Narrative Structure of Mathematical Texts

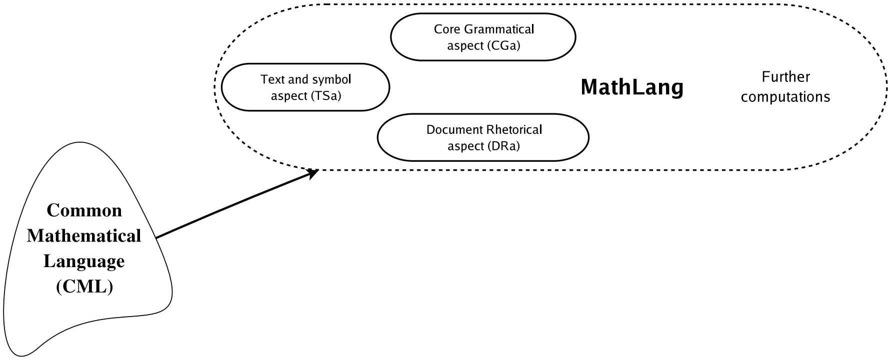
Krzysztof Retel

Joint work with

Prof. Fairouz Kamareddine, Manuel Maarek, and Dr. Joe Wells

ULTRA Group – Heriot-Watt University
`http://www.macs.hw.ac.uk/ultra/`

June 30, 2007
Mathematical Knowledge Management 2007
RISC, Hagenberg, Austria


Core Grammatical aspect (CGa)
Text and symbol aspect (TSa)
MathLang
Further computations
Document Rhetorical aspect (DRa)
Common Mathematical Language (CML)

Motivations

1. *To handle the structure of a mathematical document as it appears on paper and at the same time allowing further computerisation and analysis.*
2. *To allow the presentation of a text with different layouts.*
3. *To allow further formalisation.*

Different styles of writing mathematics

- Different font styles used to emphasize important parts of text.
- Naming sections with common mathematical labels, e.g. definition, theorem etc.
- Clear annotation of sections, definitions, theorems etc.
- Relations between mathematical labels and/or structural sections.

Examples

- Wacław Sierpiński
  Elementary theory of numbers
  Chapter V. Congruences
  §1. Congruences and their simplest properties
- The proof of Pythagoras theorem
  H. Barendregt's textual version of the original proof written by G. H. Hardy and E. M. Wright.
  It is said to be *"informal"* in contrast to the formal versions of theorem provers (see the book *The Seventeen Provers of the World* by F. Wiedijk).

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \tag{2}$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other.* [...]

W.Sierpiński

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \quad (2)$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other*. [...]

W.Sierpiński

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \quad (2)$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other.* [...]

W.Sierpiński

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \tag{2}$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other.* [...]

W.Sierpiński

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \tag{2}$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other*. [...]

W.Sierpiński

We prove that *two congruences can be added or subtracted from each other provided both have the same modulus.*

Let

$$a \equiv b \pmod{m} \text{ and } c \equiv d \pmod{m}. \tag{2}$$

In order to prove that $a + c \equiv b + d \pmod{m}$ and $a - c \equiv b - d \pmod{m}$ it is sufficient to apply the identities

$$a + c - (b + d) = (a - b) + (c - d) \quad \text{and} \quad (a - c) - (b - d) = (a - b) - (c - d).$$

Similarly, using the identity

$$ac - bd = (a - b)c + (c - d)b,$$

we prove that congruences (2) imply the congruence

$$ac \equiv bd \pmod{m}.$$

Consequently, we see that *two congruences having the same modulus can be multiplied by each other*. [...]

W.Sierpiński

Lemma 1. For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$.

Proof. Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n.m^2 = 2n^2 \ \& \ m > 0.$$

Claim. $P(m) \implies \exists m' < m.P(m')$. Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have

$$2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$$

Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}.\neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction. Therefore $m = 0$. But then also $n = 0$. □

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$.

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

H. Barendregt

Document's components

1. **Structural components** like *chapter, section, subsection, etc.*
2. **Mathematical components** like *theorem, corollary, definition, proof, etc.*
3. **Relations** between above components.

Why is it important?

- Enhance readability of a document.
- Makes the navigation of a document more enjoyable.
- Plays the narration role throughout the theory presented in a document.

Document's components

1. **Structural components** like *chapter, section, subsection, etc.*
2. **Mathematical components** like *theorem, corollary, definition, proof, etc.*
3. **Relations** between above components.

Why is it important?

