Poisson Line Processes and Spatial Transportation Networks

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Syllabus of lectures

Part 1 Stochastic geometry: a brief introduction:
The basic building blocks of stochastic geometry.

Part 2 Poisson line processes and Network Efficiency:
Curious results arising from careful investigation of measures of effectiveness of networks.

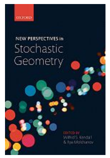
Part 3 Traffic flow in a Poissonian city:
Using Poisson line processes to model traffic intensity in a random city.

Part 4 Scale-invariant random spatial networks (SIRSN):
Novel random metric spaces arising from the methods used to analyze effectiveness of networks.

Part 5 Random flights on the SIRSN.
How can we move about on a SIRSN modelled on Poisson line processes?

Useful references for stochastic geometry

- Kingman (1993): concise and elegant introduction to Poisson processes;
- Stoyan, WSK, and Mecke (1995): now in an extended 3rd edition (Chiu, Stoyan, WSK, and Mecke, 2013);
- Schneider and Weil (2008): magisterial introduction to the links between stochastic and integral geometry;
- WSK and Molchanov (2010): collection of essays on developments in stochastic geometry.
- Last and Penrose (2017): more on Poisson point processes.
- Molchanov (1996b): Boolean models, a way to build random sets from Poisson point processes.



A short chronology of stochastic geometry

(The study of random pattern)

1. Georges-Louis Leclerc, Comte de Buffon
   (7 September, 1707 – 16 April, 1788);
2. Władysław Hugo Dionizy Steinhaus
   (14 January, 1887 – 25 February, 1972);
3. Luís Antoni Santaló Sors
   (9 October, 1911 – 22 November, 2001);
4. Rollo Davidson
   (8 October, 1944 – 29 July, 1970);
5. D.G. Kendall and K. Krickeberg coined the phrase "stochastic geometry" when preparing an Oberwolfach workshop, 1969 (also Frisch and Hammersley, 1963);
6. Mathematical morphology (G. Matheron and J. Serra);
7. Point processes arising from queueing theory
   (J. Kerstan, K. Matthes, . . .);
8. Combinatorial geometry (R.V. Ambartzumian).

Lecture I

Poisson Point Processes and Friends

Poisson point processes

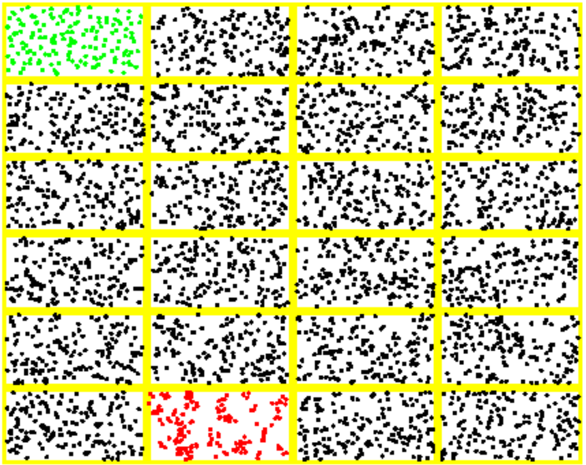
Humans have evolved to recognize patterns . . .

. . . we are less good at recognizing complete randomness.

Puzzle

Using 10^6 years of accumulated evolutionary programming . . .

. . . which of these patterns fail to be completely random?



Constructive definition of Poisson point process

Definition: unit intensity Poisson point pattern in a unit square

1. Draw a $\text{Poisson}(1)$ random variable N;
2. Conditional on $N = n$, plant n independent random points uniformly in $[0, 1]^2$.

This can be viewed as a *binomial point process* of N points, with total number of points N being Poisson-randomized.

Replace unit square $[0, 1]^2$ by region R, and $\text{Poisson}(1)$ by $\text{Poisson}(\text{area}(R))$, to produce a unit intensity Poisson point pattern in a more general planar region R.

If R is of infinite area, break it up into disjoint sub-regions of finite area and work independently in each sub-region.

Replace $\text{Poisson}(\text{area}(R))$ by $\text{Poisson}(\lambda \times \text{area}(R))$ to produce a Poisson point pattern of intensity λ in general region R.

Axiomatic definition of Poisson point process

Definition: Poisson point (alternative abstract definition)

A (stationary) point pattern is *Poisson* ("truly random") if:

1. number of points in any prescribed region has a *Poisson distribution* with mean defined by intensity measure ν (NB: ν is diffuse). EG: $\nu(\text{region}) = \lambda \times (\text{area of region})$;
2. numbers of points in disjoint regions are independent.

Basic construction: each very small region independently has a very small chance of holding a point.

Remark: Powerful theoretical result

Properties 1 and 2 are both automatically implied if the void probability of a measurable region *failing* to hold any points at all is $\exp(-\nu(\text{region}))$ (consequence of result of Choquet).

Computations: $\mathbb{P}[N(A) = k] = \frac{(\nu(A))^k e^{-\nu(A)}}{k!}$, $\mathbb{E}[N(A)] = \nu(A)$.
Much more theory: Kingman (1993), Chiu et al. (2013).

General point processes

1. We can think of a point process Φ as a random measure taking non-negative integer values (attention focussed on $\Phi(A)$ the number of points in a Borel set A);
2. or (equivalently) as a random closed set which is locally finite (attention focussed on whether the set hits compact test sets K, so whether $\Phi(K) > 0$.
3. The variety of general point processes is huge. Nevertheless, the distribution of a point process is determined by its system of void probabilities $\mathbb{P}[\Phi(K) = 0]$ for all compact K.
4. Indeed this holds more generally for random closed sets Ξ: the distribution of Ξ is determined by the system of void probabilities $\mathbb{P}[\Xi \cap K = \emptyset]$ for all compact K.

Palm distributions (I)

Consider a general point process Φ on $\mathbb{R}^2$ with (σ-finite) intensity measure ν. How to condition on specified $x \in \Phi$?

1. Intensity measure $\nu(B) = \mathbb{E}[\#\{x \in B \cap \Phi\}] = \mathbb{E}[\sum_{x\in\Phi} \mathbb{I}[x \in B]] = \mathbb{E}[\sum_{x\in B\cap\Phi} 1]$;
2. Let $\mathcal{Y}$ be some (measurable) set of possible point patterns ϕ (EG: $\mathcal{Y} = \mathcal{Y}_K = \{\phi : \phi(K) = \emptyset\}$ for given compact K). Campbell measure $C(B \times \mathcal{Y}) = \mathbb{E}[\sum_{x\in B\cap\Phi} \mathbb{I}[\Phi \in \mathcal{Y}]]$;
3. Radon-Nikodym theorem: $C(B \times \mathcal{Y}) = \int_B P_x(\mathcal{Y})\nu(\mathrm{d}x)$;
4. Use theory of regular conditional probabilities: $P_x(\cdot)$ can be made into a probability measure;
5. We interpret $P_x(\cdot)$ as the Palm distribution at x.
6. The reduced Palm distribution is obtained by deleting the point x from the Palm distribution.

Palm distributions (II)

Definition: Palm distribution of a general point process

If Φ is a general point process on $\mathbb{R}^2$ with σ-finite intensity measure ν, then its Palm distribution P_x at $x \in \mathbb{R}^2$ satisfies

$$\mathbb{E}\left[\sum_{x\in\Phi} h(x,\Phi)\right] = \int_{\mathbb{R}^2}\int h(x,\phi)P_x(\mathrm{d}\phi)\nu(\mathrm{d}x),$$

for all bounded measurable $h(x,\phi)$, for $x \in \mathbb{R}^2$ and ϕ a locally finite point pattern on $\mathbb{R}^2$.

Palm distribution of a Poisson process

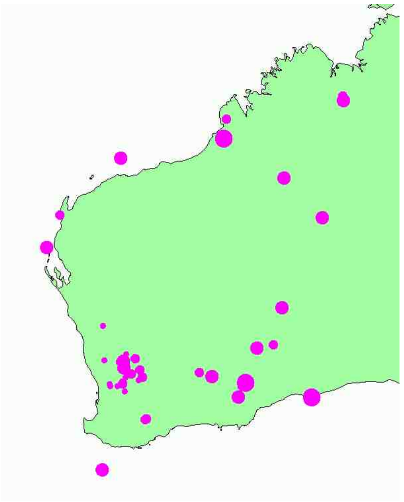
Theorem: Slivnyak

The Palm distribution of a planar Poisson point process Φ at $x \in \mathbb{R}^2$ is the superposition $\delta_x * P(\mathrm{d}\phi)$, namely Φ with added point at x.

Proof.
Consider an arbitrary (compact) set $K \subset \mathbb{R}^2$ and a bounded Borel set $B \subset \mathbb{R}^2$.
Let V_K be the set of patterns placing no points in K.
Consider $C(B, V_K) = \mathbb{E}\left[\sum_{x \in B} \mathbb{I}\left[\Phi \cap K = \emptyset\right]\right]$.
But $C(B \cap K, V_K) = 0$, while $C(B \setminus K, V_K) = \int_{B \setminus K} P(V_K) \nu(\mathrm{d}x)$, when P is the original Poisson distribution (independence of Poisson process in disjoint regions).
This agrees with the results obtained by replacing $C(\mathrm{d}x, \mathrm{d}\xi)$ by $\nu(\mathrm{d}x) \times (\delta_x * P(\mathrm{d}\xi))$.
Choquet's capacitability theorem implies measures agree.

Application: When is a point process Poisson?

Is the randomness of a pattern "unstructured"?

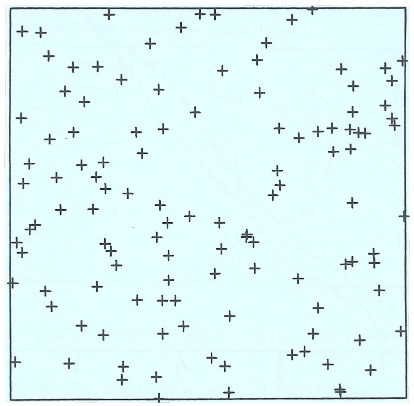


Earthquake locations in Western Australia.

- Use statistical tests for Poisson distribution of point counts;
- Deciding whether a pattern is *completely* random is relatively easy (answer is usually no!). Deciding what sort of pattern can be hard.
- Compare Ripley (1977)
  (a) First-contact distribution $H(r)$ of distance from fixed location to nearest point,
  (b) Nearest-neighbour distribution $D(r)$.

Example: Alpha particles

Centres of traces of α-particles on a detector in a 60×60 square (unit of $2\mu\, m$).



Locations of centres of traces (Stoyan et al., 1995, Figure 2.5ff).

Mephistopheles and the bus company

Classic bus paradox: mean time between Poisson buses is twice the mean length of interval between bus arrivals.
This follows immediately from Slivnyak's theorem!
For renewal point processes the picture can be stranger.

1. The aspirant devil Mephistopheles is forced to take a job with Hades Regional Bus Company.
2. He is responsible for operation of the N°12 bus service between city centre and university campus.
3. Contract requires him to deliver mean time between bus arrivals of 10 minutes.
4. How bad can he make the service while still (statistically speaking) fulfilling the contract?

In mean-value terms, Mephistopheles can make matters infinitely bad ...

Marked point processes

Many constructions and applications use the notion that points of the pattern carry "marks".

1. In applications the points of a pattern are often accompanied by extra measurements (diameters of trees, yields of ore deposits, ...).
2. A marked point process on $\mathbb{R}^2$ can be viewed as a point process on $\mathbb{R}^2 \times \mathbb{M}$, where $\mathbb{M}$ is mark space.
3. Under stationarity we can factorize intensity measure as $\nu(A \times B) = \lambda \operatorname{Leb}(A) \times \mathcal{M}(B)$, where $\mathcal{M}$ is the mark distribution.
4. If additionally marks are independent then factorization carries through to distribution of process.
5. An independently marked Poisson process is a Poisson process on $\mathbb{R}^2 \times \mathbb{M}$.

Palm distribution for marks (sketch)

- Independently marked Poisson point process Ψ, typical marked point $[x, m]$, independent mark distribution Uniform$[0, 1]$.
- We want to impose (Palm) conditioning by requiring a point to lie at x, but *without* conditioning the corresponding mark m.
- Distribution of Ψ can be obtained by
  1. Constructing *un-marked* point process Φ from Ψ;
  2. Ordering points of Φ (eg: lexicographical);
  3. Assigning marks from an i.i.d. sequence.
- Campbell equation for marked process: $\mathbb{E}\left[\sum_{x \in B \cap \Phi} \mathbb{I}\left[\Psi \in \mathcal{Y}\right]\right] = \int_B P_x(\mathcal{Y}) \nu(\mathrm{d}\, x)$.
- Integrate out the marks and apply Slivnyak to $\mathbb{E}\left[\sum_{x \in \Phi} \widetilde{h}(x, \Psi)\right] = \int \int \widetilde{h}(x, \psi) P_x(\mathrm{d}\, \psi) \nu(\mathrm{d}\, x)$.
- Recognize Palm distribution as superposition of (a) Ψ, and (b) point x with independent random mark m.

Palm distribution for marked Poisson point process

Theorem: Slivnyak for independent marks

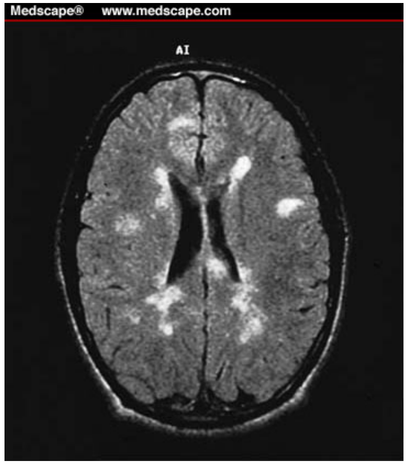
The Palm distribution of a planar marked Poisson point process Ψ at $x \in \mathbb{R}^2$ (independently marked with identically distributed marks of distribution $\mathcal{M}$) is the superposition $\delta_{[x;M]} * P(\mathrm{d}\, \psi)$, namely Ψ with added point at x furnished with independent random mark M of distribution $\mathcal{M}$.

