta\rangle.\langle\alpha : *\rangle\beta \vdash_{re} (\lambda_{x:\alpha}.x)y : \alpha$
$\langle\beta : *\rangle.\langle y : \beta\rangle \vdash_{re} (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y)\beta : \alpha[\alpha := \beta]$
$\langle\beta : *\rangle.\langle y : \beta\rangle \vdash_{re} (\lambda_{\alpha:*}.(\lambda_{x:\alpha}.x)y)\beta : \beta$

Remark 4.5 In [BKN 96], we introduced a notion of generalised definitions which, like abbreviations, bind a name to a complex expression. In [BKN 96] however, definitions were nested. Such nesting is unnecessary for the reductions we are using in the present paper. In [BKN 96], reduction was generalised due to the use of a useful notation (see [KN 95a]). With that generalisation of reduction (which may contract some redex r before other redexes upon which this r depends have been contracted), definitions had to be nested to mirror this generalised reduction.

In the present paper, we do not consider generalised reduction or nested definitions. Instead, we study ordinary (non-nested) definitions combined with Π-reduction. We note that any abbreviation in the sense of the present paper is also a definition in the sense of [BKN 96] when the notation is changed. Furthermore any type derivation with abbreviations in this paper (not involving Π-reduction) is also a type derivation with definitions in [BKN 96]. That is, if $\Gamma \vdash_{\beta e} A : B$ then $\mathcal{I}(\Gamma) \vdash_e \mathcal{I}(A) : \mathcal{I}(B)$ where $\vdash_e$ is the type derivation of [BKN 96] and $\mathcal{I}$ translates terms to the notation of [KN 95a].

Now, we go through the usual properties of the λ-cube showing that they hold for $\vdash_{re}$.

Lemma 4.6 *(Free variable Lemma for $\vdash_{re}$)*
Let Γ be a legal context such that $\Gamma \vdash_{re} B : C$. Then the following holds:

1. *If d and d' are two different elements of Γ-$\mathtt{decl} \cup \Gamma$-$\mathtt{abb}$, then $\mathtt{var}(d) \not\equiv \mathtt{var}(d')$.*
2. *$FV(B), FV(C) \subseteq dom(\Gamma)$.*
3. *If $\Gamma \equiv \Gamma_1.d.\Gamma_2$ then $FV(d) \subseteq dom(\Gamma_1)$.*

Proof: *All by induction on the derivation of $\Gamma \vdash_{re} B : C$.* $\Box$

Lemma 4.7 *(Start Lemma for $\vdash_{re}$)*
Let Γ be a legal context. Then $\Gamma \vdash_{re} * : \Box$ and $\forall d \in \Gamma[\Gamma \vdash_{re} d]$.

Proof: *Γ legal $\Rightarrow \exists B, C[\Gamma \vdash_{re} B : C]$; now use induction on the derivation of $\Gamma \vdash_{re} B : C$.* $\Box$

Lemma 4.8 *(Transitivity Lemma for $\vdash_{re}$)*
Let Γ and Δ be legal contexts such that $\Gamma \vdash \Delta$.

1. *If $\Delta \vdash_{re} A =_{\mathtt{ab}} B$ then $\Gamma \vdash_{re} A =_{\mathtt{ab}} B$.*
2. *If $\Delta \vdash_{re} A : B$ then $\Gamma \vdash_{re} A : B$.*

Proof:

1. *For all $d \in \Delta$, $\Gamma \vdash_{re} \mathtt{var}(d) =_{\mathtt{ab}} \mathtt{ab}(d)$. If A' results from A by replacing one free occurrence of $\mathtt{var}(d)$ in A by $\mathtt{ab}(d)$, then $\Gamma \vdash_{re} A =_{\mathtt{ab}} A'$ by repeating the process of proving $\Gamma \vdash_{re} \mathtt{var}(d) =_{\mathtt{ab}} \mathtt{ab}(a)$ on the particular occurrence of $\mathtt{var}(d)$ in A.*
2. *Induction on the derivation $\Delta \vdash_{re} A : B$, using 1 in the case of the conversion rule.* $\Box$

Note in the following lemmas how definitions behave well in thinning and substitution.

