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On Directed Random Graphs and Greedy Walks on Point Processes

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Abstract

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This thesis consists of an introduction and five papers, of which two contribute to the theory of directed random graphs and three to the theory of greedy walks on point processes. We consider a directed random graph on a partially ordered vertex set, with an edge between any two comparable vertices present with probability p, independently of all other edges, and each edge is directed from the vertex with smaller label to the vertex with larger label. In Paper I we consider a directed random graph on $\mathbb{Z}^2$ with the vertices ordered according to the product order and we show that the limiting distribution of the centered and rescaled length of the longest path from (0,0) to $(n, \lfloor n^a \rfloor)$, $a<3/14$, is the Tracy-Widom distribution. In Paper II we show that, under a suitable rescaling, the closure of vertex 0 of a directed random graph on $\mathbb{Z}$ with edge probability n^{-1} converges in distribution to the Poisson-weighted infinite tree. Moreover, we derive limit theorems for the length of the longest path of the Poisson-weighted infinite tree.

The greedy walk is a deterministic walk on a point process that always moves from its current position to the nearest not yet visited point. Since the greedy walk on a homogeneous Poisson process on the real line, starting from 0, almost surely does not visit all points, in Paper III we find the distribution of the number of visited points on the negative half-line and the distribution of the index at which the walk achieves its minimum. In Paper IV we place homogeneous Poisson processes first on two intersecting lines and then on two parallel lines and we study whether the greedy walk visits all points of the processes. In Paper V we consider the greedy walk on an inhomogeneous Poisson process on the real line and we determine sufficient and necessary conditions on the mean measure of the process for the walk to visit all points.

Keywords: Directed random graphs, Tracy-Widom distribution, Poisson-weighted infinite tree, Greedy walk, Point processes

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To Markus and Lukas

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Konstantopoulos, T. and Trinajstić, K. (2013). Convergence to the Tracy-Widom distribution for longest paths in a directed random graph. *ALEA Lat. Am. J. Probab. Math. Stat.* **10**, 711–730.

II Gabrysch, K. (2016). Convergence of directed random graphs to the Poisson-weighted infinite tree. *J. Appl. Probab.* **53**, 463–474.

III Gabrysch, K. (2016). Distribution of the smallest visited point in a greedy walk on the line. *J. Appl. Probab.* **53**, 880–887.

IV Gabrysch, K. (2016). Greedy walks on two lines. Submitted for publication.

V Gabrysch, K. and Thörnblad, E. (2016). The greedy walk on an inhomogeneous Poisson process. Submitted for publication.

Contents

1. Introduction

This thesis contributes to two models in probability theory: directed random graphs and greedy walks. All models studied in this thesis are related to point processes. As they play an important role in the proofs, we give a brief overview of point processes in Section 1.1. The first two papers included in the thesis study models of directed random graphs, which are introduced in Section 1.2. In Paper I we look at the longest path in a long and thin rectangle and prove that the length of such a path, properly rescaled and centered, converges to the Tracy-Widom distribution. To be able to show this, we observe that there are special points in the graphs, called skeleton points, which are defined in Section 1.3. The Tracy-Widom distribution is described in Section 1.4. The last three papers study greedy walks defined on various point processes. The greedy walk model is presented in Section 1.5.

1.1 Point processes

In this section we define point processes in one dimension and explain the concept of a stationary and ergodic point process. We also present two examples of point processes. Point processes (or some models of point processes) are studied in many textbooks in probability theory and stochastic processes. For a broad survey of the theory of point processes we refer to [15].