- Enhance readability of a document.
- Makes the navigation of a document more enjoyable.
- Plays the narration role throughout the theory presented in a document.

Document's components

1. **Structural components** like *chapter, section, subsection, etc.*
2. **Mathematical components** like *theorem, corollary, definition, proof, etc.*
3. **Relations** between above components.

Why is it important?

- Enhance readability of a document.
- Makes the navigation of a document more enjoyable.
- Plays the narration role throughout the theory presented in a document.

Document's components

1. **Structural components** like *chapter, section, subsection, etc.*
2. **Mathematical components** like *theorem, corollary, definition, proof, etc.*
3. **Relations** between above components.

Why is it important?

- Enhance readability of a document.
- Makes the navigation of a document more enjoyable.
- Plays the narration role throughout the theory presented in a document.

Both components in DRa system

What is the difference between structural and mathematical components?

- Visible difference in the font styles, headings, indentation etc.
- Make boundaries of chunks of text explicit.
- There are other differences, e.g., *"chapter" might play the role of external library for the following "section"; "definition" introduce new concept within a theory.*

- All are instances of the same class – StructuredUnit
- We differentiate these components by the role they play in mathematical texts:
  1. StructuralRhetoricalRole like *chapter* or *section*
  2. MathematicalRhetoricalRole like *lemma* or *proof*

Both components in DRa system

What is the difference between structural and mathematical components?

- Visible difference in the font styles, headings, indentation etc.
- Make boundaries of chunks of text explicit.
- There are other differences, e.g., *"chapter" might play the role of external library for the following "section"; "definition" introduce new concept within a theory.*

- All are instances of the same class – StructuredUnit
- We differentiate these components by the role they play in mathematical texts:
  1. StructuralRhetoricalRole like *chapter* or *section*
  2. MathematicalRhetoricalRole like *lemma* or *proof*

Both components in DRa system

What is the difference between structural and mathematical components?

- Visible difference in the font styles, headings, indentation etc.
- Make boundaries of chunks of text explicit.
- There are other differences, e.g., *"chapter" might play the role of external library for the following "section"; "definition" introduce new concept within a theory.*

- All are instances of the same class – StructuredUnit
- We differentiate these components by the role they play in mathematical texts:
  1. StructuralRhetoricalRole like *chapter* or *section*
  2. MathematicalRhetoricalRole like *lemma* or *proof*

Relations

- RELATESTO
- USES
- JUSTIFIES
- SUBPARTOF
- INCONSISTENTWITH
- EXEMPLIFIES

Description
Instances of the STRUCTURALRHETORICALROLE *class:* preamble, part, chapter, section, paragraph, *etc.*
Instances of the MATHEMATICALRHETORICALROLE *class:* lemma, corollary, theorem, conjecture, definition, axiom, claim, proposition, assertion, proof, exercise, example, problem, solution, *etc.*

Relation
Types of relations: RELATESTO, USES, JUSTIFIES, SUBPARTOF, INCONSISTENTWITH, EXEMPLIFIES

Relations

- RELATESTO
- USES
- JUSTIFIES
- SUBPARTOF
- INCONSISTENTWITH
- EXEMPLIFIES

Description
Instances of the STRUCTURALRHETORICALROLE *class:* preamble, part, chapter, section, paragraph, *etc.*
Instances of the MATHEMATICALRHETORICALROLE *class:* lemma, corollary, theorem, conjecture, definition, axiom, claim, proposition, assertion, proof, exercise, example, problem, solution, *etc.*

Relation
Types of relations: RELATESTO, USES, JUSTIFIES, SUBPARTOF, INCONSISTENTWITH, EXEMPLIFIES

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$

Proof.
Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \& \ m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

Therefore $m = 0$. But then also $n = 0$.

□

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

Original view of Pythagoras proof written by H. Barendregt.

Lemma 1.

For $m, n \in \mathbb{N}$ *one has:* $m^2 = 2n^2 \implies m = n = 0$

Proof.
Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n.m^2 = 2n^2 \ \&\ m > 0.$$

Claim. $P(m) \implies \exists m' < m.P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}.\neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

Therefore $m = 0$. But then also $n = 0$.

□

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

What does the mathematician have to do?

- He wraps chunks of text with boxes and uniquely names each box.
- He assigns to each box the structural and/or mathematical rhetorical roles this box plays.
- He indicates the relations between wrapped chunks of texts using the relation names.

Lemma 1.

[A] *For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$*

[B] **Proof.**

[E] Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \& \ m > 0.$$

Claim. [F] $P(m) \implies \exists m' < m. P(m')$.