The Boolean model

Suppose we need a model for a random set rather than a random point pattern.

Simplest approach that might work: plant a random set on top of each point in a Poisson process and take the union.

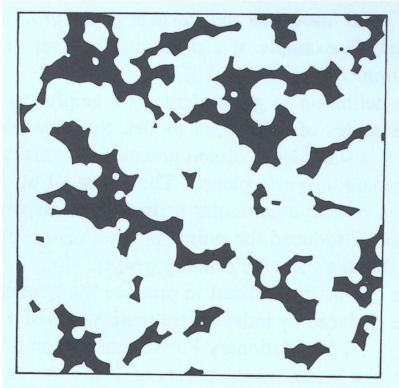
Theory: Hall (1988), Molchanov (1996b).



MS lesions (white areas) in human brain.

Boolean model notation

- *Germs* defined by points of a Poisson process;
- *Grains* are (independent random compact) sets placed on each grain (control size to ensure vacant regions exist);
- the Boolean model is the union of all grains.
- Boolean models can also be used to define interacting point processes (Baddeley and van Lieshout, 1995; WSK, van Lieshout, and Baddeley, 1999; WSK, 1997).



Simulated Boolean model matched to image of potassium deposit (grains are disks of random radius).

A result which bridges to the next lecture

- Suppose we have a high-intensity Boolean model using disks for grains. What is the approximate statistical behaviour of the vacant regions?
- At high intensity, vacancies are small and rare. In the limit, boundaries are locally straight: the vacancy will look approximately like a cell formed by dividing up space by many random lines (Hall, 1988).
- Similar results under much weaker conditions, and in higher dimensions too (Hall, 1988; Molchanov, 1996a).
- But what do we *mean* by a random collection of lines?

Lecture II

Poisson Line Processes and Network Efficiency

Georges-Louis Leclerc, Comte de Buffon

(7 September, 1707 – 16 April, 1788)

- French encyclopædist, corresponded with Daniel Bernoulli;
- Published "Essai d'Arithmétique Morale" (Comte de Buffon, 1829)

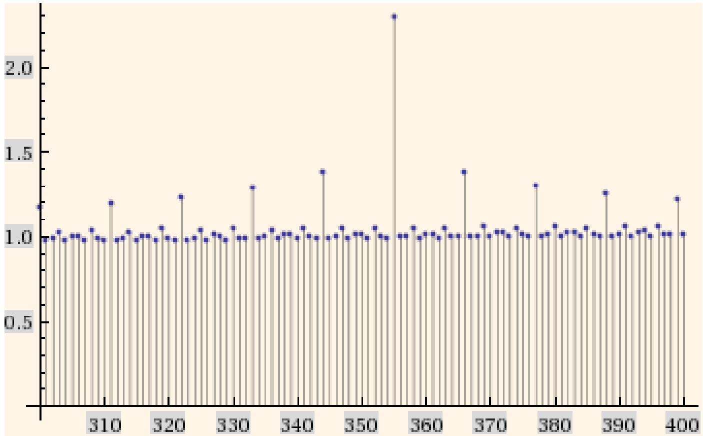
  (web translation by Hey, Neugebauer, and Pasca, 2010);
- Measured and valued uncertainty;
- Striking mixture of empiricism and rationalism;
- "Calculate π: drop needle randomly on ruled plane, count mean proportion of hits."

A statistical puzzle

- Since de Buffon, several people have published (recreational!) attempts to estimate π this way. Stigler (1991) gives a list, eg citing Lazzarini (1901).
- Suppose you have to design such an experiment.
- Unit-length needle, unit-ruled floor. Say you must choose the number of trials n.
- The number of successes X is Binomial(n, $2/\pi$).
- Measure success by $1/\sqrt{|2/\pi - X/n|}$.
- Choose $n \in [300, 400]$, say.

  Which choice would you make?

Estimating π with Buffon



Log of mean of $1/\sqrt{|2/\pi - X/n|}$
for X with distribution Binomial(n, $2/\pi$).

Continued fraction for π

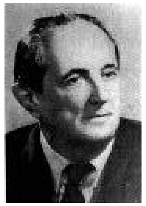
$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}$$

$$\approx 3 + \cfrac{1}{7 + \cfrac{1}{15+1}} = \frac{355}{113}.$$

So consider

$$\frac{2}{\pi} - \frac{X}{355} \qquad \ldots$$

Dualizing Buffon, with Steinhaus



Hugo Steinhaus (1887-1972) observed something which we can view as the dual of the Buffon phenomenon;

- Replace needle by convex polygon, or convex curve, perimeter 2ℓ, diameter less than 1. Randomize lines not polygon! Probability of hitting ruled line is $2\ell/\pi$ (see, eg, Gnedenko, 1998, Ch. 2.4, Example 5).
- There is a notion of invariant measure on the space of lines in the plane;
- Invariant measure of set of lines hitting a regularizable curve is proportional to the length of the curve (Steinhaus, 1930).
- Mean number of random lines hitting curve?

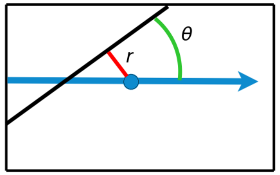
Random lines

How to build random patterns of lines?

1. We could use the theory of Random Closed Sets: characterize a random closed set Ξ by void-probabilities $\mathbb{P}[\Xi \not\uparrow A] = \mathbb{P}[\Xi \cap A = \emptyset]$.
2. More constructively, we could try the Boolean model: take union of (possibly random) sets placed on Poisson points.
3. However, the Boolean model does not adapt well to random lines, since random lines are not localized.
4. Simple solution: parametrize lines ℓ by signed perpendicular distance r and angle θ:
    4.1 each line ℓ corresponds to a point (r, θ) in representation space;
    4.2 view Poisson line process as Poisson point process in representation space, μ being invariant line measure.
5. The representation space is a cylinder with twist: in fact an infinite Möbius strip, or punctured projective plane.
6. Calculations: often reduce to probabilities that there are no lines of particular forms.

Illustration of invariant line measure

The key to understanding invariant line measure:
use a good parametrization of (undirected) lines in the plane.



Parametrize the line ℓ by fixing a reference point **o** and a reference line through the point. Define:

(a) (signed) perpendicular distance r of ℓ from **o**;
(b) angle $\theta \in (0, \pi]$ between ℓ and reference line.

Invariant line measure is then given by

$$\frac{1}{2}\ \mathrm{d}r\ \mathrm{d}\theta\,,$$

(Normalization by $\frac{1}{2}$:
unit measure for set of lines hitting unit segment).

Poisson Line Process

Definition: Poisson line process

A (stationary) Poisson (undirected) line process of intensity $\lambda > 0$ is obtained from a Poisson point process on $\mathbb{R} \times [0, \pi)$ with intensity $\frac{1}{2}\lambda\,\mathrm{d}r\,\mathrm{d}\theta$ as follows: for each point (r, θ) in the point process, a line is generated at signed distance r from the origin and making an angle θ with the x-axis.

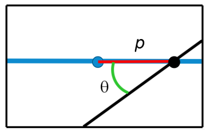
(Stationary) Poisson *directed* line processes can be obtained from undirected Poisson line processes by assigning a direction to each line. A natural parameter space would then be $\mathbb{R} \times [0, 2\pi)$.

Tools for line processes

Buffon The length of a curve equals the mean number of hits by a unit-intensity Poisson line process (intensity $\frac{1}{2}\ \mathrm{d}r\ \mathrm{d}\theta$);

Slivnyak Suppose we condition a Poisson process on placing a “point” z at a specified location. The conditioned process is again a Poisson process with added z;

Angles Generate a planar line process from a unit-intensity Poisson point process on a reference line ℓ, by constructing lines through the points p whose angles $\theta \in [0, \pi)$ to ℓ are independent with density $\frac{1}{2}\sin\theta$. The result is a unit-intensity Poisson line process. Intensity measure in these coordinates: $\frac{1}{2}\sin\theta\ \mathrm{d}p\ \mathrm{d}\theta$.



To 3-space and beyond

The notion of line processes in higher dimensions is important in generalizations.

Poisson line processes in $\mathbb{R}^d$:

- Parametrize by ϖ "direction" of (undirected) line (point on hemisphere: more properly $\mathbb{R}P^{d-1}$),

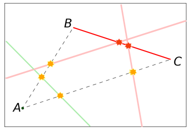
  and x location on perpendicular hyperplane.
- Invariant measure now $c_d \, \mathrm{d}x \times \nu_d(\mathrm{d}\varpi)$.

  Coordinate x is "twisted" by ϖ: measure theory doesn't see this.
- Variant parametrization replaces x by p, intersection of ℓ with reference hyperplane.

  Invariant measure now $c_d \sin\theta \;\, \mathrm{d}p \times \nu_{d-1}(\mathrm{d}\varpi)$.

A Useful Exercise for planar lines:

- Compute probability that point A is separated from line segment BC by a unit intensity Poisson line process.



Let T be triangle ABC. Let $[T]$ be the set of lines intersecting T. Let each line $\ell \in [T]$ score $+1$ for each intersection with AB or AC, -1 for intersection with BC.

- Integrating score against invariant line measure μ, count contributions from intersections with AB, AC and BC: $\mu([AB]) = |AB|$, $\mu([AC]) = |AC|$, $-\mu([BC]) = -|BC|$.
- Lines intersecting AB and BC, or AC and BC, score zero. Lines intersecting AB and AC score 2. (Ignore measure-zero cases of vertex intersections.)
- Deduce invariant line measure of lines intersecting AB and AC is $\frac{1}{2}(|AB| + |AC| - |BC|)$.
- Accordingly required probability is $1 - \exp\left(-\frac{1}{2}(|AB| + |AC| - |BC|)\right)$.

Some challenging facts about line processes

1. The Steinhaus method of determining length by the line measure of intercepts can be viewed as lying at the heart of modern stereology: take a random slice (line or plane), count geometrical features along the slice, deduce estimates of 3d geometric structure.

   The resulting geometric calculations can be complicated!
2. D.G. Kendall conjecture. Consider a plane tessellated by an invariant Poisson line process. Consider the cell containing the origin. Suppose that we condition on this cell having large area. What is its limiting shape?

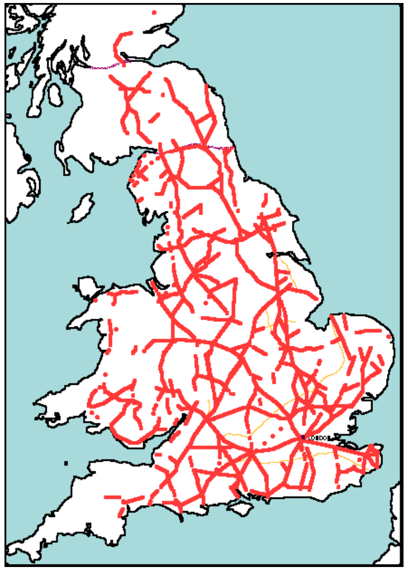
   Answer: Circular (Miles, Kovalenko, Schneider and others . . .).
3. Rollo Davidson's conjecture. Consider an invariant line process with finite mean-square counts on compact sets, and with no parallel lines. Must it be a Cox process? (Poisson process with randomized intensity measure.)

   Answer: No. Kallenberg and Kingman describe a beautiful construction based on a random lattice.

Transportation Network Questions

Mastiles Lane, Yorkshire Dales

An ancient optimization problem



A Roman Emperor's dilemma:

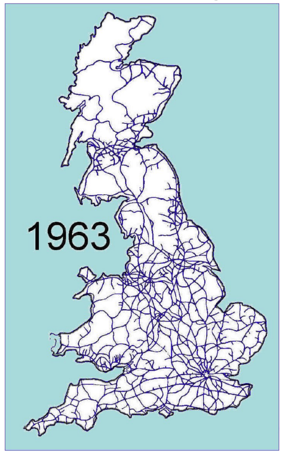
PRO: Roads are needed to move legions quickly around the country;

CON: Roads are expensive to build and maintain;

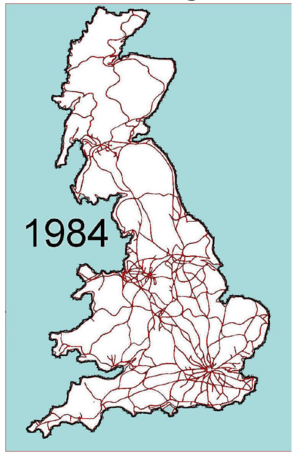
Pro optimo quod faciendum est?

Modern variants

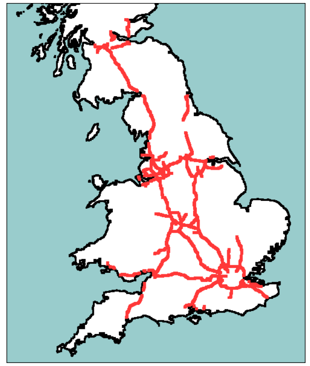
British Railway network before Beeching



British Railway network after Beeching



UK Motorways:



A Mathematical Idealization

(a problem in frustrated optimization)

Consider N cities $x^{(N)} = \{x_1, \ldots, x_N\}$ in square side $\sqrt{N}$.
Assess road network $G = G(x^{(N)})$ connecting cities by:

- network total road length $\mathrm{len}(G)$
(minimized by Steiner minimum tree $\mathrm{ST}(x^{(N)})$);
versus
- average network distance between two random cities,

$$\mathrm{average}(G) = \frac{1}{N(N-1)} \sum_{i \neq j} \sum \mathrm{dist}_G(x_i, x_j),$$

(minimized by laying tarmac for complete graph).
- Perhaps the average ratio would be a good measure of performance?

$$\frac{1}{N(N-1)} \sum_{i \neq j} \sum \frac{\mathrm{dist}_G(x_i, x_j)}{\|x_i - x_j\|}$$

Main Question

Aldous and WSK (2008) provide answers for the

Question

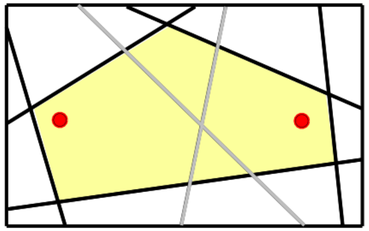
Consider a configuration $x^{(N)}$ of N cities in $[0, \sqrt{N}]^2$ as above, and a well-chosen connecting network $G = G(x^{(N)})$. How does large-N trade-off between $\mathrm{len}(G)$ and $\mathrm{average}(G)$ behave?