Lemma 4.9 *(Thinning Lemma for $\vdash_{re}$)*

1. *If* $\Gamma_1.\Gamma_2 \vdash_{re} A =_{\mathtt{ab}} B$, $\Gamma_1.\Delta.\Gamma_2$ *is a legal context, then* $\Gamma_1.\Delta.\Gamma_2 \vdash_{re} A =_{\mathtt{ab}} B$.

2. *If* Γ *and* Δ *are legal contexts such that* $\Gamma \subseteq' \Delta$ *and if* $\Gamma \vdash_{re} A : B$, *then* $\Delta \vdash_{re} A : B$.

Proof: *1. is by induction on the derivation* $\Gamma_1.\Gamma_2 \vdash_{re} A =_{\mathtt{ab}} B$. *2. is as follows:*

- *If* $\Gamma.\Delta \vdash_{re} A : B$, $\Gamma \vdash_{re} C : S$, x *is fresh, then also* $\Gamma.\langle x : C\rangle.\Delta \vdash_{re} A : B$. *We show this by induction on the derivation* $\Gamma.\Delta \vdash_{re} A : B$ *using 1. for conversion.*

- *If* $\Gamma.\Delta \vdash_{re} A : B$, $\Gamma \vdash_{re} C : D : S$, x *is fresh, then also* $\Gamma.\langle x : D\rangle C.\Delta \vdash_{re} A : B$. *We show this by induction on the derivation* $\Gamma.\Delta \vdash_{re} A : B$.

- *If* $\Gamma.\langle x : A\rangle.\Delta \vdash_{re} B : C$, $\Gamma \vdash_{re} D : A$, *then* $\Gamma.\langle x : A\rangle D.\Delta \vdash_{re} B : C$ *is shown by induction on the derivation* $\Gamma.\langle x : A\rangle.\Delta \vdash_{re} B : C$ *(for conversion, use 1.; note that* $\Gamma.\langle x : A\rangle.\Delta \vdash_{re} B_1 =_{\mathtt{ab}} B_2$ *is equivalent to* $\Gamma.\Delta \vdash_{re} B_1 =_{\mathtt{ab}} B_2$*).* $\Box$

Lemma 4.10 *(Substitution Lemma for* $\vdash_{re}$*)*
Let $d = \langle x : C\rangle D$, $\Delta_d = \Delta[x := D]$, $A_d = A[x := D]$ *and* $B_d = B[x := D]$.

1. *If* $\Gamma.d.\Delta \vdash_{re} A =_{\mathtt{ab}} B$, A *and* B *are* $\Gamma.d.\Delta$*-legal, then* $\Gamma.\Delta_d \vdash_{re} A_d =_{\mathtt{ab}} B_d$.

2. *If* B *is a* $\Gamma.d$*-legal term, then* $\Gamma.d \vdash_{re} B =_{\mathtt{ab}} B_d$.

3. *If* $\Gamma.d.\Delta \vdash_{re} A : B$, *then* $\Gamma.\Delta_d \vdash_{re} A_d : B_d$.

4. *If* $\Gamma.\langle x : C\rangle.\Delta \vdash_{re} A : B$ *and* $\Gamma \vdash_{re} D : C$, *then* $\Gamma.\Delta_d \vdash_{re} A_d : B_d$.

Proof: *1. Induction on the derivation rules of* $=_{\mathtt{ab}}$. *2. Induction on the structure of B. 3. Induction on the derivation rules, using 1., 2. and thinning. 4. Idem.* $\Box$

Lemma 4.11 *(Generation Lemma for* $\vdash_{re}$*)*

1. $\Gamma \vdash_{re} S : C \Rightarrow S \equiv *, \Gamma \vdash_{re} C =_{\mathtt{ab}} \Box$, *and if* $C \not\equiv \Box$ *then* $\Gamma \vdash_{re} C : S'$ *for some sort* S'.

2. *If* $\Gamma \vdash_{re} x : A$ *then for some* $d \in \Gamma$, $x \equiv \mathtt{var}(d)$, $\Gamma \vdash_{re} A =_{\mathtt{ab}} \mathtt{type}(d)$ *and* $\Gamma \vdash_{re} A : S$ *for some sort* S.