[G] Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

[H] with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

[I] Therefore $m = 0$. But then also $n = 0$. □

Corollary 1. [C] $\sqrt{2} \notin \mathbb{Q}$

[D] **Proof.** Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

What does the mathematician have to do?

- He wraps chunks of text with boxes and uniquely names each box.
- He assigns to each box the structural and/or mathematical rhetorical roles this box plays.
- He indicates the relations between wrapped chunks of texts using the relation names.

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$ **A**

Proof.

Define on $\mathbb{N}$ the predicate: **E**

$$P(m) \iff \exists n.m^2 = 2n^2 \ \&\ m > 0.$$

Claim. $P(m) \implies \exists m' < m.P(m')$. **F**

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$. **G** **B**

By the claim $\forall m \in \mathbb{N}.\neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction. **H**

Therefore $m = 0$. But then also $n = 0$. **I** □

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$ **C**

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. **D** □

What does the mathematician have to do?

- He wraps chunks of text with boxes and uniquely names each box.
- He assigns to each box the structural and/or mathematical rhetorical roles this box plays.
- He indicates the relations between wrapped chunks of texts using the relation names.

Lemma 1.

[A] *For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$*

Proof. [B]

[E] Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \&\ m > 0.$$

Claim. [F] $P(m) \implies \exists m' < m. P(m')$.

[G] Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds squared to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$

[H] with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

[I] Therefore $m = 0$. But then also $n = 0$. □

Corollary 1. [C] $\sqrt{2} \notin \mathbb{Q}$

[D] **Proof.** Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

(A, hasMathematicalRhetoricalRole, *lemma*)

(E, hasMathematicalRhetoricalRole, *definition*)

(F, hasMathematicalRhetoricalRole, *claim*)

(G, hasMathematicalRhetoricalRole, *proof*)

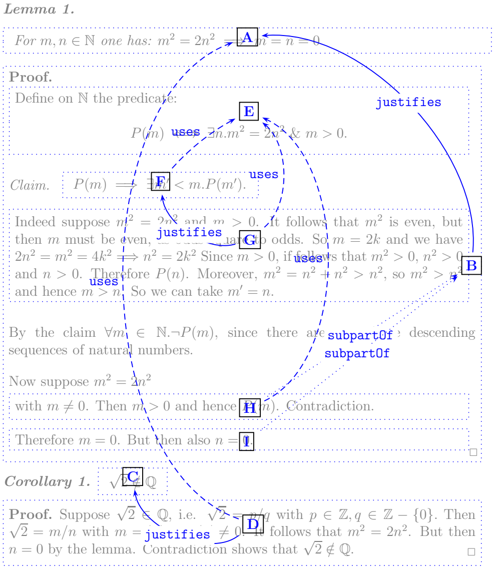
(B, hasMathematicalRhetoricalRole, *proof*)

(H, hasOtherMathematicalRhetoricalRole, *case*)

(I, hasOtherMathematicalRhetoricalRole, *case*)

(C, hasMathematicalRhetoricalRole, *corollary*)

(D, hasMathematicalRhetoricalRole, *proof*)



What does the mathematician have to do?

- He wraps chunks of text with boxes and uniquely names each box.
- He assigns to each box structural and/or mathematical rhetorical roles this box plays.
- He indicates the relations between wrapped chunks of texts using the relation names.

Lemma 1.

For $m, n \in \mathbb{N}$ one has: $m^2 = 2n^2 \implies m = n = 0$

Proof. Define on $\mathbb{N}$ the predicate:

$$P(m) \iff \exists n. m^2 = 2n^2 \ \&\ m > 0.$$

Claim. $P(m) \implies \exists m' < m. P(m')$.

Indeed suppose $m^2 = 2n^2$ and $m > 0$. It follows that m^2 is even, but then m must be even, as odds square to odds. So $m = 2k$ and we have $2n^2 = m^2 = 4k^2 \implies n^2 = 2k^2$ Since $m > 0$, if follows that $m^2 > 0$, $n^2 > 0$ and $n > 0$. Therefore $P(n)$. Moreover, $m^2 = n^2 + n^2 > n^2$, so $m^2 > n^2$ and hence $m > n$. So we can take $m' = n$.

By the claim $\forall m \in \mathbb{N}. \neg P(m)$, since there are no infinite descending sequences of natural numbers.

Now suppose $m^2 = 2n^2$ with $m \neq 0$. Then $m > 0$ and hence $P(m)$. Contradiction.