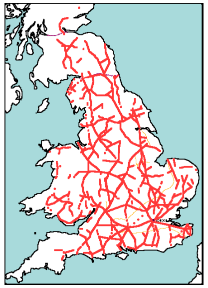
(And how clever do we have to be to get a good trade-off?)

Note:

- $\mathrm{len}(\mathrm{ST}(x^{(N)})) \leq O(N)$ (eg, Steele, 1997, §2.2);
- Average Euclidean distance between two randomly chosen cities is at most $\sqrt{2N}$;
- Perhaps increasing total network length by const $\times N^{\alpha}$ might achieve average network distance no more than order N^{β} longer than average Euclidean distance?

Answer to main question (I)



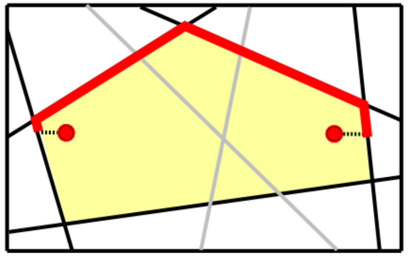


- Augment Steiner tree by a low-intensity invariant Poisson line process Π_1.
- Unit intensity is $\frac{1}{2}\,\mathrm{d}\,r\;\mathrm{d}\,\theta$: we will use this and scale.
- Pick two cities x and y at distance $n = \sqrt{N}$ units apart.

  Remove lines separating the two cities;

  focus on cell $C_{x,y}$ containing the two cities.

Answer to main question (II)

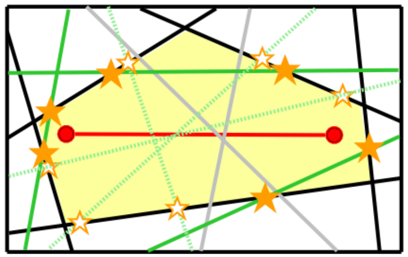


- Upper-bound "network distance" between two cities by mean semi-perimeter of cell,

$$\frac{1}{2}\,\mathbb{E}\left[\operatorname{len}\partial C_{x,y}\right]$$

- Aldous and WSK (2008) answer Main Question using this, and use other methods from stochastic geometry to show that the resolution is nearly optimal.

The key construction



(Remember, line process renormalized to unit intensity.)

- Compute mean length of $\partial C_{x,y}$ by use of independent unit-intensity invariant Poisson line process Π_2, and determine the mean number of hits.
- It is convenient to form Π_2^* by deleting from Π_2 those lines separating x from y. (Mean number of hits contributed by these lines: $2|x-y| = 2n$.)

Some stochastic geometry (I)

- We have

$$\mathbb{E}\left[\operatorname{len}\partial C_{x,y}\right] - 2|x-y| \quad = \quad \mathbb{E}\left[\#\left(\Pi_2^* \cap \partial C_{x,y}\right)\right].$$

- This is the total intensity of the intersection point process $\Pi_1^* \cap \Pi_2^*$ thinned by removing $z \in \Pi_1^* \cap \Pi_2^*$ when z is separated from both x and y by Π_1^*.
- We can appeal to a variant of Slivnyak's theorem: the retention probability for z is the probability that no line of Π_1^* hits both of segments $\overline{xz}$ and $\overline{yz}$, namely

$$\exp\left(-\tfrac{1}{2}\left(|x-z| + |y-z| - |x-y|\right)\right) \;=\; \exp\left(-\tfrac{1}{2}\left(\eta - n\right)\right),$$

where $\eta = |x-z| + |y-z|$.

Some stochastic geometry (II)

- The intensity of the completely unthinned intersection process $\Pi_1 \cap \Pi_2$ is $\frac{\pi}{2}$.
- The intensity of $\Pi_1^* \cap \Pi_2^*$ is obtained by careful computation of the probability that the intersection lines of a point of $\Pi_1 \cap \Pi_2$ do not hit $\overline{xy}$, using the "Angle" construction from given above. The resulting intensity is:

$$\frac{\pi}{2} \times \frac{\alpha - \sin\alpha}{\pi} \quad = \quad \frac{\alpha - \sin\alpha}{2},$$

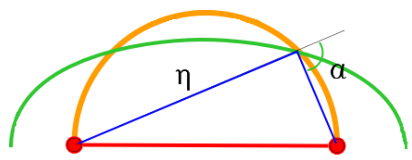
where α is the exterior angle of the triangle Δxyz at z.

Mean perimeter length as a double integral

Theorem: Mean perimeter length

$$\mathbb{E}\left[\operatorname{len}\partial C_{x,y}\right] - 2|x-y| \quad = \quad \frac{1}{2}\iint_{\mathbb{R}^2} (\alpha - \sin\alpha)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\,\mathrm{d}z$$

- Note that $\alpha = \alpha(z)$ and $\eta = \eta(z)$ both depend on z.



- Fixed α: locus of z is circle.
- Fixed η: locus of z is ellipse.

Asymptotics

Theorem: Asymptotic cell perimeter

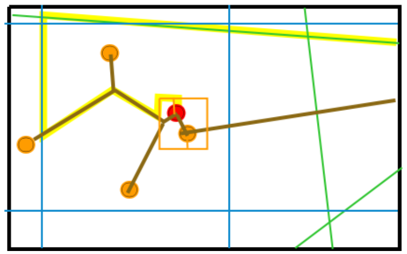
Careful asymptotics for $n \to \infty$ show that

$$\mathbb{E}\left[\tfrac{1}{2}\operatorname{len}\partial C_{x,y}\right] \quad = \quad n + \tfrac{1}{4}\iint_{\mathbb{R}^2}(\alpha - \sin\alpha)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\,\mathrm{d}z \quad \approx \quad n + \frac{4}{3}\left(\log n + \gamma + \frac{5}{3}\right)$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

Thus a unit-intensity invariant Poisson line process is within $O(\log n)$ of providing connections which are as efficient as Euclidean connections.

A Transportation Hierarchy



Use a hierarchy of:

1. a (sparse) Poisson line process;
2. a rectangular grid at a moderately large length scale;
3. the Steiner minimum tree $\mathrm{ST}(x^{(N)})$;
4. a few boxes from a grid at a small length scale, to avoid potential "hot-spots" where cities are close (boxes are connected to the cities).

Answering the main question

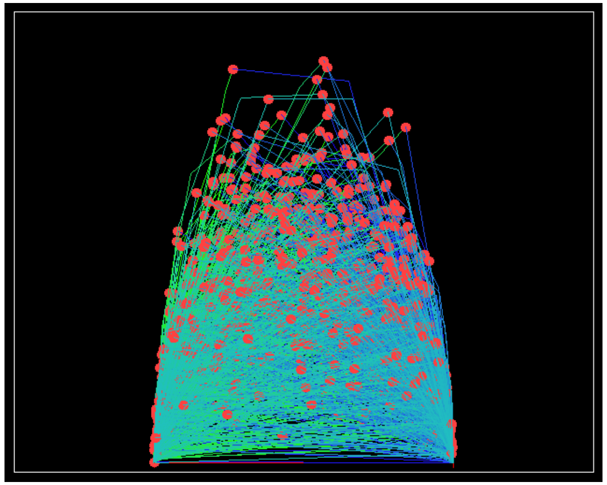
Theorem: Main efficiency result

For any configuration $x^{(N)}$ in square side $\sqrt{N}$ and for any sequence $w_N \to \infty$ there are connecting networks G_N such that:

$$\begin{aligned} \text{len}(G_N) &= \text{len}(\text{ST}(x^{(N)})) + o(N) \\ \text{average}(G_N) &= \frac{1}{N(N-1)} \sum\sum_{i \neq j} \|x_i - x_j\| + o(w_N \log N) \end{aligned}$$

The sequence $\{w_N\}$ can tend to infinity arbitrarily slowly.

Simulations (example)



1000 simulations at $n = 1000000$: average 21.22, s.e. 0.23, asymptotic 21.413.

Vertical exaggeration: $\sqrt{n}$

A Complementary Result

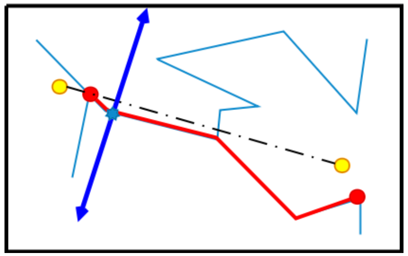
Theorem: Complementary Efficiency Result

Given a configuration of N cities in $[0, \sqrt{N}]^2$ which is $L_N = o(\sqrt{\log N})$-equidistributed: random choice X_N of city can be coupled to uniformly random point Y_N so that

$$\mathbb{E}\left[\min\left\{1, \frac{|X_N - Y_N|}{L_N}\right\}\right] \longrightarrow 0;$$

then any connecting network G_N with length bounded above by a multiple of N connects the cities with average connection length exceeding average Euclidean connection length by at least $\Omega(\sqrt{\log N})$.

Main ideas of proof



Use tension between two facts:

(a) efficient connection of a random pair of cities forces a path which is almost parallel to the Euclidean path, *and*

(b) the coupling means such a random pair is almost an independent uniform draw from $[0, \sqrt{N}]^2$ (equidistribution),

so a random perpendicular to the Euclidean path is almost a uniformly random line.

Further Questions

Question

Fluctuations: Given a good compromise between $\text{average}(G)$ and $\text{len}(G)$, how might the variance behave?

Question

True geodesics: The upper bound is obtained by controlling non-geodesic paths. How might true geodesics behave?

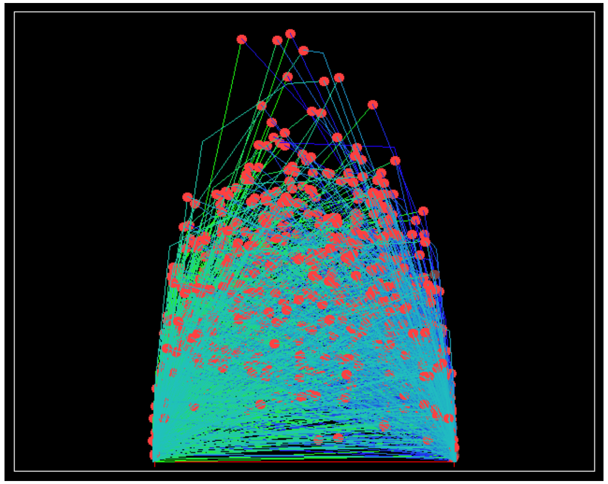
Question

Flows: Consider a network which exhibits good trade-offs. What can be said about flows of traffic in this network?

Question

Scale-invariance: Can we generate scale-invariant random structures resembling suitable compromise networks?

Location of maximal lateral deviation (I)



Red dots indicate maximum lateral deviation.
(NB: vertical exaggeration!)

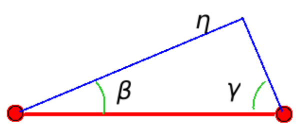
Location of maximal lateral deviation (II)

Remark: Density / intensity

Using $u = \frac{2}{n}x \in (-1,1)$ and $v = \frac{y}{\sqrt{n}} > 0$,

$$\frac{1}{4}\left(\sin\beta + \sin\gamma - \sin(\beta+\gamma)\right)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\,\mathrm{d}x\;\mathrm{d}y$$

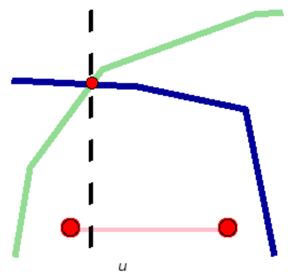
$$\approx \quad \frac{v^3}{(1-u^2)^2}\exp\left(-\frac{v^2}{1-u^2}\right)\,\mathrm{d}u\;\mathrm{d}v$$



ASYMPTOTICALLY

- Location of maximum is uniformly distributed;
- Conditional height of maximum is length of Gaussian 4-vector.

Variance and growth process (I)



- Consider cell boundary as path with x-coordinate X_τ, parametrized by τ (excess over geodesic distance).
- Let Θ be angle with positive x-axis;
- Θ jumps at Poisson point process intensity $\frac{1}{2}$;
- A jump $-\Delta\Theta = \Theta - \Theta_-$ obeys

$$\mathbb{P}\left[\Theta_- - \Theta \le \theta \mid \Theta_-\right] \quad = \quad \frac{1-\cos\theta}{1-\cos\Theta_-}\,.$$

Variance and growth process (II)

Sketch of argument:

1. $X_t \approx X_0 + 2\Theta_0^{-2} \int_0^t \frac{\Theta_0^2}{\Theta_s^2}\,\mathrm{d}s$;
2. $\log(\Theta_0/\Theta_t) \approx \xi_t$ a non-decreasing Lévy process;
3. $M_t = \xi_t - \frac{3}{4}t$ is an L^2 martingale.
4. Define a stopping time by $\sigma(n)$ by $X_{\sigma(n)} = n$. So $n = X_{\sigma(n)} \approx X_0 + 2\Theta_0^{-2} \int_0^{\sigma(n)} \Theta_0^2/\Theta_s^2\,\mathrm{d}s$.
5. Now $\exp(2\xi_{\sigma(n)})$ is (approximately) a self-similar process, about which much information is available;
6. Analyze the tautology $\sigma(n) \approx \frac{2}{3}\left(\log n - 2M_{\sigma(n)} + \log\left(\exp(2\xi_{\sigma(n)})/n\right)\right)$.
7. Control $\log\left(\exp(2\xi_{\sigma(n)})/n\right)$ using work of Bertoin and Yor (2005);
8. Deduce $\sigma(n) \approx \frac{2}{3}(\log n + O(1) - 2M_{\sigma(n)})$; now $M_{\sigma(n)})$ can be shown to be L^2.