3. *If* $\Gamma \vdash_{re} \lambda_{x:A}.B : C$ *then for some* D *and sort* S*:* $\Gamma.\langle x : A\rangle \vdash_{re} B : D$, $\Gamma \vdash_{re} \Pi_{x:A}.D : S$, $\Gamma \vdash_{re} \Pi_{x:A}.D =_{\mathtt{ab}} C$ *and if* $\Pi_{x:A}.D \not\equiv C$ *then* $\Gamma \vdash_{re} C : S'$ *for some sort* S'.

4. *If* $\Gamma \vdash_{re} \Pi_{x:A}.B : C$ *then for some sorts* S_1, S_2*:* $\Gamma \vdash_{re} A : S_1$, $\Gamma.\langle x : A\rangle \vdash_{re} B : S_2$, (S_1, S_2) *is a rule,* $\Gamma \vdash_{re} C =_{\mathtt{ab}} S_2$ *and if* $S_2 \not\equiv C$ *then* $\Gamma \vdash_{re} C : S$ *for some sort* S.

5. *If* $\Gamma \vdash_{re} Fa : C$, $F \not\equiv \pi_{x:A}.B$, *then for some* D, E*:* $\Gamma \vdash_{re} a : D$, $\Gamma \vdash_{re} F : \Pi_{x:D}.E$, $\Gamma \vdash_{re} T =_{\mathtt{ab}} C$ *and if* $T \not\equiv C$ *then* $\Gamma \vdash_{re} C : S$ *for some* S, *where* $T \equiv (\Pi_{x:D}.E)a$ *if* $r = \beta\Pi$ *and* $T \equiv E[x := a]$ *if* $r = \beta$.

6. *If* $\Gamma \vdash_{re} (\pi_{x:A}.D)B : C$, *then* $\Gamma.\langle x : A\rangle B \vdash_{re} D : C$

Proof: *1., 2., 3., 4. and 5. follow by induction on the derivations (use Thinning). As to 6., an easy induction on the derivation rules shows that one of the following holds:*

- $\Gamma.\langle x : A\rangle B \vdash_{re} D : E, \Gamma \vdash_{re} E[x := B] =_{\mathtt{ab}} C$ *and* $E[x := B] \not\equiv C \Rightarrow \exists S.\Gamma \vdash_{re} C : S$.

- $\Gamma \vdash_{re} B : F$, $\Gamma \vdash_{re} \lambda_{x:A}.D : \Pi_{y:F}.G$, $\Gamma \vdash_{re} C =_{\texttt{ab}} T$ *and if* $T \not\equiv C$ *where* $T \equiv (\Pi_{y:F}.G)B$ *if* $r = \beta\Pi$ *and* $T \equiv G[y := B]$ *if* $r = \beta$, *then* $\Gamma \vdash_{re} C : S$ *for some sort* S.

In both cases use thinning and conversion; in the second case use also 3. □

Now, recall that correctness of types fails for $\vdash_{\beta\Pi}$ but holds for $\vdash_{\beta}$. Here we show it for $\vdash_{re}$.

Lemma 4.12 *(Correctness (and hence well-formedness) of Types for* $\vdash_{re}$*)*
If $\Gamma \vdash_{re} A : B$ *then* $B \equiv \Box$ *or* $\Gamma \vdash_{re} B : S$ *for some sort* S.
Proof: *By induction on the derivation rules. The interesting cases are:*

- *Abbreviation: If* $\Gamma \vdash_{re} (\pi_{x:A}.D)B : C[x := B]$ *where* $\Gamma.\langle x : A\rangle B \vdash_{re} D : C$, *then by IH,* $C \equiv \Box$ *or* $\exists S, \Gamma.\langle x : A\rangle B \vdash_{re} C : S$. *If* $C \equiv \Box$ *then* $C[x := B] \equiv \Box$*; else, by Substitution Lemma* $\Gamma \vdash_{re} C[x := B] : S[x := B] \equiv S$.

