Probability approximations using Stein's method

Tutorial 1

1. For random variables X and Y, show that $d_{TV}(X,Y) \leq \mathbb{P}(X \neq Y)$.

2. Let $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\mu)$, where $\mu > \lambda$. By writing down a coupling of (X,Y), show that
$$d_{TV}(X,Y) \leq 1 - e^{\lambda-\mu}\,.$$

3. Let $W \sim \text{Bin}(n,p)$ have a binomial distribution with parameters n and p. Let $Z \sim \text{Po}(np)$. Compute an upper bound on $d_{TV}(W,Z)$ in the following cases:

   (a) $n = 100$, $p = 0.1$; and

   (b) $n = 1000$, $p = 0.01$.

4. **Stein's method for geometric approximation, part I:** A random variable Z taking values in the non-negative integers has a geometric distribution with parameter $p \in (0,1)$, written $Z \sim \text{Geom}(p)$, if $\mathbb{P}(Z = k) = p(1-p)^k$ for $k = 0, 1, 2, \ldots$. This random variable is often introduced as the waiting time until the first event in a sequence of IID trials, where an event occurs at each time $0, 1, 2, \ldots$ with probability p independently of the occurrence of events at other times.

   (a) Show that, for a bounded function $f : \mathbb{Z}^+ \to \mathbb{R}$ with $f(0) = 0$, we have
   $$\mathbb{E}[f(Z)] = (1-p)\mathbb{E}[f(Z+1)]\,.$$

   (b) We can use the formula in part (a) to characterize Z and as a starting point in Stein's method for geometric approximation. Show that for a given $A \subseteq \mathbb{Z}^+$, the solution $f = f_A$ of the Stein equation
   $$(1-p)f(k+1) - f(k) = I(k \in A) - \mathbb{P}(Z \in A)$$
   with $f(0) = 0$ is given by
   $$f(k) = \sum_{i \in A}(1-p)^i - \sum_{i \in A, i \geq k}(1-p)^{i-k}\,,$$
   and satisfies $\sup_{j,k \in \mathbb{Z}^+} |f(j) - f(k)| \leq \frac{1}{p}$.

   (c) Let $Z' \sim \text{Geom}(p')$ for $p' < p$. Use Stein's method for geometric approximation to show that
   $$d_{TV}(Z, Z') \leq \frac{p-p'}{p}\,.$$