Let E be a complete separable metric space and let $\mathscr{B}(E)$ be the Borel σ-field generated by the open balls of E. A *counting measure* m on E is a measure on $(E,\mathscr{B}(E))$ such that $m(C) \in \{0,1,2,\dots\} \cup \{\infty\}$ for all $C \subset \mathscr{B}(E)$ and $m(C) < \infty$ for all bounded $C \subset \mathscr{B}(E)$. The counting measure m is *simple* if $m(\{x\})$ is 0 or 1 for all $x \in E$. Let M be the set of all counting measures on E and let $\mathscr{M}$ be the σ-field of M generated by the functions $m \longmapsto m(C)$, $C \in \mathscr{B}(E)$. A counting measure m can be expressed as

$$m(\cdot) = \sum_{i\in\mathbb{N}} k_i \delta_{x_i}(\cdot),$$

where $k_i \in \{0,1,2,\dots\}$, $\{x_i : i \in \mathbb{N}\} \subset E$ and δ_x denotes the Dirac measure. If m is a simple counting measure, then $k_i = 1$ for all $i \in \mathbb{N}$.

A *point process* is a measurable mapping from a probability space $(\Omega,\mathscr{F},\mathbb{P})$ into $(M,\mathscr{M})$. The point process is simple if it is a simple counting measure with probability 1. Instead of thinking of a simple point process Π as a random measure, we may think of Π as a random discrete subset of E. We write $|\Pi \cap A|$ for the number of points of Π in the set A and $x \in \Pi$ for $|\Pi \cap \{x\}| = 1$.

A point process Π is *stationary* if, for all $k \geq 1$ and for all bounded Borel sets $A_1, A_2, \ldots, A_k \subset \mathscr{B}(E)$, the joint distribution

$$\{|\Pi \cap (A_1 + t)|, |\Pi \cap (A_2 + t)|, \ldots, |\Pi \cap (A_k + t)|\}$$

does not depend on the choice of $t \in E$.

For $t \in E$, define the shift operator $\theta_t : M \to M$ by $\theta_t m(A) = m(A+t)$ for all $A \in \mathscr{B}(E)$. Let $\mathscr{I}$ be the set of all $I \in \mathscr{B}(E)$ such that $\theta_t^{-1} I = I$ for all $t \in E$. We say that a stationary point process is *ergodic* if $\mathbb{P}(\Pi \in I) = 0$ or 1 for any $I \in \mathscr{I}$.

If $E = \mathbb{R}$, then the *rate* of a stationary point process is defined as $\rho = \mathbb{E}\left[\left|\Pi \cap (0,1]\right|\right]$ (more generally, for a stationary point process on any E the rate is the expected number of points in a set of measure 1). If the rate m is finite, then the limit

$$\psi = \lim_{x \to \infty} \frac{|\Pi \cap (0,x]|}{x}$$

exist almost surely and $\mathbb{E}\psi = \rho$. If a stationary point process with finite rate m is *ergodic*, then

$$\mathbb{P}(\psi = \rho) = 1.$$

Two standard examples of point processes appearing also in this thesis are Bernoulli processes and Poisson processes. Let $\{X_i\}_{i \in \mathbb{Z}}$ be a sequence of independent random variables with Bernoulli distribution with parameter p, that is, for all $i \in \mathbb{Z}$, $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p$. The law of $\phi = \{i \in \mathbb{Z} : X_i = 1\}$ is called *Bernoulli process* with parameter p.

Let $\psi_n = \{i/n : X_i = 1\}$ be the rescaled Bernoulli process with parameter n^{-1}. The number of points of ψ_n in any interval $(a,b]$ has binomial distribution with parameters $\lfloor n(b-a) \rfloor$ and n^{-1} and this distribution converges to the Poisson distribution with mean $(b-a)$. Moreover, the number of points of ψ_n in disjoint sets are independent and this is preserved in the limit. The law of the limit of the processes ψ_n is called the *homogeneous Poisson process* with rate 1. Both, a Bernoulli process and a homogeneous Poisson process, are stationary and ergodic point processes.