Therefore $m = 0$. But then also $n = 0$. □

Corollary 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = p/q$ with $p \in \mathbb{Z}, q \in \mathbb{Z} - \{0\}$. Then $\sqrt{2} = m/n$ with $m = |p|, n = |q| \neq 0$. It follows that $m^2 = 2n^2$. But then $n = 0$ by the lemma. Contradiction shows that $\sqrt{2} \notin \mathbb{Q}$. □

(B, JUSTIFIES, A)

(D, JUSTIFIES, C)

(D, USES, A)

(G, USES, E)

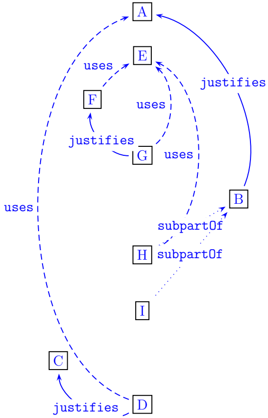
(F, USES, E)

(H, USES, E)

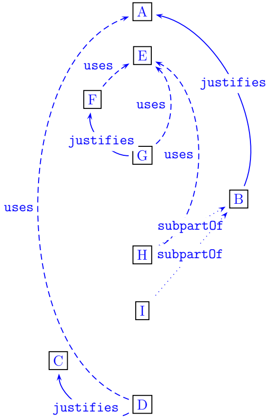
(H, SUBPARTOF, B)

(H, SUBPARTOF, I)

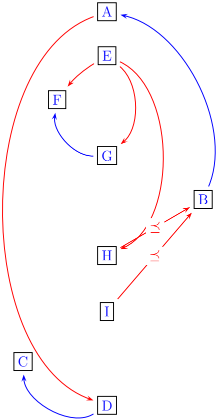
Dependency Graph (DG)



Dependency Graph (DG)



Graph of Logical Precedences (GoLP)



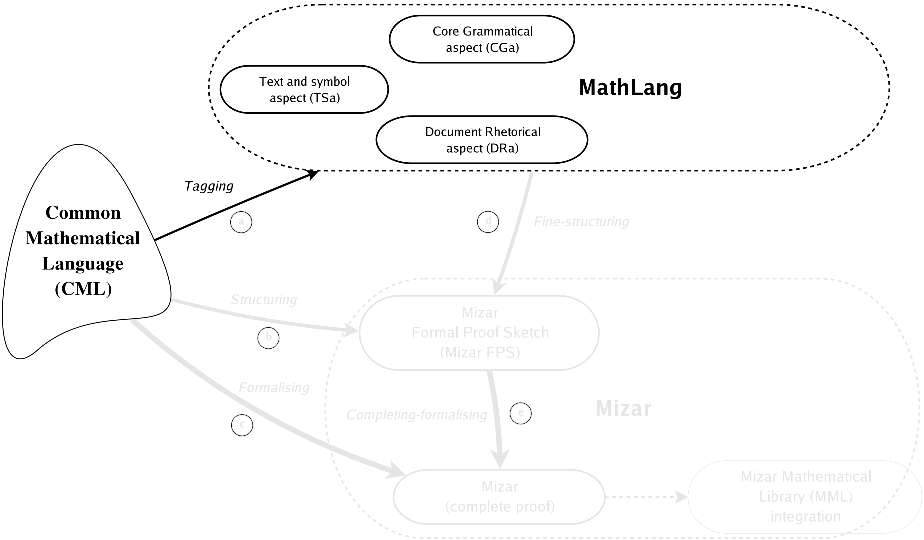
What can we check?