Theorem: Near-geodesic fluctuations

Let $\sigma(n)$ be the geodesic distance between the half-x-axis $(-\infty, 0]$ and $x = n$. Then

$$\begin{aligned} \mathbb{E}[\sigma(n)] &= \frac{2}{3}\log n + O(1) \\ \mathrm{Var}[\sigma(n)] &= \frac{20}{27}\log n + O\left(\sqrt{\log n}\right). \end{aligned}$$

Hence perimeter length fluctuations are $O(\sqrt{\log n})$.

Check: Earlier today, we calculated mean semi-perimeter-length as $n + \frac{4}{3}\log n + \ldots$. Each semi-perimeter excess contains contributions from two of these paths (growth from left, growth from right), so this stochastic calculus approach is in agreement with the stochastic geometry calculations.

What about true geodesics?

- The above can be used to show true geodesics typically have smaller excess than our paths, nevertheless
- the number of "top-to-bottom" crossings of a true geodesic is stochastically bounded in any region $[na, nb] \subseteq (0, n)$, so all but a stochastically bounded number of "short-cuts" must be within $O(n/\log n)$ of start or end, affecting coefficient of $\log(n)$ but no more; indeed

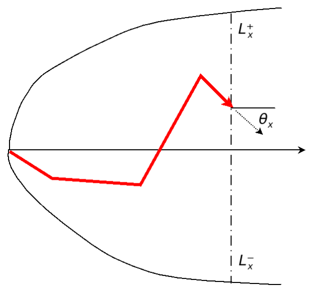
Theorem: Lower bound on mean excess

Any path of length n built using the Poisson lines must have mean excess at least

$$\left(\log 4 - \frac{5}{4}\right)\log n + o(\log n)\,.$$

-

How much better can a true geodesic be?



Key tools:
Measure excess with

$$\mathbb{E}\left[\int_1^n (\sec\theta_x - 1)\,\mathrm{d}x\right] \geq \frac{1}{2}\int_1^n \mathbb{E}\left[\theta_x^2\right]\mathrm{d}x$$

$$\mathbb{P}[|\theta_x| \geq u] \quad \geq \quad \left(\mathbb{E}\left[\exp(-uL_x^+)\right]\right)^2$$

Approximate by integrals involving Gaussian kernels . . .

$$\frac{2}{\sqrt{p^2+4}+p} \quad \leq \quad e^{p^2/2}\int_p^\infty e^{-s^2/2}\,\mathrm{d}s \quad \leq \quad \frac{4}{\sqrt{p^2+8}+3p}$$

Birnbaum (1942) Sampford (1953)

Lecture III

Traffic Flow in a Poissonian City

Traffic in the City

Calculations for traffic distribution

Improper line processes

Effective simulation of limiting distribution

Empirical comparisons

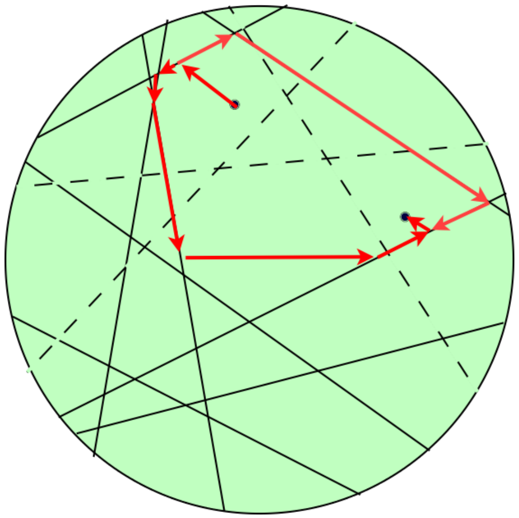
Conclusion

Traffic in a random city

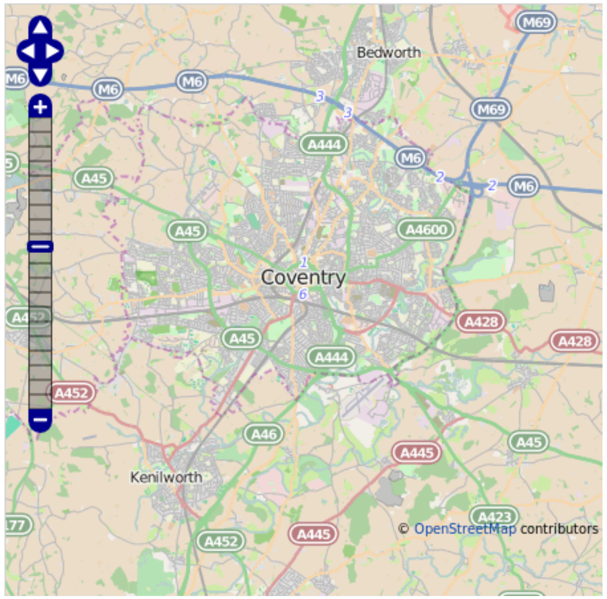
The Poissonian City (I)

Design an idealized city!

Complete the 2 routes using off-road sections.



Real city plan



Palm distributions

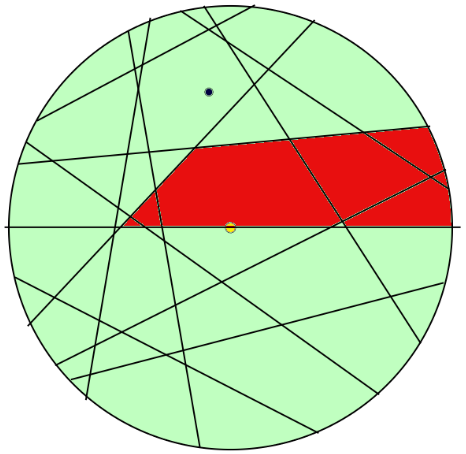
- Recall Slivynak's theorem: if we condition on a Poisson process placing a point at the origin . . .
- . . . then the *reduced Palm distribution*, the residual point pattern (*minus* the conditioned point), is again Poisson with the same intensity.
- Best understood in the same way as we understand abstract conditional expectation: plugging in the Palm distribution in certain formulae gives the correct answer. (Reductionist point of view: just "integration by parts".)
- Similarly for Poisson line processes:

  If we condition on a line through the centre, then the remainder of the line process is again Poisson.
- So what could we then say about traffic at the centre?

The Poissonian City (II)

Aldous idea: Every point-pair (x, y) contributes infinitesimal $\mathrm{d}x\,\mathrm{d}y$ split between two near-geodesics.

Compute 4-volume of random polytope!



More sophisticated approaches (I)

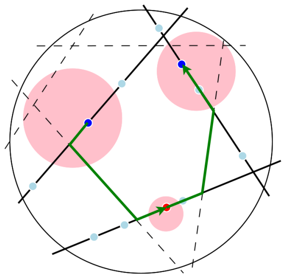
Options:

Up to constant factors (readily computable using Slivnyak's theorem) we get essentially the same mean value behaviour if instead traffic is generated

- uniformly between pairs of points lying on the lines only;
- between points of a point pattern that is Poisson along the lines (using Lebesgue length measure);
- between the points of intersections of lines;
- the process used to generate traffic along fibres or at intersections can be random if it is of finite mean and does not depend on location.

More sophisticated approaches (II):

Poisson locations scattered along Π



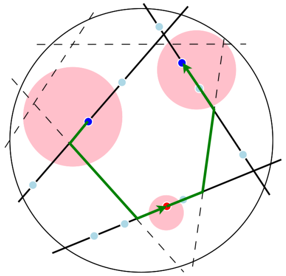
We consider a Poissonian city formed by unit intensity Poisson line process Π. Independently, source / destination points form a Poisson point process whose intensity is Lebesgue linear measure along Π.

Let $2 f_{x_0,x_1,x_2}(\Pi)$ indicate whether x_1 lies on near-geodesic $x_1 \to x_2$, with $x_0 \in \mathrm{ball}(z_0, \epsilon)$, $x_1 \in \mathrm{ball}(z_1, \rho_1)$.
Our task is to compute

$$\mathbb{E}\left[\sum_{x_0 \in \Psi} \sum_{x_1 \in \Psi} \sum_{x_2 \in \Psi} f_{x_0,x_1,x_2}(\Pi)\right]$$

More sophisticated approaches (III):

Palm theory for lines of Π

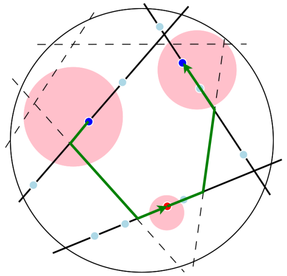


We consider a Poissonian city formed by unit intensity Poisson line process Π. Independently, source / destination points form a Poisson point process whose intensity is Lebesgue linear measure along Π.

Palm theory for ℓ_2, change coordinates, cutoff for x_2:

$$\left(\frac{\pi}{2}\right)^3 \int_{\text{ball}(z_0,\varepsilon)} \int_{\text{ball}(z_1,\rho_1)} \int_{\text{ball}(z_2,\rho_2)} \mathbb{E}\left[f_{x_0,x_1,x_2}(\Pi \cup \{\ell_*\})\right] \, \mathrm{d}\,x_2\,\mathrm{d}\,x_1\,\mathrm{d}x_0 \quad \text{asymptotically as } n \to \infty$$

Results for more sophisticated approaches (I)

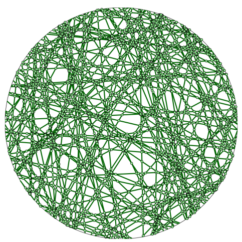


We consider a Poissonian city formed by unit intensity Poisson line process Π. Independently, source / destination points form a Poisson point process whose intensity is Lebesgue linear measure along Π.

Theorem: Mean traffic between Poisson points

If source / destination points are scattered by a (conditionally independent) Poisson process along Π then mean traffic through a “typical” point x_0 is given by integrating $\left(\frac{\pi}{2}\right)^3 \mathbb{E}\left[f_{x_0,x_1,x_2}(\Pi \cup \{\ell_*\})\right]$ against x_1 and x_2, conditional on x_0 belonging to Π *via* a randomly oriented line ℓ_*.

Results for more sophisticated approaches (II)

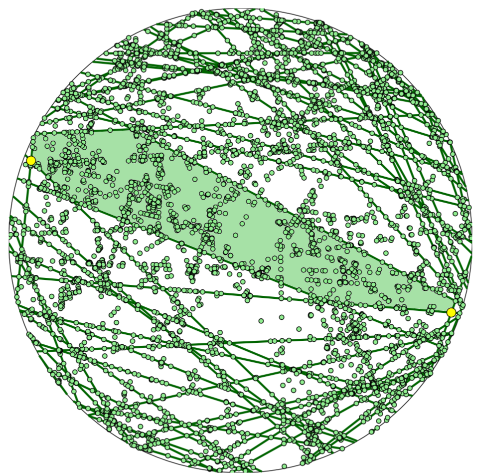


Consider again a Poissonian city formed by unit intensity Poisson line process Π.
Source / destination points are formed by the line intersections of Π.

Theorem: Mean traffic between intersections

If source / destination points are formed by the intersections of the lines of Π then mean traffic through a “typical” point x_0 is given by integrating $2\left(\frac{\pi}{4}\right)^3 \mathbb{E}\left[f_{x_0,x_1,x_2}(\Pi \cup \{\ell_*\})\right]$ against x_1 and x_2, conditional on x_0 belonging to Π *via* a randomly oriented line ℓ_*.

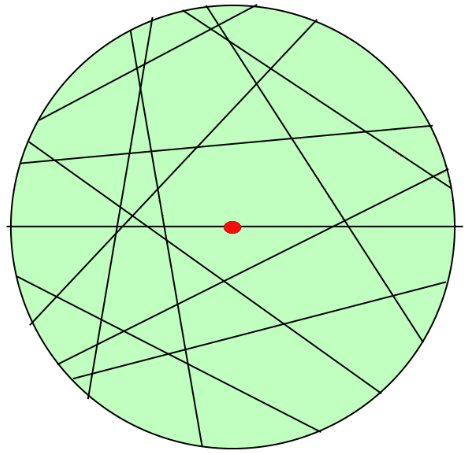
Illustration of traffic routes between intersection points



Measure congestion?

Questions:

1. What is the mean traffic flow through the centre?
2. Is there a limiting scaled distribution?
3. Is the limiting distribution accessible to computation?



Mean traffic computation

Mean traffic flow through centre can be obtained by direct arguments from stochastic geometry:

Theorem: Mean traffic flow through centre

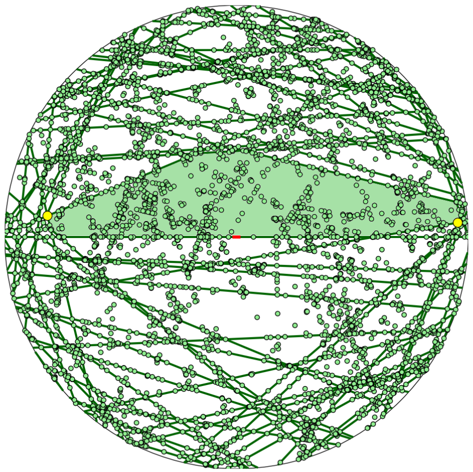
For $\rho = \sqrt{r^2 + s^2 - 2rs\cos\theta}$, the traffic flow T_n through the centre of a Poissonian city built on a disk of radius n has mean value

$$\mathbb{E}\left[T_n\right] \quad = \quad \int_0^\pi \int_0^n \int_0^n \exp\left(-\tfrac{1}{2}(r+s-\rho)\right) r \, \mathrm{d}r \, s \, \mathrm{d}s \, \theta \, \mathrm{d}\theta \, .$$

Asymptotically for large city radius ($n \to \infty$),

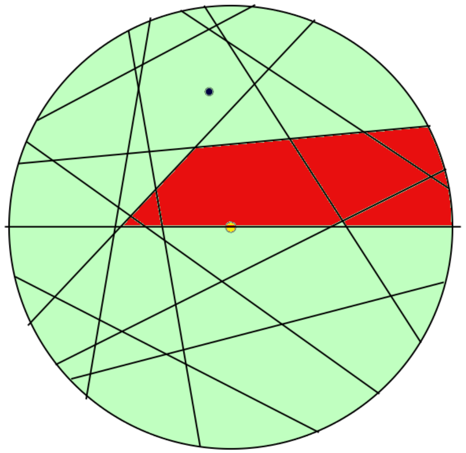
$$\mathbb{E}\left[T_n\right] \quad = \quad 2n^3 + O\left(n^2\sqrt{n}\right) .$$

Where does most of the traffic come from?