In general, a Poisson process on $\mathbb{R}$ is defined as follows. A *Poisson process* Π with mean measure μ is a random countable subset of $\mathbb{R}$ such that the number of points in disjoint Borel sets $A_1, A_2, \ldots, A_n$ are independent and so the number of points in a Borel set A has the Poisson distribution with mean $\mu(A)$, that is,

$$\mathbb{P}(|\Pi \cap A| = k) = \mathrm{e}^{-\mu(A)} \frac{\mu(A)^k}{k!}, \quad k \geq 0.$$

The mean density μ of a Poisson process might be given also in terms of an intensity function λ, where $\lambda : \mathbb{R} \to [0, \infty)$ is measurable, so that

$$\mu(A) = \int_A \lambda(x) \mathrm{d}x,$$

for any Borel set $A \subset \mathbb{R}$. A Poisson process with a constant intensity function λ is called a *homogeneous Poisson process*. A Poisson process with any other mean measure is referred to as an *inhomogeneous Poisson process*.

Paper I contains yet another example of a stationary and ergodic point process formed by special points in the graph called skeleton points. These points are defined in Section 1.3.

1.2 Random graphs

One of the fundamental models of random graphs is the graph $G(n, p)$ on the vertex set $V = \{1, 2, \dots, n\}$ with each pair of distinct vertices $\{i, j\}$ connected by an edge with probability p, $0 < p < 1$, independently of the other pairs. This random graph is usually referred to as the Erdős-Rényi graph. The probability p of connecting two vertices is called *edge probability*.

1.2.1 Directed random graphs

The *directed random graph* considered in this thesis is obtained from the Erdős-Rényi graph on the vertex set $\{1, 2, \dots, n\}$ with edge probability p by directing all edges from the vertex with smaller label to the vertex with larger label.

Directed random graphs have applications in computer science, biology and physics. One such example from biology [14] is the modelling of food webs, where the vertices $\{1, 2, \dots, n\}$ represent different species and the presence of a directed edge (i, j), $i < j$, indicates that species i is the predator and species j the prey. In computer science we can find an example of a model of parallel computation [19] where the directed edge (i, j) indicates that task i has to be done before task j.

Directed random graphs have also been referred to in the literature as *random acyclic directed graphs* and *random graph orders*. The name random acyclic directed graph relates to the fact that for every acyclic directed graph there exists a permutation of the vertices such that all of the edges are directed from the vertex with the smaller label to the vertex with the larger label. The other name, random graph orders, stems from a partial ordered set induced by the edges in the graph, that is, $i \prec j$ whenever $i < j$ and there is an edge between i and j. Various properties of directed random graphs have been studied in terms of partial orders, for example width (the longest antichain) [7], height (the longest chain) [2, 10, 17, 16], the number of linear extensions [4, 10] and the number of incomparable pairs [10, 23].

In Paper I we consider an extension of the directed random graph to $\mathbb{Z}^2$ with labels of the vertices ordered according to the product order $\prec$, that is, $(i_1, j_1) \prec (i_2, j_2)$ if the two pairs are distinct, $i_1 \leq i_2$ and $j_1 \leq j_2$. The graph has an edge between any two comparable vertices present with probability p, independently of the other edges, and the edge is directed from the vertex with the smaller label to the vertex with the larger label. There are no edges between the vertices which are not comparable. We are interested in the asymptotic behaviour of the height (the length of the longest path) of $n \times m$ subgraphs in this graph.

An extension of the directed random graph to $\mathbb{Z}$ is considered in Paper II, where we let $p \to 0$ and look at the height of the limit graph. A vertex i of the directed random graph on $\mathbb{Z}$ with edge probability p is connected via directed edges to the vertices $j_1, j_2, j_3, \ldots$, $i < j_1 < j_2 < \ldots$, that form a Bernoulli process $\{j_1 - i, j_2 - i, j_3 - i, \ldots\}$ with rate p starting from 0. As mentioned in Section 1.1, a Bernoulli process on $n^{-1}\mathbb{Z}$ with parameter n^{-1} converges to the Poisson process with rate 1 as $n \to \infty$. It is therefore natural to guess that we can find a similar connection also when we appropriately rescale the directed random graph on $\mathbb{Z}$ and simultaneously let $p \to 0$. To be able to do this, we need to introduce rooted geometric graphs.