- *Application: If* $\Gamma \vdash_{\beta\Pi e} Fa : (\Pi_{x:A}.B)a$ *where* $\Gamma \vdash_{\beta\Pi e} F : \Pi_{x:A}.B$ *and* $\Gamma \vdash_{\beta\Pi e} a : A$, *then by IH,* $\exists S, \Gamma \vdash_{\beta\Pi e} \Pi_{x:A}.B : S$. *By Generation* $\Gamma.\langle x : A\rangle \vdash_{\beta\Pi e} B : S$. *By Thinning* $\Gamma.\langle x : A\rangle a \vdash_{\beta\Pi e} B : S$ *and by the (abb rule)* $\Gamma \vdash_{\beta\Pi e} (\Pi_{x:A}.B)a : S[x := a] \equiv S$. □

From correctness of types for $\vdash_{re}$, we can establish its subject reduction.

Theorem 4.13 *(Subject Reduction for* $\vdash_{re}$ *and* $\twoheadrightarrow_r$*)*
If $\Gamma \vdash_{re} A : B$ *and* $A \twoheadrightarrow_r A'$ *then* $\Gamma \vdash_{re} A' : B$.
Proof: *We prove by simultaneous induction on the derivation rules:*

1. *If* $\Gamma \vdash_{re} A : B$ *and* Γ' *results from contracting one of the terms in the declarations and abbreviations of* Γ *by a one step r-reduction, then* $\Gamma' \vdash_{re} A : B$

2. *If* $\Gamma \vdash_{re} A : B$ *and* $A \rightarrow_r A'$ *then* $\Gamma \vdash_{re} A' : B$

We will only treat the case $r = \beta\Pi$. *If (axiom): easy. If (start rule): we consider the case* $d \equiv \langle x : A\rangle B$, $A \rightarrow_{\beta\Pi} A'$. *The other cases are similar or easy. We have:* $\Gamma.\langle x : A\rangle B \vdash_{\beta\Pi e} x : A$ *where* $\Gamma \prec \langle x : A\rangle B$, *i.e.* $\Gamma \vdash_{\beta\Pi e} B : A : S$. *By IH,* $\Gamma \vdash_{\beta\Pi e} A' : S$. *By IH and conversion* $\Gamma \vdash_{\beta\Pi e} B : A'$. *Hence* $\Gamma.\langle x : A'\rangle B \vdash_{\beta\Pi e} x : A'$ *and again by conversion* $\Gamma.\langle x : A'\rangle B \vdash_{\beta\Pi e} x : A$. *If (weak), (formation), (conversion) or (abstraction): use IH (and conversion for (abstraction)). Now we treat the rest:*

- *(abbreviation):* $\Gamma \vdash_{\beta\Pi e} (\pi_{x:A}.D)B : C[x := B]$ *where* $\Gamma.\langle x : A\rangle B \vdash_{\beta\Pi e} D : C$. *Now* $\Gamma' \vdash_{\beta\Pi e} (\pi_{x:A}.D)B : C[x := B]$, $\Gamma \vdash_{\beta\Pi e} (\pi_{x:A}.D')B : C[x := B]$ *and* $\Gamma \vdash_{\beta\Pi e} (\pi_{x:A'}.D)B : C[x := B]$ *by IH. Furthermore, if* $B \rightarrow_{\beta\Pi} B'$ *then* $\Gamma \vdash_{\beta\Pi e} C[x := B] =_{\texttt{ab}} C[x := B']$ *and by IH and the (abb rule) we get* $\Gamma \vdash_{\beta\Pi e} (\pi_{x:A}.D)B' : C[x := B']$. *Now by Lemma 4.12,* $C \equiv \Box$ *or* $\exists S, \Gamma.\langle x : A\rangle B \vdash_{\beta\Pi e} C : S$. *If* $C \equiv \Box$ *then* $C[x := B] \equiv C \equiv C[x := B']$. *Else, by the Substitution Lemma* $\Gamma \vdash_{\beta\Pi e} C[x := B] : S[x := B] \equiv S$, *so by conversion* $\Gamma \vdash_{\beta\Pi e} (\pi_{x:A}.D)B' : C[x := B]$.