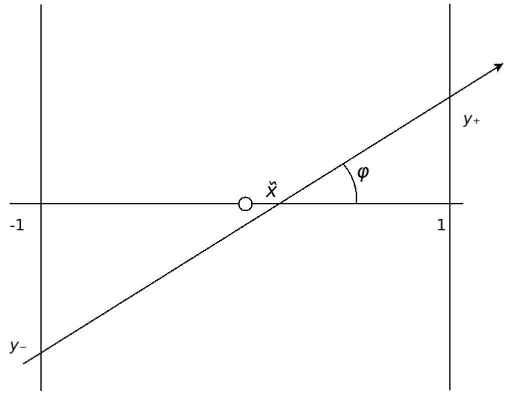
Existence of limiting distribution?

Is there a limiting distribution for T_n/n^3 as $n \to \infty$?



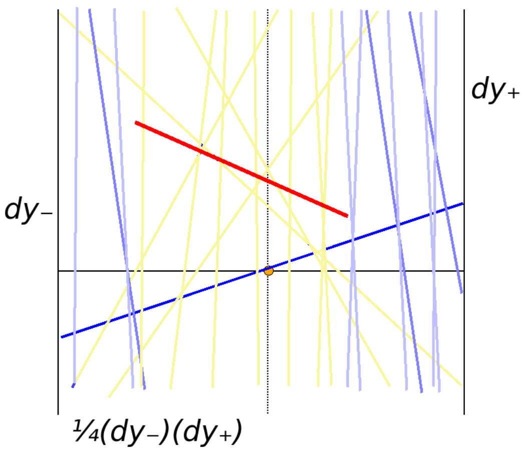
Improper line processes to the rescue (I)

- Consider *stretching* the city.
- Vertically, rescale by $1/\sqrt{n}$. Horizontally, rescale by $1/n$. (Affine transformation.)
- New coordinates y_- and y_+.
- New (improper) intensity (as $n \to \infty$) is $\frac{1}{4}\,\mathrm{d}y_-\,\mathrm{d}y_+$,



Improper line processes to the rescue (II)

Infinitely many nearly vertical lines running near any point.



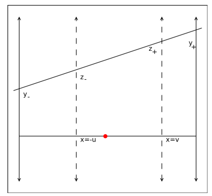
Centre sees no traffic when separated off by lines.
Coupling argument: limiting distribution non-degenerate.

Computing mean traffic (I)

Change coordinates to heights z_- and z_+ of intercepts on $x = -u$ and $x = v$.

$z_- = \frac{1}{2}((1+u)y_- + (1-u)y_+)$;

$z_+ = \frac{1}{2}((1-v)y_- + (1+v)y_+)$.



Compute $\mathrm{d}z_- \wedge \mathrm{d}z_+$ using formalism of differential forms:

$$\begin{aligned}\mathrm{d}z_- \wedge \mathrm{d}z_+ &= \frac{1+u}{2}\frac{1+v}{2}\,\mathrm{d}y_- \wedge \mathrm{d}y_+ + \frac{1-u}{2}\frac{1-v}{2}\,\mathrm{d}y_+ \wedge \mathrm{d}y_- \\ &= \frac{u+v}{2}\,\mathrm{d}y_- \wedge \mathrm{d}y_+\,,\end{aligned}$$

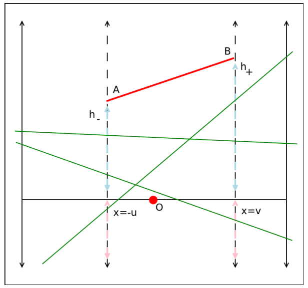
hence

$$\frac{1}{4}\,\mathrm{d}y_-\,\mathrm{d}y_+ = \frac{1}{2(u+v)}\,\mathrm{d}z_-\,\mathrm{d}z_+\,.$$

Computing mean traffic (II)

Measure of lines separating AB from O:

- Area of a rectangle $h_{-}h_{+}/(2(u+v))$;
- plus area of a triangle $\frac{1}{2}(u/v)h_{+} \times h_{+}/(2(u+v))$;
- plus area of another triangle $\frac{1}{2}(v/u)h_{-} \times h_{-}/(2(u+v))$.



$$\frac{h_{-}h_{+}}{2(u+v)} + \frac{u}{v}\frac{h_{+}h_{+}}{4(u+v)} + \frac{v}{u}\frac{h_{-}h_{-}}{4(u+v)} = \frac{1}{4(u+v)}\left(\sqrt{\frac{u}{v}}h_{+} + \sqrt{\frac{v}{u}}h_{-}\right)^2 .$$

Computing mean traffic (III)

We now compute the mean 4-volume of (A, B) above $(-u, v)$ such that AB is separated from O:

$$\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{4(u+v)}\left(\sqrt{\frac{u}{v}}h_{+} + \sqrt{\frac{v}{u}}h_{-}\right)^2\right) \mathrm{d}h_{-}\,\mathrm{d}h_{+} = 2(u+v) .$$

Now we can integrate out $u > 0$ and $v > 0$. Suppose $(-u, v)$ is contained in the region $(-u_0, v_0) \subseteq (-1, 1)$: total mean traffic through the origin is given by

$$\frac{1}{2} \times 2 \times \int_0^{u_0} \int_0^{v_0} 2(u+v)\,\mathrm{d}u\,\mathrm{d}v = u_0 v_0 (u_0 + v_0) .$$

Mean traffic flow in regular region

Theorem: Mean traffic flow

Consider a general convex body C, and traffic derived from a Poissonian city based on a Poisson line process of intensity n. Consider a point x in the region, and Palm condition on a line ℓ passing through this point with further conditioning on the direction of the line. Suppose that the two intervals $(\ell(\theta) \cap C) \setminus \{x\}$ have lengths $w_{-}(\theta)$ and $w_{+}(\theta)$. The traffic flow T_n through x under this conditioning has mean value asymptotic to

$$\mathbb{E}[T_n] \sim w_{-}(\theta)w_{+}(\theta)(w_{-}(\theta) + w_{+}(\theta))n^3 .$$

Mean traffic flow over the disk

Theorem: Mean traffic flow (case of disk)

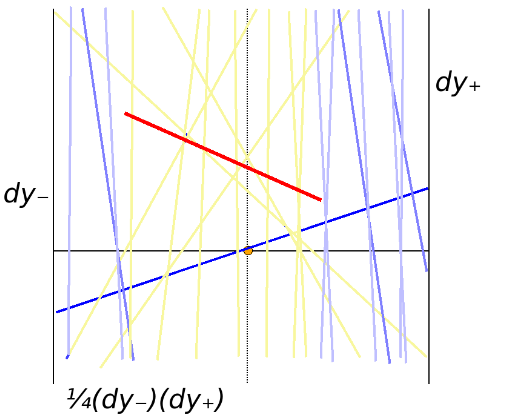
Consider traffic derived from a Poissonian city based on a Poisson line process of intensity n in a unit disk. Consider a point x in the region, and Palm condition on a line ℓ passing through this point *without* further conditioning on the direction of the line. The traffic flow T_n through x under this conditioning has mean value asymptotic to

$$\mathbb{E}[T_n] \sim \frac{2}{\pi}(1-r^2)\int_0^\pi \sqrt{1 - r^2 \sin^2\theta}\,\mathrm{d}\theta = \frac{4}{\pi}(1-r^2)E(r) .$$

where $r \in [0, 1]$ is the distance of x from the disk centre.

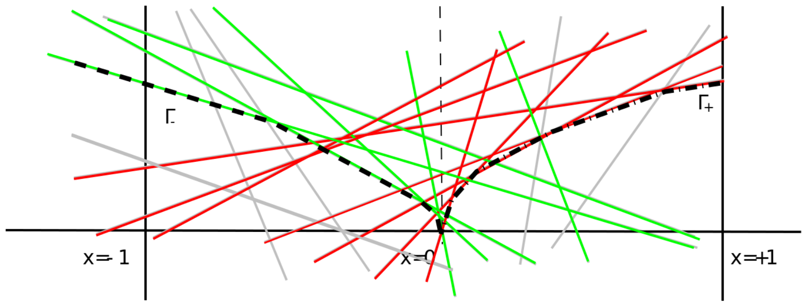
There is a limiting distribution, but how to access it?

- Can't find analytic expression for the limiting distribution $\lim_{n\to\infty} \mathcal{L}\left(T_n/n^3\right)$.
- We now can recognize T_n as 4-dimensional volume of random polytope (use $\frac{1}{4}\,\mathrm{d}y_-\;\mathrm{d}y_+$).
- Can we produce achievable approximating simulations?



Seminal curves

WSK (2014):



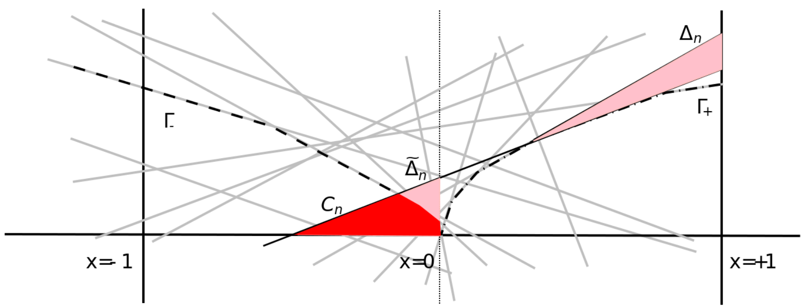
- Focus on upper half-plane (quadrants $Q_\pm$) and subset of source/destinations in $Q_- \times Q_+$ not separated from $\mathbf{o}$.
- Identify two sub-families of improper line process Π_∞:

$$\Pi_{\infty,\pm} \quad = \quad \{\ell \in \Pi_\infty : \text{slope}(\ell) = \pm, \ell \text{ intercepts } \mp x\text{-axis}\,\}\,.$$

- Define *seminal curves* $\Gamma_\pm$ as the concave lower envelopes of the unions of lines in $\Pi_{\infty,\pm}$.

A series expression (I)

WSK (2014):



Use regions in figure to bound 4-volume ...

$$\left(\int_0^1 \Gamma_-(-s)\;\mathrm{d}s\right) \times \left(\int_0^1 \Gamma_+(s)\;\mathrm{d}s\right)$$

$$+ \sum_{n=0}^{\infty} \mathrm{Leb}_2(C_n^+)\,\mathrm{Leb}_2(\Delta_n^+) + \sum_{n=0}^{\infty} \mathrm{Leb}_2(C_n^-)\,\mathrm{Leb}_2(\Delta_n^-)$$

Recursive representation of $\Gamma_\pm$

Theorem: Representing the seminal curves

Consider times of slope-change of $\Gamma = \Gamma_+$ in reversed time: $1 = S_0 > S_1 > S_2 > \ldots > 0$.
Successive tangents $\ell_0, \ell_1, \ell_2, \ldots, \ell_n$: slopes $\Gamma'(s)$ for $S_n \geq s > S_{n+1}$.
Let $Y_n = \Gamma(S_n) - S_n\Gamma'(S_n)$ be intercept of ℓ_n on y axis,

$$\frac{1}{S_{n+1}} \quad = \quad \frac{1}{S_n} + \frac{4}{Y_n^2}E_{n+1}\,,$$

$$\Gamma'(S_{n+1}) \quad = \quad \Gamma'(S_n) + \frac{Y_n}{S_{n+1}}\sqrt{U_{n+1}}\,.$$

for $E_n \sim$ Exponential(1), $U_n \sim$ Uniform$[0,1]$, independent of each other and $\Gamma(S_0) = \Gamma(1)$, $\Gamma'(S_0) = \Gamma'(1)$.
($\Gamma(1)$, $\Gamma'(1)$ joint distribution is computable.)

Simulation Error Estimation

Theorem: L^1 simulation error

The L^1 error of the 4-volume approximation

$$\int_{Q_+}\int_{Q_-} \mathbb{I}_{[\mathbf{o}\in\partial C(\mathbf{x},\mathbf{y})]}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \approx \left(\int_0^1 \Gamma_-(-s)\,\mathrm{d}s\right)\times\left(\int_0^1 \Gamma_+(s)\,\mathrm{d}s\right) + \sum_{n=0}^{N} \mathrm{Leb}_2(C_n^+)\,\mathrm{Leb}_2(\Delta_n^+) + \sum_{n=0}^{N} \mathrm{Leb}_2(C_n^-)\,\mathrm{Leb}_2(\Delta_n^-)$$

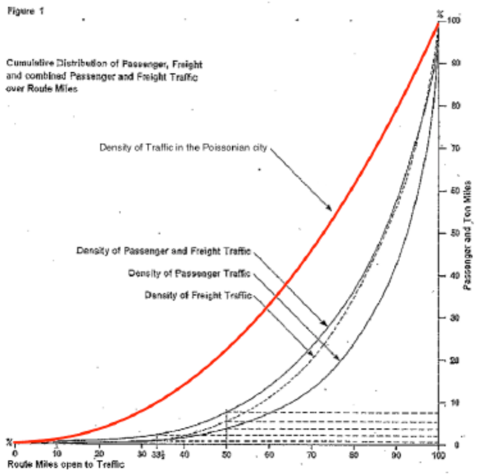
is bounded above by

$$\frac{20}{7}\times 3^{-N} + \frac{20}{27}\times 6^{-N}\,.$$

Empirical comparisons

- We can now compare traffic distribution to traffic on the UK railway system before Beeching (1963) (approximate UK by ellipse . . .).
- density computable using elliptic integrals, or numerically.
- Remarkably, mean traffic distribution appears to have an unexpected approximate affine invariance!