1.2.2 Rooted geometric graphs

In a survey paper, Aldous and Steele [3] define rooted geometric graphs as follows: Let $G = (V,E)$ be a connected graph with a finite or countably infinite vertex set V, edge set E and associated function $w : E \to (0,\infty]$ for weights of the edges of G. The distance between any two vertices u and v is defined as the infimum over all paths from u to v of the sum of the weights of the edges in the path. The graph G is called a *geometric graph* if the weight function makes G locally finite in the sense that for each vertex v and each $\rho > 0$ the number of vertices within distance ρ from v is finite. If, in addition, there is a distinguished vertex v^*, we say that G is a *rooted geometric graph* with root v^*. The set of rooted geometric graphs will be denoted by $\mathscr{G}_*$.

In Paper II we look at three rooted geometric random graph models. The first model arises from the directed random graph on vertices $\{0,1,2,\ldots\}$ with edge probability n^{-1}. We look at a subgraph of the directed random graph which consists of all vertices that are connected via a directed path to 0 and of all edges between these vertices. This subgraph is also called the *closure of vertex* 0. We take the vertex 0 to be the root and to each edge $e = (i, j)$ we assign the weight $w_n(e) = n^{-1}|j-i|$.

The other two models, the *Bernoulli-weighted infinite tree with parameter* n^{-1} (BWIT_n) and the *Poisson-weighted infinite tree* (PWIT), are infinite trees with vertex set $U = \{\varnothing\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k$ such that $\varnothing$ is the root and any vertex $u \in U$ of the tree has a countably infinite number of children with labels $u1, u2, u3, \ldots$. The weight functions for the BWIT_n and the PWIT are denoted by $\widehat{w}_n$ and w, respectively. We define $\widehat{w}_n$ as follows: To each $u \in U$ assign an independent Bernoulli process with parameter n^{-1} and let $\{\zeta_j^u, j \in \mathbb{N}\}$ be the arrival times of that Bernoulli process. Let then, for each $u \in U$ and $j \in \mathbb{N}$, the weight $\widehat{w}_n$ of the edge (u,uj) be $\hat{w}_n(u,uj) = n^{-1}\zeta_j^u$. Similarly, for the PWIT, assign to each vertex $u \in U$ an independent Poisson process with rate 1 and let $\{\xi_j^u, j \in \mathbb{N}\}$ be the arrival times of that Poisson process. Then the weight function w of the edges from the vertex u is given by $w(u,uj) = \xi_j^u$, $j \in \mathbb{N}$.

Aldous and Steele [3] also define the convergence of rooted geometric graphs. Before defining the convergence, we need a definition of the isomorphism between two rooted geometric graphs. A *graph isomorphism* between rooted geometric graphs $G = (V,E)$ and $G' = (V',E')$ is a bijection $\Phi : V \to V'$ such that $(\Phi(u),\Phi(v)) \in E'$ if and only if $(u,v) \in E$ and Φ maps the root of G to the root of G'. A *geometric isomorphism* between rooted geometric graphs G and G' is a graph isomorphism Φ between G and G' which preserves edge weights, that is $w'(\Phi(e)) = w(e)$ for all $e \in E$ (where $\Phi(e)$ denotes $(\Phi(u),\Phi(v))$ for $e = (u,v)$).

When comparing two infinite rooted geometric graphs, we look at their subgraphs with all vertices at finite distance ρ from the root. Thus, define first $N_\rho(G)$, where $G \in \mathscr{G}_*$ and $\rho > 0$, to be the graph whose vertex set $V_\rho(G)$ is the set of vertices of G that are at a distance of at most ρ from the root and whose edge set consists of precisely those edges of G that have both vertices in $V_\rho(G)$. We say that ρ is a continuity point of G if no vertex of G is exactly at a distance ρ from the root of G.

We say that G_n *converges* (locally) to G in $\mathscr{G}_*$ if for each continuity point ρ of G there is an $n_0 = n_0(\rho,G)$ such that for all $n \geq n_0$ there exists a graph isomorphism $\Phi_{\rho,n}$ from the rooted geometric graph $N_\rho(G)$ to the rooted geometric graph $N_\rho(G_n)$ such that for each edge e of $N_\rho(G)$ the weight of $\Phi_{\rho,n}(e)$ converges to the weight of e.