  For the last possibility, $(\pi_{x:A}.D)B \rightarrow_{\beta\Pi} D[x := B]$, *we remark that by the Substitution Lemma we get out of* $\Gamma.\langle x : A\rangle B \vdash_{\beta\Pi e} D : C$ *that* $\Gamma \vdash_{\beta\Pi e} D[x := B] : C[x := B]$.

- *(application):* $\Gamma \vdash_{\beta\Pi e} Fa : (\Pi_{x:A}.B)a$ *where* $\Gamma \vdash_{\beta\Pi e} F : \Pi_{x:A}.B$ *and* $\Gamma \vdash_{\beta\Pi e} a : A$. *Then* $\Gamma' \vdash_{\beta\Pi e} Fa : (\Pi_{x:A}.B)a$ *and* $\Gamma \vdash_{\beta\Pi e} F'a : (\Pi_{x:A}.B)a$ *by IH, and* $\Gamma \vdash_{\beta\Pi e} Fa' : (\Pi_{x:A}.B)a$ *because by IH* $\Gamma \vdash_{\beta\Pi e} Fa' : (\Pi_{x:A}.B)a'$, *by Lemma 4.12* $\exists S, \Gamma \vdash_{\beta\Pi e} (\Pi_{x:A}.B)a : S$, *so by conversion* $\Gamma \vdash_{\beta\Pi e} Fa' : (\Pi_{x:A}.B)a$.

Now the crucial case: $F \equiv (\pi_{y:C}.D)$, $Fa \to_{\beta\Pi} D[y := a]$. *Then* $\Gamma \vdash_{\beta\Pi e} (\pi_{y:C}.D)a : (\Pi_{x:A}.B)a$ *so by Generation* $\Gamma.\langle y : C\rangle a \vdash_{\beta\Pi e} D : (\Pi_{x:A}.B)a$, *now by Substitution* $\Gamma \vdash_{\beta\Pi e} D[y := a] : ((\Pi_{x:A}.B)a)[y := a]$, *but by BC* $((\Pi_{x:A}.B)a)[y := a] \equiv (\Pi_{x:A}.B)a$. $\square$

The proof of strong normalisation (SN) is based on SN of the λ-cube extended with abbreviations as in [BKN 96]. $\mathrm{SN}_{\to_r}(A)$ denotes A strongly normalising with respect to $\to_r$.

Theorem 4.14 *(Strong Normalisation for the λ-cube with respect to $\vdash_{\beta e}$ and $\to_\beta$)*
If A is a $\vdash_{\beta e}$-legal term then $SN_{\to_\beta}(A)$.

Proof: *By Remark 4.5, $\vdash_{\beta e}$ is a subset of $\vdash_e$ of [BKN 96] in that if $\Gamma \vdash_{\beta e} A : B$ then $\Gamma \vdash_e A : B$. Now, SN for $\vdash_{\beta e}$ follows from that of $\vdash_e$ (see [BKN 96] for the lengthy but standard proof (similar to that of [Geu 95]) of SN for $\vdash_e$ which can be adapted to $\vdash_{\beta e}$).* $\square$

SN of $\vdash_{\beta\Pi e}$ is a consequence of that of $\vdash_{\beta e}$. First we change Π-redexes into λ-redexes.

Definition 4.15

- *For all pseudo-expressions A we define $\widetilde{A}$ to be the term A where in all Π-redexes the Π-symbol has been changed into a λ-symbol, creating a lambda-redex instead.*
- *For a context $\Gamma \equiv d_1.\cdots.d_n$ we define $\widetilde{\Gamma}$ to be $\widetilde{d_1}.\cdots.\widetilde{d_n}$, where $\widetilde{\langle x : A\rangle} \equiv \langle x : \widetilde{A}\rangle$ and $\widetilde{\langle x : A\rangle B} \equiv \langle x : \widetilde{A}\rangle \widetilde{B}$.*

Lemma 4.16 *If $\Gamma \vdash_{\beta\Pi e} A : B$ then $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{A} : \widetilde{B}$.*

Proof: *Induction on the derivation rules of $\vdash_{\beta\Pi e}$. All rules except the* (new application rule) *are trivial since they are also rules in $\vdash_{\beta e}$.*

If $\Gamma \vdash_{\beta\Pi e} Fa : (\Pi_{x:A}.B)a$ results from $\Gamma \vdash_{\beta\Pi e} F : \Pi_{x:A}.B$ and $\Gamma \vdash_{\beta\Pi e} a : A$. Then by IH $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{F} : \Pi_{x:\widetilde{A}}.\widetilde{B}$ and $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{a} : \widetilde{A}$, so by application of $\vdash_{\beta e}$, $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{F}\widetilde{a} : \widetilde{B}[x := \widetilde{a}]$.