UK railway traffic and Poissonian cities



Pre-Beeching rail traffic is more highly concentrated than Poissonian city traffic.

Data from Beeching (1963); annotation by Gameros Leal (2017).

Conclusion

Traffic at the centre of the Poissonian city:

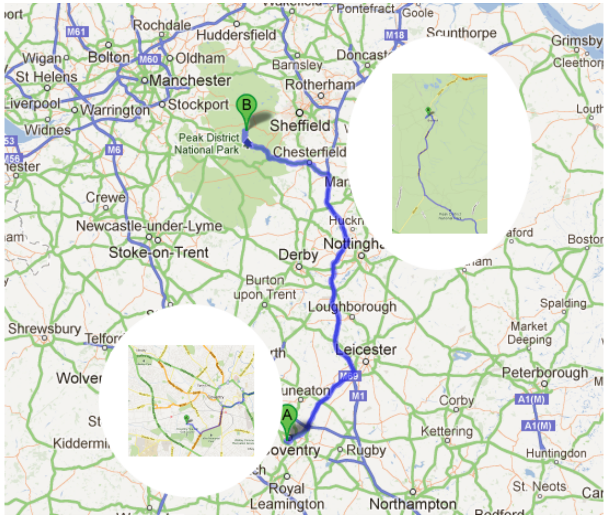
- There is a limiting distribution under scaling.
- There is a theoretical basis for developing an effective simulation algorithm.
- How to build an actual implementation?
- Analytical representation?
- Perfect simulation?
- Links to Gamma distributions?
- “Grid”-based cities.
- Replacing lines by fibre fragments?
- Is there a scale-invariant approach?

Lecture IV

Scale-Invariant Spatial Transportation Networks

Google holidays

for cats



SIRSN axioms

What would a scale-invariant (random) network look like? (Aldous, 2014; Aldous and Ganesan, 2013)

Input: set of nodes $x_1, \ldots, x_n$;

Output: random network $N(x_1, \ldots, x_n)$ connecting nodes.

(1) Scale-invariance: for each Euclidean similarity λ, $\mathcal{L}\left(N(\lambda x_1, \ldots, \lambda x_n)\right) = \mathcal{L}\left(\lambda N(x_1, \ldots, x_n)\right)$.

(2) If D_1 is length of fastest route between two points at unit distance apart then $\mathbb{E}\left[D_1\right] < \infty$.

(3) Weak SIRSN property: the network connecting points of (independent) unit intensity Poisson point process has finite average length per unit area.
(Strong) SIRSN property: the network connecting points of *dense* Poisson point process has finite average length per unit area of "long-range" part of network (more than distance 1 from source or destination).

Formal definition

Definition: (Strong) SIRSN

A SIRSN is a random mechanism for generating routes $\mathrm{Route}(x, y)$ between locations $x, y \in \mathbb{R}^2$, such that

1. Given x, y the route $\mathrm{Route}(x, y)$ is almost surely uniquely defined;
2. Let $N(x_1, \ldots, x_n)$ be the union of routes between various locations x_i. For each Euclidean similarity λ, $\mathcal{L}\left(N(\lambda x_1, \ldots, \lambda x_n)\right) = \mathcal{L}\left(\lambda N(x_1, \ldots, x_n)\right)$;
3. $D(x, y) = \mathbb{E}\left[\mathrm{len}\,\mathrm{Route}(x, y)\right] < \infty$.
4. (Strong form of SIRSN property) re-use of routes means that the "fibre process" $\bigcup_{x \neq y \in \Phi}(\mathrm{Route}(x, y) \setminus (\mathrm{ball}(x, 1) \cup \mathrm{ball}(y, 1)))$ of long-range routes has finite mean length per unit area for a "dense Poisson process" Φ.

Model for these axioms: Aldous' dyadic SIRSN

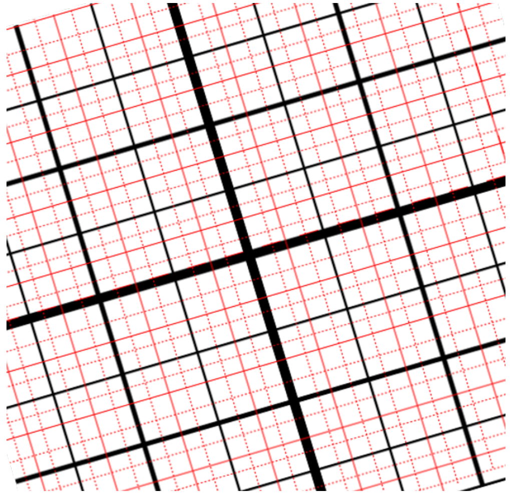
Aldous (2014):

1. Consider integer lattice $\mathbb{Z}^2$. Verticals at $x = (2k+1)2^h$, horizontals at $y = (2k+1)2^h$, have height h.
2. Fix $\gamma \in (\frac{1}{2}, 1)$. A path along the lattice travels at speed γ^h when on a segment of height h.
3. Mark segments randomly to help choose between non-unique geodesics.
4. Extend to dyadic lattice $\bigcup_r 2^{-r}\mathbb{Z}^2$.
5. Force stationarity by taking weak limit of random translations.
6. Force isotropy by taking random rotation.
7. Force scaling symmetry by taking scaling by random factor $c \in (1, 2)$.

Warwick Statistics

Aldous' dyadic SIRSN

SIRSN by randomization



Warwick Statistics

The Poisson line process SIRSN candidate

Proposed by Aldous (2014).

1. Fill plane with dense speed-marked Poisson line process.

   (Parametrize each line by signed perpendicular distance from origin r, angle $\theta \in [0, \pi)$, and positive speed v: use intensity measure $\frac{1}{2}(\gamma - 1)v^{-\gamma}\, \mathrm{d}v\, \mathrm{d}r\, \mathrm{d}\theta$.)
2. "Π-paths": Lipschitz paths integrating resulting measurable orientation field, (almost always) obeying speed-limit.
3. Connect pairs of points by fastest routes ("Π-geodesics") travelling entirely on line pattern at maximum speed.
4. For $\gamma > 2$, almost surely, all pairs are connected. Almost surely, fastest connection for a specified pair is unique.
5. Strong SIRSN axioms hold (WSK, 2017; Kahn, 2016).

Warwick Statistics

Impropriety and marks

- Need to take care when defining random point or line processes which are dense!
- There are strange measure-theoretic issues: (Aldous and Barlow, 1981; WSK, 2000).
- Such processes are often described as "improper": the intensity measure ν is not locally finite.
- Typical way to resolve the problem: add marks, for example lines with speed-limits:
  - We want to define a Poisson line process that is "dense everywhere".
  - Introduce a notion of speed v, a random speed-limit mark for each line ℓ.
  - View the process as a point process in an extended representation space, typical point (v, r, θ).
  - For intensity measure use $f(v)\, \mathrm{d}v\ \mathrm{d}r\ \mathrm{d}\theta$, locally finite on v-r-θ space even if it does not project down to local finite measure on r-θ space.

Warwick Statistics

About Π-paths (I)

Seek shortest-time paths ("temporal geodesics" or Π-geodesics) built using line process Π.
Require $\gamma > 1$, or fast lines will go everywhere.

- Introduce maximum speed limit, upper-semi-continuous $V : \mathbb{R}^d \to [0, \infty)$.
- A Π-path is locally Lipschitz, integrates measurable orientation field determined by Π, obeys speed limit.
- If $\gamma > d$ then:
  - there is an *a priori* random bound on distance travelled by Π-path in fixed time;
  - space of paths up to time T is closed, weakly closed in Sobolev space $L^{1,2}([0, T) \to \mathbb{R}^d)$,
  - paths up to time T, beginning in a compact set, together form a weakly compact set.

When infinity is infinitely far away

Theorem: An *a priori* upper bound for path space

Suppose $\gamma \geq d \geq 2$. Fix $T < \infty$, consider Π-path ξ begun in unit ball. There is $C < \infty$ depending on realization of Π such that almost surely $|\xi(t)| \leq C$ for all $t \leq T$.

Proof.
Set $\overline{V}(r)$ to be speed of fastest line within r of origin.
Map line $(\ell, v) \mapsto (|r|^{d-1}, v^{-(\gamma-1)})$ to a point in $[0, \infty)^2$: this maps Π to a planar Poisson process!
$S = (\overline{V}(r))^{-(\gamma-1)}$ changes at $|r|^{d-1} = P_0 < P_1 < P_2 < \ldots$.

$P_n - P_{n-1} = 1/(S_{n-1}T_n)$ for Exponential T_n;
$S_n = U_n S_{n-1}$ for Uniform$(0, 1)$ U_n.

$X_n = S_n P_n$ is a perpetuity.
Compute $\mathbb{P}[$ infinite time to get to infinity $|\Pi] = 1$. □

We can now apply standard compactness arguments for the Sobolev space $L^{1,2}([0, T) \to \mathbb{R}^d)$:

Lemma: Π-geodesics exist if Π-paths exist

Suppose Π is a speed-marked Poisson line process in $\mathbb{R}^d$ with intensity measure $\frac{1}{2}(\gamma - 1)v^{-\gamma}\, \mathrm{d}v\, \mathrm{d}r\, \mathrm{d}\theta$. If $\gamma > 1$ then almost surely Π-geodesics exist between all point pairs ξ_1 and ξ_2 in $\mathbb{R}^d$ if Π-paths exist.

About Π-paths (II)

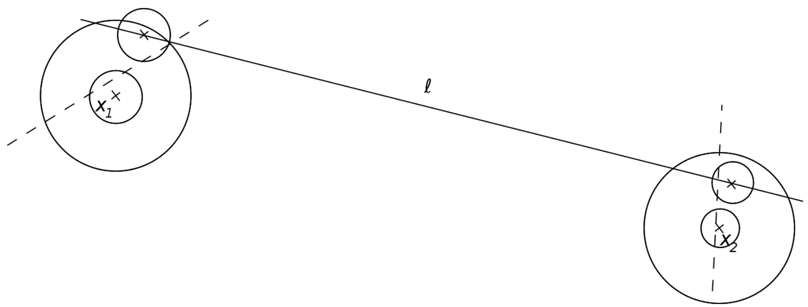
Suppose one wishes to connect two points ξ_1 and ξ_2 in $\mathbb{R}^d$ by a Π-path. Suppose $\gamma > d$ (turns out to be essential).

- Construct small balls around ξ_1 and ξ_2;
- Connect balls by fastest line ℓ intersecting both balls;
- Construct daughter nodes on ℓ closest to ξ_1 and ξ_2;
- Recurse.
- Borel-Cantelli, *et cetera*: establish almost sure existence of resulting path.
- This yields a binary tree representation of the path. Note that this is unavoidable if $d > 2$!

  A similar but more complicated argument almost surely allows simultaneous construction of paths between all possible pairs ξ_1 and ξ_2 in $\mathbb{R}^d$.

  Exercise: Figure out a way to visualize Π-paths in case $d = 3$.

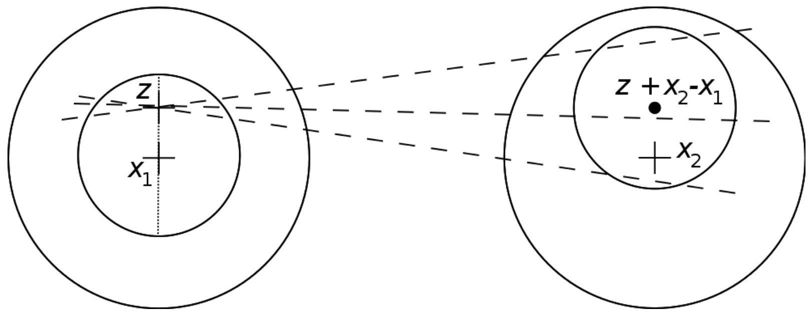
Construction to connect two points



First find the fastest line passing reasonably close to both source and destination.

Then recurse.

Lower bound on measure of set of lines hitting 2 balls



Reduction of hitting set $[\text{ball}(x_1, \alpha^{-1}r)] \cap [\text{ball}(x_2, \alpha^{-1}r)]$ to a smaller hitting set for which the line measure is more easily computable yet which still provides a useful lower bound.

Now use resulting lower bound

$$\mu_d\left([\text{ball}(\xi_1, \alpha^{-1}r)] \cap [\text{ball}(\xi_2, \alpha^{-1}r)]\right) \quad \geq \quad \kappa_{d-1}\left(\frac{r}{4\alpha^2}\right)^{d-1}$$

and a Borel-Cantelli argument to control the amount of time spent traversing recursively defined paths between any pair of points.

Lemma: Existence of Π-paths

Suppose Π is a speed-marked Poisson line process in $\mathbb{R}^d$ with intensity measure $\frac{1}{2}(\gamma - 1)v^{-\gamma}\, \mathrm{d}v\, \mathrm{d}r\, \mathrm{d}\theta$. If $\gamma > d$ then Π-paths exist between all point pairs ξ_1 and ξ_2 in $\mathbb{R}^d$ and no Π-paths of finite length can reach infinity.

Remark: Random metrics for Euclidean space

So if $\gamma > d$ then Π determines a random metric on $\mathbb{R}^d$.