In Paper II we propose the following distance function d on $\mathscr{G}_*$: Let $G_1, G_2 \in \mathscr{G}_*$ and define

$$d(G_1,G_2) = \int_0^\infty \frac{R(N_\rho(G_1),N_\rho(G_2))}{\mathrm{e}^\rho}\mathrm{d}\rho,$$

where

$$R(N_\rho(G_1),N_\rho(G_2)) = \min\{1, \min_{\substack{\{\Phi:\ \Phi:V_\rho(G_1)\to V_\rho(G_2)\\ \text{graph isomorphism}\}}} \max_{\substack{\{e:\ e\text{ edge}\\ \text{of } N_\rho(G_1)\}}} |w(e) - w(\Phi(e))|\}.$$

This distance function makes $\mathscr{G}_*$ into a complete separable metric space. Moreover, the notion of convergence in this metric space is equivalent with the definition of convergence given by Aldous and Steele [3].

1.3 The longest path and skeleton points

In this section we define the longest path and skeleton points of a directed random graph on the vertex set $\mathbb{Z}$ and we describe why the skeleton points are an important tool in the evaluation of the length of the longest path.

A path in a directed random graph is a sequence of vertices $(i_0, i_1, \ldots, i_\ell)$, $i_0 < i_1 < \cdots < i_\ell$, such that every consecutive pair of vertices in the sequence is connected by an edge. The length of a path is the number of edges traversed along the path. The path between vertices i and j with maximal length is called the longest path and is denoted by $L[i,j]$. In the example with food webs, mentioned in Subsection 1.2.1, the maximal path represents the longest food chain, while in parallel computation, if we assume that we have enough processors and each task takes one unit of time, the maximum path length is the total time needed for processing. In [2] and [17] it is shown that there exists a constant $C = C(p)$ such that

$$\lim_{n\to\infty} \frac{L[1,n]}{n} = C \quad \text{a.s.}$$

and also upper and lower bounds on C are provided.

A version of the central limit theorem for $L[1,n]$ is studied in [10] and [16]. The first paper investigates the case that the edge probability p is a function of n such that p tends to 0 slower than $(\log n)^{-1}$, while the second paper considers the case that the edge probability p is constant. Both papers introduce skeleton points as a tool to find an asymptotic distribution of $L[1,n]$. A vertex i is a *skeleton point* if for all $j,\ j < i$, there exists a path from vertex j to vertex i and for all $j,\ j > i$, there exists a path from vertex i to vertex j. Alon *et al.* [4] showed that the probability of vertex i being a skeleton point is given by

$$\lambda = \prod_{k=0}^{\infty} (1-(1-p)^k)^2.$$

They further prove that the sequence of the skeleton points in a directed random graph forms a stationary renewal process with the distances between two successive skeleton points having all moments finite. Denote the skeleton points by

$$\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \ldots$$

and let $(\Phi(n) : n \in \mathbb{Z})$ be a counting process such that $\Phi(n) = \max\{i : \Gamma_i \leq n\}$ is the index of the last skeleton point before vertex n. An important property of the skeleton points is that if γ is a skeleton point such that $1 \leq \gamma \leq n$, then a path with length $L[1,n]$ must necessarily contain γ (see Figure 1.1).

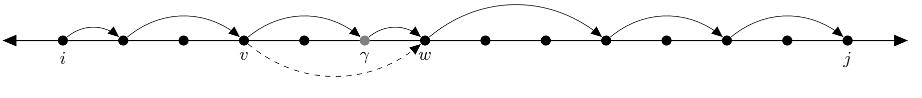

Figure 1.1. The gray point is the skeleton point γ. By the definition of the skeleton points, there is a path from vertex v to γ and there is a path from γ to w. Therefore, the longest path from vertex i to vertex j goes through skeleton point γ (marked by solid lines), instead of connecting the vertices v and w directly (marked by dashed line).