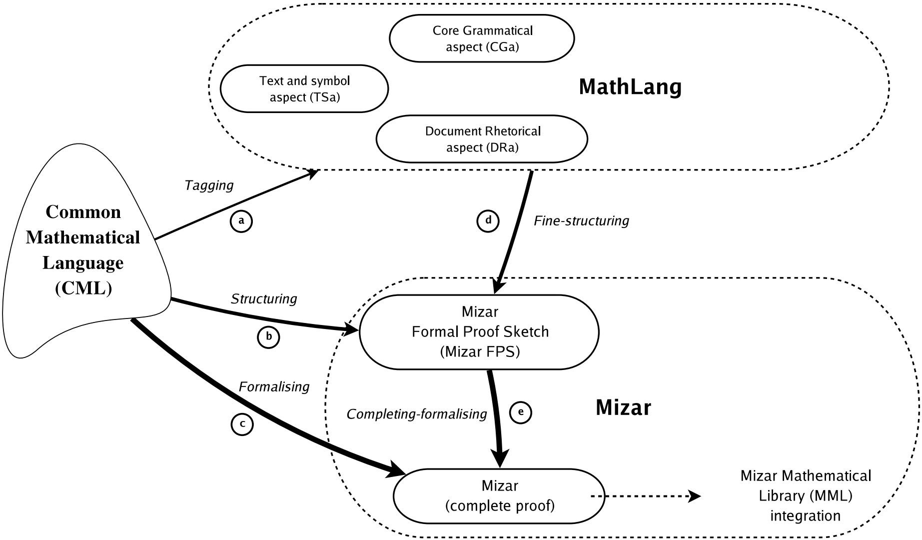
Checking *DG*

- Checking good-usage of labels and relations (e.g., that a "proof" justifies a "theorem" but cannot justify an "axiom").

Checking *GoLP*

- Checking that the *GoLP* is consistent.


Core Grammatical aspect (CGa)
Text and symbol aspect (TSa)
MathLang
Document Rhetorical aspect (DRa)
Tagging
Common Mathematical Language (CML)
a
d
Fine-structuring
Structuring
b
Mizar Formal Proof Sketch (Mizar FPS)
Formalising
c
Completing-formalising
e
Mizar
Mizar (complete proof)
Mizar Mathematical Library (MML) integration


Core Grammatical aspect (CGa)
Text and symbol aspect (TSa)
MathLang
Document Rhetorical aspect (DRa)
Common Mathematical Language (CML)
Tagging
a
d
Fine-structuring
Structuring
b
Mizar Formal Proof Sketch (Mizar FPS)
Formalising
c
Completing-formalising
e
Mizar
Mizar (complete proof)
Mizar Mathematical Library (MML) integration

MathLang document structure

Document preamble
(*A*, HASSTRUCTURALRHETORICALROLE, *preamble*)
Document body
(*B*, HASSTRUCTURALRHETORICALROLE, *body*)

environ

Environment

begin

Text-Proper

MathLang document structure

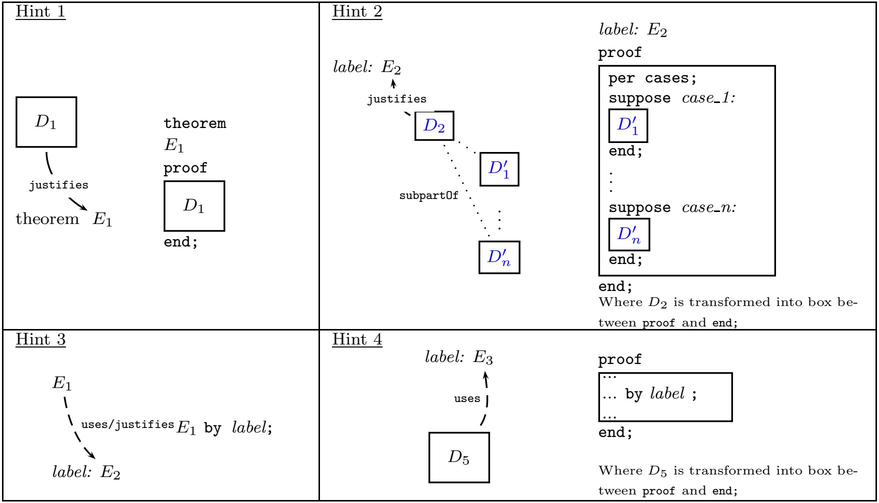
Document preamble (*A*, HASSTRUCTURALRHETORICALROLE, *preamble*)
Document body (*B*, HASSTRUCTURALRHETORICALROLE, *body*)