About Π-paths (III)

Exponential moments for powers of Π-diameters

Kahn (2016), improving on WSK (2017, Theorem 3.6) has proved results of which the following is a simple example:

- Consider $\text{ball}(\mathbf{o}, R)$, a ball of Euclidean radius R.
- Let T_R be the supremum of the minimum times for a Π-path to pass from one point of $\text{ball}(\mathbf{o}, R)$ to another;

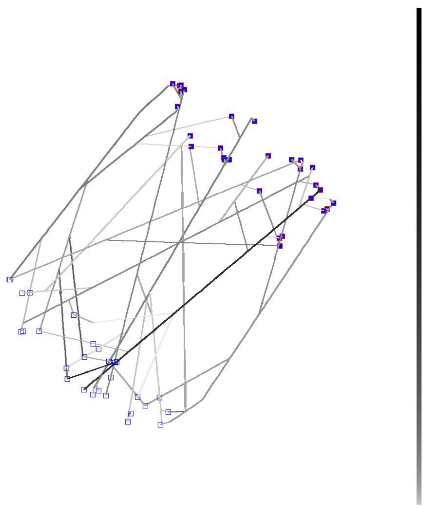
$$T_R = \sup_{x,y \in \text{ball}(\mathbf{o},R)} \inf\{T :$$

$$\text{some } \Pi\text{-path } \xi : [0, T] \to B \text{ satisfies } \xi(0) = x, \xi(T) = y\}.$$

  We call T_R the Π-diameter of $\text{ball}(\mathbf{o}, R)$.
- For explicit $\delta_R > 0$, $K_R < \infty$,

$$\mathbb{E}\left[\exp\left(\delta_R T_R^{\gamma-1}\right)\right] \quad \leq \quad K_R \quad < \quad \infty\,.$$

Simulations (approximate!) of a typical set of routes



Case $\gamma = 16$

Are Π-geodesics unique? (I)

Suppose now $d = 2$ and $\gamma > 2$, and we fix ξ_1 and $\xi_2 \in \mathbb{R}^2$. If Π is to generate a network between a finite set of points, then we need to know the Π-geodesic between ξ_1 and ξ_2 is almost surely unique.

Theorem: "line meets line"

All non-singleton intersections of a Π-geodesic with lines ℓ of Π are of the form "line meets line".

- First, reduce to case of ℓ being fastest line in region, with speed w.
- Now change focus from high speed v to low "cost", where
$$\text{“cost”} \quad = \quad \frac{\csc\theta}{v} - \frac{\cot\theta}{w}.$$
where θ is angle of line with ℓ.
- Argue that Π-geodesic hits ℓ using line of finite cost.

Are Π-geodesics unique? (II)

So Π-geodesics between ξ_1 and ξ_2 are made up of countable collection of intervals of lines of Π.

- Fix a given ℓ from Π, and consider the set S of such intervals lying in ℓ.
  - Consider two different finite collections $S_1 \subset S$ and $S_2 \subset S$, each composed of non-overlapping intervals.
  - Probability density argument: the total lengths of S_1 and S_2 have a joint density, unless one is empty.
  - Conditioning on time spent off ℓ, almost surely two Π-paths using S_1 and S_2 respectively must have different total travel times.
  - Almost surely two Π-geodesics between ξ_1 and ξ_2 must use the same finite collection of non-overlapping intervals from each ℓ of Π.
  - But we can reconstruct the Π-geodesic uniquely from the collections of intervals of each line ℓ in Π.

Theorem: Uniqueness of Π-geodesics in planar case

Suppose Π is a speed-marked Poisson line process in $\mathbb{R}^2$ with intensity measure $\frac{1}{2}(\gamma - 1)v^{-\gamma}\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\theta$. If $\gamma > 2$ then for any point pair ξ_1 and ξ_2 in $\mathbb{R}^2$ it is almost surely the case that there is just one Π-geodesic between ξ_1 and ξ_2.

Almost surely there will exist non-unique Π-geodesics!

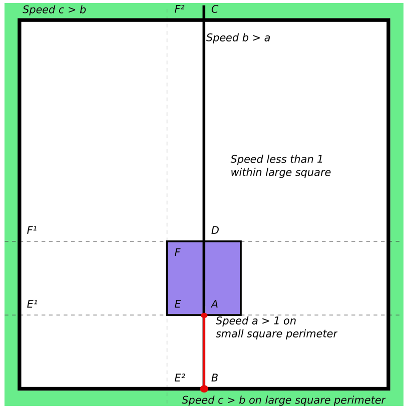
Do Π-geodesics have finite mean length?

Suppose again that $d = 2$ and $\gamma > 2$.

- Techniques for showing existence of Π-paths show finite mean of length of Π-geodesic *if lying in a fixed ball*.
- Could fast geodesics generate long lengths outside balls? (Oxford → Cambridge by motorway *via* London? or *Edinburgh*? ...)
- Time spent by Π-geodesic can be bounded above by time spent on a circuit of a "racetrack" construction around ξ_1 and ξ_2 using fastest lines.
- We can upper-bound distance travelled outside a ball by using the "idealized path" construction employed above.
- The resulting perpetuity can be combined with the "racetrack" bound to establish finite mean length.

Diagram to prove Pre-SIRSN property

- A careful construction, together with a Borel-Cantelli argument, shows that re-scaled small perturbations of the following diagram (for suitable re-scaled speeds $1 < a < b < c$) can be found at all length scales:



If $c > 10b > 59a/3 > 354/3$, red segment is only exit.

Kahn (2016) has now produced a proof of the strong SIRSN property for all dimensions d and all $\gamma > d$, using the exponential inequality mentioned above for powers of Π-diameters

$$\mathbb{E}\left[\exp\left(\delta_R T_R^{\gamma-1}\right)\right] \quad \leq \quad K_R \quad < \quad \infty\,.$$

This can be used to show that Π-geodesics must make substantial re-use of shared lines: for all Π-geodesics bridging a suitable annulus, the exponential inequality forces each Π-geodesic to make substantial use of a limited number of "fast" lines. SIRSN follows by using a measure-theoretic version of the pigeonhole principle.

Kahn (2016) deploys further ingenious arguments to obtain uniqueness (essential if above argument is to work) and finite mean-length of Π-geodesics in case $d > 2$.

Theorem: Poisson line process SIRSN

Suppose Π is a speed-marked Poisson line process in $\mathbb{R}^d$ with intensity measure $\frac{1}{2}(\gamma - 1)v^{-\gamma}\, \mathrm{d}\,v\, \mathrm{d}\,r\, \mathrm{d}\,\theta$. If $\gamma > d$ then Π is a (strong) SIRSN.

Random Metric Spaces

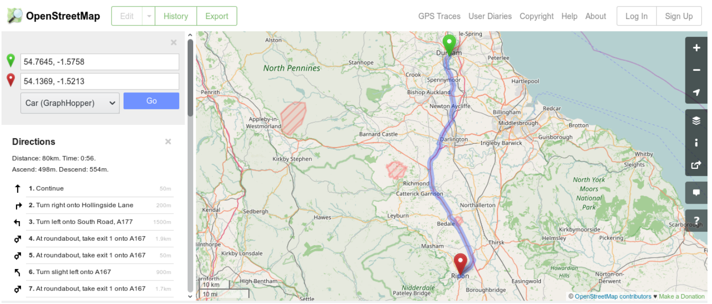
Study of metric spaces generated by line processes should form a chapter in the theory of random metric spaces:

1. Vershik (2004) builds random metric spaces out of random distance matrices (compare MDS in statistics); almost all are isometric to Urysohn's celebrated universal metric space. Not finite-dimensional.
2. Brownian map: limit of random quadrangulations of 2-sphere (eg, Le Gall, 2010). Far from flat.
3. Baccelli, Tchoumatchenko, and Zuyev (2000) link to geometric spanners; they exhibit $\frac{4}{\pi}$-spanner paths in Poisson-Delaunay triangulations.

Lecture V

Random Flights on the SIRSN

Wandering about Yorkshire and Durham



https://www.openstreetmap.org/

Tsukayama, Washington Post, 30th January 2013: Estimated annual impact of online maps: \$1.6tn. Estimated growth 30% per year (smartphones!).

Random wandering on a SIRSN

Options:
(thanks to Banerjee, Croydon for helpful conversations!)

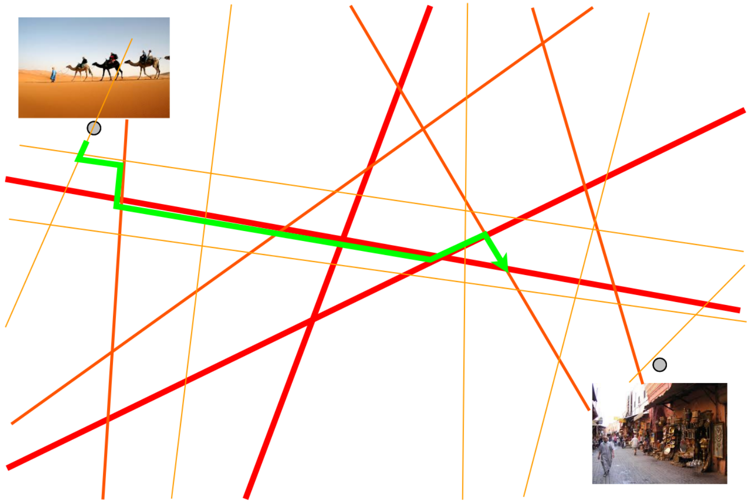
1. Random walk, connect successive jumps by Π-geodesics.
   (Con: Always stopping and starting!)
2. Brownian motion on the line pattern.
   (Con: Relationship with speed limits is not explicit.)
3. Rayleigh random flight:
   - Proceed at top speed along current line;
   - Switch to intersecting lines depending on relative speeds;
   - Choose new direction equi-probably.
4. This is SIRSN-RRF: a "randomly broken Π-geodesic".
   Can it be speed-neighbourhood-recurrent?

Scattering – an abstract approach (I)

- How to define a SIRSN-RRF?
  - Proceed at top speed along current line;
  - Switch to intersecting lines depending on relative speeds;
  - Choose new direction equi-probably.
- Suffices to sample process when it changes speed.
- Since Π is improper line process, opportunities to change speed are dense along each line!
- Convenient to take an abstract view:

  Advantages of axiomatic method
  "same as the advantages of theft over honest toil"
  (Russell, 1919, p.71).

Scattering – an abstract approach (sketch)



Scattering – an abstract approach (II)

Generalization (possibly of wider interest?):

1. A discrete-time Markov chain on countable state-space is an abstract scattering process if $p_{ab} = \omega_{ab} s_b$ for (symmetric) transmission probabilities ω_{ab} and scattering probabilities s_b (set $\omega_{a,\tilde{a}} = 0$, $s_a > 0$ for all a).
2. (Consider a matrix (ω_{ab}): if row-vectors $(\omega_{a\cdot})$ all lie in ℓ^1 then the Hahn-Banach theorem can be used to characterize whether (ω_{ab}) is transmission matrix.)
3. Require dynamically reversibility for measure π:

   Involution $a \leftrightarrow \tilde{a}$ preserving measure π,
   with $\pi_a p_{a,\tilde{b}} = \pi_b p_{b,\tilde{a}}$.
4. Relate $\pi_a / s_{\tilde{a}}$ to scatter-equivalence classes ("lines"): if there is a chain $a = b_0, b_1, \ldots, b_n = c$ with $\omega_{b_{m-1},\tilde{b}_m} > 0$, then $\pi_a / s_{\tilde{a}} = \pi_c / s_{\tilde{c}}$.

Scattering – an abstract approach (III)

5. Adopt "Metropolis-Hastings recipe": divide state-space into equivalence classes using ω,
   - set $\pi_a = \min\{\kappa, \kappa'\}$ where we choose κ, κ' as positive constants belonging to classes of a and $\tilde{a}$,
   - set scattering probability $s_a = \min\{1, \kappa/\kappa'\}$ (dynamic reversibility is then automatic!).
6. In case of a suitable total ordering $\prec$ for each "line", transmission probabilities are functions of scattering probabilities.
   For each $a \prec b$, there are $\omega_{a,\pm}$ summing to 1 with

$$p_{a,\tilde{b}} \quad = \quad \omega_{a,\tilde{b}} s_{\tilde{b}} \quad = \quad \omega_{a,+} \left(\prod_{a \prec c \prec b} (1 - s_{\tilde{c}}) \right) s_{\tilde{b}} \,,$$

   and similar for $p_{b,\tilde{a}}$ using $\omega_{b,-}$.
   All follows from choice of the class constants κ
   (and say equiprobable choice of direction $\omega_{a,\pm} = \frac{1}{2}$).

Application

Define the SIRSN-RRF, sampled at changes in direction, by specifying equilibrium probabilities at intersections of lines.

1. Scaling invariance: $\pi_{(\ell_1,\ell_2)} = \min\{v_1^\alpha, v_2^\alpha\}$, parameter α.
2. Scattering probability: $s_{(\ell_1,\ell_2)} = \min\{1, (v_2/v_1)^\alpha\}$.
3. Dynamical reversibility: non-symmetric Dirichlet form.
4. Apply Campbell-Slivnyak-Mecke theorem (twice!) to identify (translated, rotated, scaled) "environment viewed from particle" *via* reduced non-symmetric Dirichlet form.
5. Resulting log-relative-speed-changes V_1, V_2, ... form a stationary process.
6. Analyze using RWRE theory (but state-space is not a lattice!).

Dynamical reversibility

Let $f(x,\Pi)$ be bounded, measurable, x an intersection of lines $\mathcal{L}_1$, $\mathcal{L}_2$ in Π. Set $\tilde{f}(\mathcal{L}_1,\mathcal{L}_2;\Pi) = f(\mathcal{L}_2,\mathcal{L}_1;\Pi)$. Consider the non-symmetric form

$$B(f,g) \quad = \quad \mathbb{E}\left[\mathbb{E}\left[\sum_{\mathcal{L}_1\neq\mathcal{L}_2\in\Pi}\mathbb{E}\left[\tilde{f}(\tilde{Z}_0;\Pi)\times g(Z_1;\Pi)\times\pi_x \;\middle|\; Z_0 = x = (\mathcal{L}_1,\mathcal{L}_2)\right] \;\middle|\; \Pi\right]\right].$$

Using Campbell-Mecke-Slivnyak theory twice, this can be reduced (taking out translations, rotations, scale-changes) to the study of

$$\mathbb{E}\left[\sum_{\mathcal{L}_3\in\Pi} f^{(2)}(\mathcal{L}_1^*;\Pi\cup\{\mathcal{L}_1^*\})s_{(\tilde{\mathcal{L}}_0,\mathcal{L}_1^*)}\left(\prod_{\substack{\mathcal{L}\in\Pi:\ \mathcal{L}\text{ separates}\\ \text{origin from }\tilde{\mathcal{L}}_0\cap\mathcal{L}_3}}(1-s_{(\tilde{\mathcal{L}}_0,\mathcal{L})})\right)\right.$$
$$\left.\times\, s_{(\tilde{\mathcal{L}}_0,\mathcal{L}_3)}g^{(2)}(\mathcal{L}_3;\Pi\cup\{\mathcal{L}_1^*\})\right]$$

Read off equilibrium distribution from reduced non-symmetric form: at critical $\alpha = 2(\gamma-1)$, the typical log-relative-speed-change X has stationary symmetric Laplace distribution.