Therefore, we can write

$$L[1,n] = L[1,\Gamma_1] + \sum_{i=2}^{\Phi(n)} L[\Gamma_{i-1},\Gamma_i] + L[\Gamma_{\Phi(n)},n].$$

The first and last terms of the right hand side above are negligible when divided by $\sqrt{n}$ and the middle term is a sum of independent and identically distributed random variables. Thus, as proved in [16, Thm. 2], the middle term carries the weak limit of $L[1,n]$ so that

$$\frac{L[1,n]-Cn}{\lambda\sigma\sqrt{n}} \xrightarrow[n\to\infty]{\text{(d)}} N(0,1), \tag{1.1}$$

where $\sigma^2 = \mathrm{var}(L[\Gamma_1,\Gamma_2] - C(\Gamma_2-\Gamma_1))$.

1.4 The Tracy-Widom distribution

One of the classic random matrix models is the *Gaussian Unitary Ensemble* (GUE). That is an ensemble of $n \times n$ Hermitian matrices M_n with entries:

$$[M_n]_{ij} = \begin{cases} X_{i,j} + iY_{i,j}, & \text{if } i < j \\ Z_i, & \text{if } i = j, \end{cases}$$

where $\{X_{i,j}, i,j \in \{1,2,\dots,n\}\}$ and $\{Y_{i,j}, i,j \in \{1,2,\dots,n\}\}$ are independent and normally distributed random variables with mean 0 and variance $1/2$ and $\{Z_i, i \in \{1,2,\dots,n\}\}$ are independent and normally distributed random variables with mean 0 and variance 1. Moreover, all three families of random variables are assumed to be independent. The term *unitary* refers to the fact that the distribution is invariant under unitary conjugation, that is, for any unitary matrix U the matrix U^*M_nU has the same distribution as M_n.

Let $\lambda_1^n, \lambda_2^n, \dots, \lambda_n^n$, with $\lambda_1^n < \lambda_2^n < \cdots < \lambda_n^n$, be the eigenvalues of M_n. The empirical distribution function of the eigenvalues, defined by $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^n/\sqrt{n}}$, converges weakly, in probability, to the semicircle law with density $\frac{1}{2\pi}\sqrt{4-x^2}$ on $[-2,2]$ [5, Thm. 2.2.1]. Fluctuations of λ_n^n around the upper bound of the density support 2 have been quantified by Tracy and Widom [24]. The limiting distribution which they found is today called the *Tracy-Widom distribution*. This distribution function describes universal limit laws for a wide range of processes arising in mathematical physics and interacting particle systems. A variety of the examples is collected in a survey by Tracy and Widom [25]. Here we present two examples, a directed last-passage percolation and a Brownian directed percolation model, which are important in the study of the length of the longest path in Paper I.

A *directed last-passage percolation model* on $\mathbb{N}_0^2$ is defined as follows. Let $\{X(i,j), i,j \in \mathbb{N}_0\}$ be independent and identically distributed random variables. To each node $(i,j) \in \mathbb{N}_0^2$ we associate the weight $X(i,j)$. Let $\Pi_{m,n}$

be the set of all up/right paths in $\mathbb{N}_0^2$ from $(0,0)$ to (m,n), that is, all sequences $((0,0),(i_1,j_1),\dots,(i_{m+n-1},j_{m+n-1}),(m,n))$ such that each successive member of the sequence has a single coordinate increased by 1. The last-passage time from $(0,0)$ to (m,n) is the maximum weight of all directed paths from $(0,0)$ to (m,n), that is,

$$T_{m,n} = \max_{\pi\in\Pi_{m,n}} \sum_{(i,j)\in\pi} X(i,j).$$

Johansson [20] showed that if the weights are geometrically or exponentially distributed, then $T_{n,\lfloor an\rfloor}$, appropriately rescaled, converges in law to the Tracy-Widom distribution as $n\to\infty$, for any $a\geq 1$. Later, Bodineau and Martin [9] proved that $T_{n,\lfloor n^a\rfloor}$, appropriately rescaled, converges to the Tracy-Widom distribution, whenever the weights have a finite moment greater than 2 and a is sufficiently small (the threshold depending on the order of the finite moment). Independently, Baik and Suidan [6] obtained the same result for weights with finite fourth moment and for $a<3/14$.