As $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{F} : \Pi_{x:\widetilde{A}}.\widetilde{B}$, we also get $\widetilde{\Gamma}.\langle x : \widetilde{A}\rangle \vdash_{\beta e} \widetilde{B} : S$ and hence by thinning and the (abb rule) for $\vdash_{\beta e}$, $\widetilde{\Gamma} \vdash_{\beta e} (\lambda_{x:\widetilde{A}}.\widetilde{B})\widetilde{a} : S$, so by conversion $\widetilde{\Gamma} \vdash_{\beta e} \widetilde{F}\widetilde{a} : (\lambda_{x:\widetilde{A}}.\widetilde{B})\widetilde{a}$.

But F cannot contain a Π-symbol which will mix with a in Fa to form a Π-redex. Otherwise, by generation on $\Gamma \vdash_{\beta\Pi e} F : \Pi_{x:A}.B$, $\exists S$, $\Gamma \vdash_{\beta\Pi e} \Pi_{x:A}.B =_{\mathtt{ab}} S$. But this is impossible, hence $\widetilde{Fa} \equiv \widetilde{F}\widetilde{a}$. $\square$

Theorem 4.17 *(Strong Normalisation for the λ-cube with respect to $\vdash_{\beta\Pi e}$ and $\to_{\beta\Pi}$)*
If A is a $\vdash_{\beta\Pi e}$-legal term then $SN_{\to_{\beta\Pi}}(A)$.

Proof: *If A is $\vdash_{\beta\Pi e}$-legal then $\widetilde{A}$ is $\vdash_{\beta e}$-legal by Lemma 4.16 and hence $SN_{\to_\beta}(\widetilde{A})$ (Theorem 4.14). Due to Subject Reduction, as no Π-redexes can be created in the course of $\to_\beta$-reduction of $\widetilde{A}$, therefore $SN_{\to_\beta}(\widetilde{A})$ implies $SN_{\to_{\beta\Pi}}(A)$.* $\square$

See [KN 95b, BKN 96] for other properties of the λ-cube with Π-reduction or abbreviations.

5 Conclusion

In type theory, abstraction is done via both λ and Π and one writes either $\lambda_{x:A}.B$ or $\Pi_{x:A}.B$. Reduction or evaluation however, usually only operates on λ-redexes. Hence, one always evaluates $(\lambda_{x:A}.B)C$ to $B[x := C]$ and almost never evaluates $(\Pi_{x:A}.B)C$ to $B[x := C]$. An exception to this is the AUTOMATH notation where the distinction between λ and Π

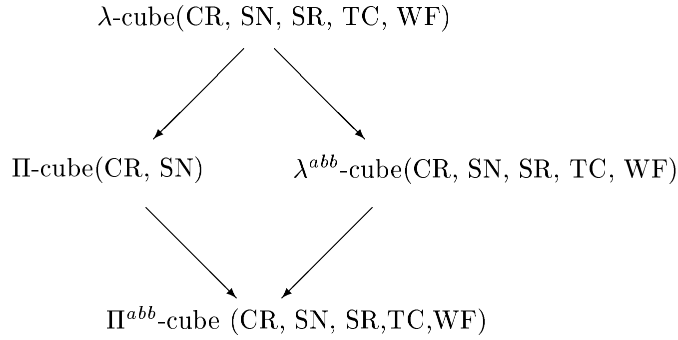

Figure 1: Properties of the λ-cube with various extensions

is absent and one writes $[x : A]B$ to express either $\lambda_{x:A}.B$ or $\Pi_{x:A}.B$. In all type systems however (including AUTOMATH and the system of this paper), there *is* a distinction between λ and Π. The various accounts differ in how large such distinction is.