Ergodicity (I)

Rayleigh random flight (RRF) on Π, unit initial speed.

- The cumulative log-speed-change $X_n = V_1 + \ldots + V_n$ can be expressed using a sum of log-relative-speed-changes $V_1, \ldots, V_n, \ldots$ which form a stationary sequence.
- If the mean log-relative-speed-change $\mu = \mathbb{E}[V_1]$ is non-zero then $X_n/n = (V_1 + \ldots + V_n)/n$ will converge to a non-identically-zero limit random variable ("non-ergodic" part of Birkhoff's ergodic theorem); therefore for any $\varepsilon > 0$ there is at least a positive chance of X_n eventually never re-visiting $(-\varepsilon, \varepsilon)$.
- If we could show $V_1, \ldots, V_n, \ldots$ to be ergodic, then the limit equals μ. In this case transience is sure if $\mu \neq 0$.
- Remarkably, the converse also holds! (Zero-mean forces neighborhood recurrence.) This is a consequence of the ideas of the Kesten-Spitzer-Whitman range theorem.

Ergodicity (II)

Theorem: Neighborhood-recurrence and ergodicity

Suppose $V_1, \ldots, V_n, \ldots$ form a stationary ergodic sequence, with $\mathbb{E}[V_1] = 0$. Set $X_n = V_1 + \ldots + V_n$. Then for all $\varepsilon > 0$ it is the case that

$$\mathbb{P}[|X_n - X_0| \leq \varepsilon \text{ infinitely often in } n] \quad = \quad 1\,.$$

Proof.

$X_n/n \to 0$ a.s. so $n^{-1}\sup\{|X_1|,\ldots,|X_n|\} \to 0$ a.s.
Set $A_n = [|X_m - X_n| > \varepsilon \text{ for all } m > n]$. Birkhoff's ergodic theorem: $n^{-1}(\mathbb{I}[A_1] + \ldots + \mathbb{I}[A_n]) \to p = \mathbb{P}[A_1]$.
Packing argument: $|X_m - X_n| > \varepsilon$ on $A_n \cap A_m$ if $m \neq n$.
Deduce
$n^{-1}(\mathbb{I}[A_1] + \ldots + \mathbb{I}[A_n]) \leq (n\varepsilon)^{-1}\sup\{|X_1|,\ldots,|X_n|\}$.
Let $n \to \infty$ and deduce $p = 0$.
Complete argument by sub-sampling ....

An argument of Kozlov type

Consider the process Ψ of the (re-scaled, shifted, rotated) environment viewed from the RRF particle. Let $\widetilde{h}$ be a bounded harmonic function on the (re-scaled, shifted, rotated) environment state-space (harmonic with respect to the process Ψ). Then $\widetilde{h}(\Psi_n)$ is almost surely constant in (discrete) time n. For consider

$$\mathbb{E}\left[(\widetilde{h}(\Psi_0) - \widetilde{h}(\Psi_1))^2\right] = 2\,\mathbb{E}\left[\widetilde{h}(\Psi_0)^2\right] - 2\,\mathbb{E}\left[\widetilde{h}(\Psi_0)h(\Psi_1)\right] = 0\,,$$

where the first step follows from stationarity and the second because $\widetilde{h}(\Psi)$ is a martingale. Thus $\mathbb{P}\left[\widetilde{h}(\Psi_1) = \widetilde{h}(\Psi_0)\right] = 1$.

Thus if E is a shift-invariant event then $T^{-1}E = E$ (up to null-sets) for T a composition of any finite sequence of transformations induced by possible moves of the RRF.

Proving ergodicity

We only need to show ergodicity of the environment viewed from the particle.

1. Given $\delta > 0$, approximate E by E_δ depending only on lines of Π in a finite speed-range $[-W, W]$ (and coming within a bounded distance R of $\mathbf{o}$):

$$\mathbb{P}\left[E \Delta E_\delta\right] \quad \leq \quad \delta\,.$$

2. Find a composition of transformations T such that T moves lines in speed-range $[-W, W]$ to a disjoint speed-range, so that E_δ and $T^{-1}E_\delta$ are independent.
3. Deduce that $|\,\mathbb{P}[E] - \mathbb{P}[E]^2\,| = |\,\mathbb{P}[E \cap E] - \mathbb{P}[E]^2\,| \leq |\,\mathbb{P}[E_\delta \cap T^{-1}E_\delta] - \mathbb{P}[E_\delta]\,\mathbb{P}[T^{-1}E_\delta]\,| + 4\delta = 4\delta$.
4. Since $\delta > 0$ is arbitrary, it follows $\mathbb{P}[E]$ must be zero or one. It follows that Ψ, and hence the log-speed-change process, is ergodic. QED

Concluding Theorem

Theorem: Transience and neighborhood-recurrence

1. Critical case ($\alpha = 2(\gamma - 1)$): SIRSN-RRF speed is neighbourhood-recurrent.
2. Sub-critical case ($\alpha < 2(\gamma - 1)$): SIRSN-RRF converges to a random limiting point in the plane (trapped by cells of tessellation of high-speed lines).
3. Super-critical case ($\alpha > 2(\gamma - 1)$): SIRSN-RRF disappears off to infinity (consider high-speed tessellation).

Conclusion: always more questions!

1. So a critical "randomly-broken Π-geodesic" does not halt *en route*. What about Π-geodesics themselves?
2. Is critical case space-(neighbourhood)-recurrent? speed-point-recurrent? (working on proof . . .)
3. Above argument should work for critical (non-SIRSN) Π for which $\gamma = 2$ and $\alpha = 2$ (not yet checked).
4. "Brownian-like" variations should follow from WSK and Westcott (1987): scale-invariant "Liouville diffusion" (Berestycki, 2015; Garban, Rhodes, and Vargas, 2013).
5. Can anything be done for the high-dimensional SIRSN supplied by case $\gamma > d > 2$?
6. Replace lines by long segments? nearly straight fibres?

References:

Aldous, D. J. (2014, April). Scale-Invariant Random Spatial Networks. *Electronic Journal of Probability 19*, no. 15, 1–41.

Aldous, D. J. and M. T. Barlow (1981). On countable dense random sets. *Lecture Notes in Mathematics 850*, 311–327.

Aldous, D. J. and K. Ganesan (2013, December). True scale-invariant random spatial networks. *Proceedings of the National Academy of Sciences of the United States of America 110*(22), 8782–8785.

Aldous, D. J. and J. Shun (2010). Connected spatial networks over random points and a route-length statistic. *Statistical Science 25*(3), 1–28.

Aldous, D. J. and WSK (2008, March). Short-length routes in low-cost networks via Poisson line patterns. *Advances in Applied Probability 40*(1), 1–21.

Ambartzumian, R. V. (1982). *Combinatorial Integral Geometry (with an appendix by A J Baddeley)*. Chichester / New York: Wiley & Sons.

Ambartzumian, R. V. (1990). *Factorization Calculus and Geometric Probability*. Cambridge: Cambridge University Press.

Baccelli, F., K. Tchoumatchenko, and S. Zuyev (2000). Markov paths on the Poisson-Delaunay graph with applications to routing in mobile networks. *Advances in Applied Probability 32*(1), 1–18.

Baddeley, A. J. and E. B. V. Jensen (2005). *Stereology For Statisticians*. CRC Press.

Baddeley, A. J. and M. N. M. van Lieshout (1995). Area-interaction point processes. *Annals of the Institute of Statistical Mathematics 47*(4), 601–619.

Bast, H., S. Funke, P. Sanders, and D. Schultes (2007, April). Fast routing in road networks with transit nodes. *Science (New York, N.Y.) 316*(5824), 566.

Beeching, R. (1963). *The reshaping of British railways*. Number 1 in The Reshaping of British Railways. Her Majesty's Stationery Office.

Berestycki, N. (2015, August). Diffusion in planar Liouville quantum gravity. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques 51*(3), 947–964.

Bertoin, J. and M. Yor (2005). Exponential functionals of Lévy processes. *Probability Surveys 2*, 191–212 (electronic).

Birnbaum, Z. W. (1942). An inequality for Mill's ratio. *Annals of Mathematical Statistics 13*, 245–246.

Böröczky, K. J. and R. Schneider (2010). The mean width of circumscribed random polytopes. *Canadian Mathematical Bulletin 53*, 614–628.

Chartier, T. (2017). Data for Good. *Math Horizons*, 18–21.

Chiu, S. N., D. Stoyan, WSK, and J. Mecke (2013, June). *Stochastic Geometry and Its Applications* (3 ed.). Chichester / New York: John Wiley & Sons Ltd.

Comte de Buffon, G. L. L. (1829). Essais d'Arithmétique Morale. *Oeuvres Complètes de Buffon, tome III Nouveau Edition*, 338–405.

Davidson, R. (1974). Line-processes, roads, and fibres. In E. F. Harding and D. G. Kendall (Eds.), *Stochastic geometry (a tribute to the memory of Rollo Davidson)*, pp. 248–251. London: John Wiley & Sons Ltd.

Frisch, H. L. and J. M. Hammersley (1963). Percolation processes and related topics. *J. Soc. Indust. Appl. Math. 11*, 894–918.

Gameros Leal, R. M. (2014). *Traffic in Poissonian Line Networks*. Msc dissertation, University of Warwick.

Gameros Leal, R. M. (2017). *Mean traffic behaviour in Poissonian cities*. Phd thesis, University of Warwick.

Garban, C., R. Rhodes, and V. Vargas (2013). Liouville Brownian motion. *arXiv 1301.2876*.

Gnedenko, B. V. (1998, May). *Theory of Probability*. CRC Press.

Hall, P. (1988). *Introduction to the theory of coverage processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. New York: John Wiley & Sons.

Hey, J. D., T. M. Neugebauer, and C. M. Pasca (2010). Georges-Louis Leclerc de Buffon's 'Essays on Moral Arithmetic' (1777).

Kahn, J. (2016, July). Improper Poisson line process as SIRSN in any dimension. *Annals of Probability 44*(4), 2694–2725.

Kingman, J. F. C. (1993). *Poisson processes*, Volume 3 of *Oxford Studies in Probability*. New York: The Clarendon Press Oxford University Press.

Kozen, D. (1992). *The design and analysis of algorithms*. Springer.

Last, G. and M. Penrose (2017). *Lectures on the Poisson Process*.

Lazzarini, M. (1901). Un'applicazione del calcolo della probabilita alla ricerca esperimentale di un valore approssimato di π. *Periodico di Matematica 4*, 49–68.

Le Gall, J.-F. (2010). Geodesics in large planar maps and in the Brownian map. *Acta mathematica 205*(2), 287–360.

Molchanov, I. S. (1996a, September). A limit theorem for scaled vacancies of the Boolean model. *Stochastics An International Journal of Probability and Stochastic Processes 58*(1-2), 45–65.

Molchanov, I. S. (1996b). *Statistics of the Boolean Models for Practitioners and Mathematicians*. Chichester: John Wiley & Sons.

Ostoja-Starzewski, M. and D. C. Stahl (2000). Random fiber networks and special elastic orthotropy of paper. *Journal of Elasticity 60*(2), 131–149.

Ripley, B. D. (1977). Modelling spatial patterns (with discussion). *Journal of the Royal Statistical Society (Series B: Methodological) 39*, 172–212.

Roy, R. (1991). Percolation of poisson sticks on the plane. *Probability Theory and Related Fields 89*(4), 503–517.

Russell, B. (1919). *Introduction to Mathematical Philosophy*. New York, London: George Allen and Unwin.

Sampford, M. R. (1953). Some inequalities on Mill's ratio and related functions. *Annals of Mathematical Statistics 24*, 130–132.

Santaló, L. A. (1976). *Integral geometry and geometric probability*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam.

Schneider, R. and W. Weil (2008). *Stochastic and Integral Geometry*. Springer.

Steele, J. M. (1997). *Probability theory and combinatorial optimization*, Volume 69 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Steinhaus, H. (1930). Zur Praxis der Rektifikation und zum Längenbegriff. *Sächsischen Akad. Wiss. Leipzig 82*, 120–130.

Stigler, S. M. (1991). Stochastic simulation in the nineteenth century. *Statistical Science 6*(1), 89–97.

Stoyan, D., WSK, and J. Mecke (1995). *Stochastic Geometry and its Applications* (Second ed.). Chichester: John Wiley & Sons.

Thorisson, H. (2000). *Coupling, stationarity, and regeneration*. New York: Springer-Verlag.

Tsukayama, H. (2013). Google asks: How much is mapping worth?

Vershik, A. M. (2004). Random and universality. In *Dynamics and randomness II*, Volume 10 of *Nonlinear Phenom. Complex Systems*, pp. 199–228. Dordrecht: Kluwer Acad. Publ.

WSK (1997). On some weighted Boolean models. In D. Jeulin (Ed.), *Advances in Theory and Applications of Random Sets*, Singapore, pp. 105–120. World Scientific.

WSK (2000, March). Stationary countable dense random sets. *Advances in Applied Probability 32*(1), 86–100.

WSK (2011, October). Geodesics and flows in a Poissonian city. *Annals of Applied Probability 21*(3), 801–842.

WSK (2014). Return to the Poissonian city. *Journal of Applied Probability 51*(A), 297–309.

WSK (2017, March). From Random Lines to Metric Spaces. *Annals of Probability 45*(1), 469–517.

WSK and I. S. Molchanov (2010, January). *New perspectives in stochastic geometry*. Oxford: Clarendon Press.

WSK, M. N. M. van Lieshout, and A. J. Baddeley (1999). Quermass-interaction processes: conditions for stability. *Advances in Applied Probability 31*(2), 315–342.

WSK and M. Westcott (1987). Scattering processes and convergence to Brownian motion. *Advances in Applied Probability 19*, 81–105.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