`environ`

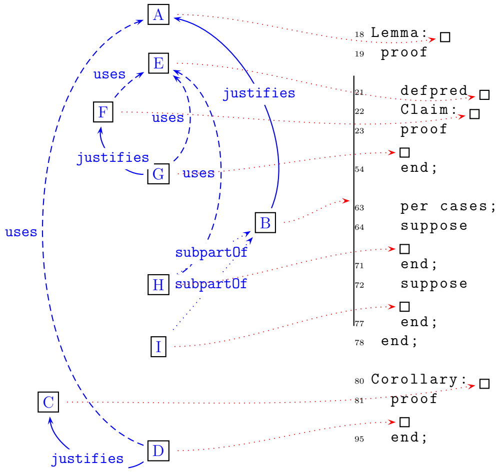
Environment

`begin`

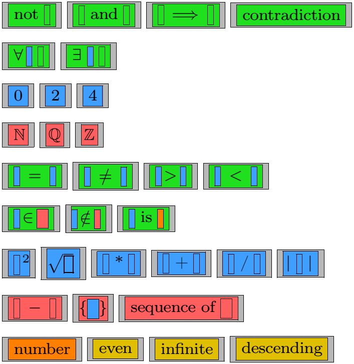
Text-Proper


Hint 1
D_1
justifies
theorem E_1
theorem
E_1
proof
D_1
end;
Hint 2
label: E_2
justifies
D_2
D'_1
subpartOf
D'_n
label: E_2
proof
per cases;
suppose case_1:
D'_1
end;
suppose case_n:
D'_n
end;
end;
Where D_2 is transformed into box between proof and end;
Hint 3
E_1
uses/justifies
E_1 by label;
label: E_2
Hint 4
label: E_3
uses
D_5
proof
...
... by label ;
...
end;
Where D_5 is transformed into box between proof and end;

The DRa annotation into Mizar document skeleton



MathLang preamble as subset of Mizar environment




```
vocabularies INT_1, SQUARE_1, MATRIX_2, IRRAT_1,
  RAT_1, ARYTM_3, ABSVALUE, SEQM_3, FINSET_1;
notations INT_1, NAT_1, SQUARE_1, XXREAL_0,
  ABIAN, RAT_1, IRRAT_1, XCMPLX_0, INT_2, SEQM_3,
  FINSET_1, REAL_1, PEPIN;
constructors INT_1, NAT_1, SQUARE_1, XXREAL_0,
  ABIAN, RAT_1, IRRAT_1,XCMPLX_0, INT_2, SEQM_3,
  FINSET_1, PEPIN;
requirements SUBSET, NUMERALS, ARITHM, BOOLE, REAL;
registrations XREAL_0, REAL_1, NAT_1, INT_1;
```

```
Lemma:  for m,n being Nat holds
                m^2 = 2*n^2 implies m = 0 & n = 0
 proof
   let m,n being Nat;
   defpred P[Nat] means
           ex n being Nat st $1^2 = 2*n^2 & $1 > 0;
   Claim: for m being Nat holds
          P[m] implies ex m' being Nat st m' < m & P[m']
   proof
     let m being Nat;
     assume P[m];
     then consider n being Nat such that
     m^2 = 2*n^2 & m > 0;
     m^2 is even ;
::>            *4
     m is even;
::>          *4
     consider k being Nat such that m = 2*k;
::>                                         *4
     2*n^2 = m^2
::>            *4
          .= 4*k^2;
::>              *4
     then n^2 = 2*k^2;
     m > 0 implies m^2 > 0 & n^2 > 0 & n > 0;
::>                                          *4,4,4
     then P[n];
::>          *4,4
     m^2 = n^2 + n^2;
::>                *4
     n^2 + n^2 >  n^2;
::>                 *4
     then m^2 > n^2;
::>               *4
     then m > n;
::>           *4
     take m' = n;
     thus thesis;
::>            *4,4
   end;
   A2: for k being Nat holds not P[k]
   proof
     not ex q being Seq_of_Nat
         st q is infinite decreasing by Claim;
::>                                      *4
     hence thesis;
::>            *4
   end;
   assume A0: m^2 = 2*n^2;
   per cases by A0;
   suppose B1: m <> 0;
     then m > 0;
::>          *4
     then P[m] by B1;
::>            *4
     then contradiction by A2;
     hence thesis;
   end;
   suppose S1: m = 0;
     then n = 0;
::>          *4
     thus thesis by S1;
::>              *4
   end;
 end;

Corollary:  sqrt 2 is irrational
 proof
   assume sqrt 2 is rational;
   then ex p,q being Integer st
   q <> 0 & sqrt 2 = p/q;
::>                   *4
   then consider m,n being Integer such that
   A0: sqrt 2 = m/n & m = abs m & n = abs n & n <> 0;
::>                                                  *4
   m^2 = 2*n^2;
::>         *4
   n = 0 by Lemma;
::>       *4
   hence contradiction;
::>                 *4
 end;

::> 4: This inference is not accepted
```

Future work

- Further development of DRa aspect:
  - Refine the DRa: to allow adding relation by the user.
  - Finish the implementation of DRa "analyser".
- Integration into Mizar:
  - Build assistant supporting transformation into Mizar.
  - Research on using the Mizar library search engines.

Thank you for your attention.