We believe that present type theory makes such a distinction too large. As we have seen, applications of type systems need Π-reduction. AUTOMATH did introduce Π-reduction, but its formal properties were only established for the first time in [KN 95b] and later the problems were reconsidered in [PM 97].

In this paper, we made Π behave more like an abstraction operator and gave it the right to be applied to a term. We did not however let a Π-abstraction be typed by another abstraction. Otherwise, one will need more abstraction operators than λ and Π and such study may lead to infinite levels of abstraction operators as discussed in Subsection 1.3. Our choice was to keep the type of a Π-term as a simple sort given by the formation rule yet, to type a Π-redex using abbreviations. To do this, we typed $(\Pi_{x:A}.C)B$ in Γ by typing C in $\Gamma.\langle x : A\rangle B$. Our addition of abbreviations (which are different in this paper from the existing notions of definitions in the literature), is simple and worth studying. Furthermore, this addition enabled us to solve the problems of the Π-cube that were left open in [KN 95b].

There are many arguments why Π-reduction and abbreviations must be considered and a system combining both of them without losing any of the nice properties of the λ-cube is certainly worth considering. Moreover, we find it intreaguing that so far in the literature, abbreviations have been added for reasons of efficiency of implementation and not because they solve theoretical problems. In this paper, we have shown that abbreviations solve the problems of the λ-cube extended with Π-reduction. In [BKN 96], we show that definitions solve the problem of subject reduction in the λ-cube extended with a notion of *generalised reduction*. The reason why abbreviations solve these problems is that they keep information in the context on the defined values of some variables. This information might have been removed when some reductions in the term take place and so keeping the abbreviation in the context preserves this information.

Hence, our paper contributes to other work on definitions not only in that it offers a simple and attractive account of definitions or abbreviations which keeps all the original properties of the λ-cube, but also shows that abbreviations are theoretically important and should hence be introduced in the λ-cube. Figure 1 summarizes our results in this paper where we use the following notational conventions: CR, SN, SR and TC stand for Church Rosser, strong normalisation, subject reduction and type correctness; WF stands for well-formedness of the type of an expression, π^{abb}-cube is the π-cube extended with abbreviations for $\pi = \lambda$ or Π.

A question may now occur to the reader: "if our paper is concerned with abstractions via

Π and λ, then why have another kind of abstraction in the contexts? Why write $\langle x : A\rangle$ and $\langle x : A\rangle B$ to describe the binding rather than use λ and Π?". We believe strongly that the binding operators λ and Π should be used in the contexts as well and indeed in many of our work we do so [KN 95b, BKN 96]. In this paper however, we refrain from this option because we do not want to give the false impression that a binding in the context using either λ or Π has anything to do with solving the problems of Π-reduction that we are tackling. If the reader however is interested in using λ's and Π's in the context and hence in writing $\lambda_{x:A}$ or $\Pi_{x:A}$, for declarations and $(\lambda_{x:A}.-)B$ or $(\Pi_{x:A}.-)B$ for abbreviations, then he/she may like to know that with such an account and the following new version of the (abb rule):

$$(\pi\text{-abb rule}) \qquad \frac{\Gamma.(\pi_{x:A}.-)B \vdash C : D}{\Gamma \vdash (\pi_{x:A}.C)B : D[x := B]} \text{ where } \pi \in \{\lambda, \Pi\}$$

one can show the following lemma whose proof is a simple induction on the derivation rules:

Lemma 5.1 *($\lambda\Pi$-exchanging) The following holds:*

1. $\Gamma.\lambda_{x:A}.\Delta \vdash_{\beta\Pi e} C : D \Longleftrightarrow \Gamma.\Pi_{x:A}.\Delta \vdash_{\beta\Pi e} C : D$
2. $\Gamma.(\lambda_{x:A}.-)B.\Delta \vdash_{\beta\Pi e} C : D \Longleftrightarrow \Gamma.(\Pi_{x:A}.-)B.\Delta \vdash_{\beta\Pi e} C : D$

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