A *Brownian directed percolation model* is defined on $[0,\infty)\times\{1,2,\dots,n\}$ with a standard Brownian motion $(B_t^{(j)}, t\geq 0)$ associated with every half-line $[0,\infty)\times\{j\}$. Then a last-passage time at $t\geq 0$ is defined as

$$Z_{t,n} = \sup_{0=t_0<t_1\cdots<t_{m-1}<t_n=t} \sum_{j=1}^{n} [B_{t_j}^{(j)} - B_{t_{j-1}}^{(j)}].$$

Baryshnikov [8] showed that $Z_{1,n}$ has the same distribution as the largest eigenvalue of a random $n\times n$ matrix from the GUE, and thus, appropriately scaled, converges in distribution to the Tracy-Widom distribution as $n\to\infty$.

1.5 Greedy walks

Consider a simple point process Π in a metric space (E,d). We think of Π as a collection of points (the support of the measure) and we assume that there are no accumulation points in E. We define a *greedy walk* $(S_n)_{n\geq 0}$ on Π recursively as follows. The walk starts from some point $S_0\in E$ and

$$S_{n+1} = \arg\min\left\{d(X,S_n) : X\in\Pi,\ X\notin\{S_0,S_1,\dots,S_n\}\right\}, \qquad (1.2)$$

that is, the greedy walk always moves on the points of Π by picking the nearest not yet visited point.

The greedy walk is a model in queueing systems where the points of the process represent positions of customers and the walk represents a server moving towards customers. Applications of such a system can be found, for example, in telecommunications, computer networks and transportation. As described in [11], the model of a greedy walk on a point process can be defined in various ways and on different spaces. For example, Coffman and Gilbert [13] and

Leskelä and Unger [21] study a dynamic version of the greedy walk on a circle with new customers arriving to the system according to a Poisson process.

Bordenave *et al.* [11] and Rolla *et al.* [22] state that one can show, using the Borel-Cantelli lemma, that the greedy walk on a homogeneous Poisson process on the real line does not visit all the points of Π. Since none of their papers gives a detailed proof of this claim, we include it here.

Theorem 1. *Let* Π *be a homogeneous Poisson process with rate* 1 *on the real line. Then the greedy walk, with* $S_0 = 0$*, almost surely does not visit all points of* Π.

Proof. We show that, almost surely, the walk jumps over S_0 only finitely many times and, thus, visits only finitely many points on one side of S_0. For $\ell \geq 1$ define the events

$$A_\ell = \{\exists\, n : \ell \leq S_n < \ell + 1 \text{ and } S_{n+1} < 0\}$$

that the walk jumps over 0 after visiting a point in $[\ell, \ell+1)$. If A_ℓ occurs, then from the definition of the greedy walk (1.2) it follows that all points of Π in the open interval $(S_{n+1},\ 2S_n - S_{n+1})$ have been visited in the first n steps. In particular, Π has no points in the subinterval $[\ell+1,\ 2\ell) \subset (S_n,\ 2S_n - S_{n+1})$. Therefore,

$$A_\ell \subset \{\Pi \cap [\ell,\ \ell+1) \neq 0\} \cap \{\Pi \cap [\ell+1,\ 2\ell) = 0\}$$

and

$$\mathbb{P}(A_\ell) \leq (1 - e^{-1})e^{-\ell+1}.$$

Since

$$\sum_{\ell=1}^{\infty} \mathbb{P}(A_\ell) \leq \sum_{\ell=1}^{\infty} (1 - e^{-1})e^{-\ell+1} = 1,$$

the Borel-Cantelli lemma implies that

$$\mathbb{P}(A_\ell \text{ for infinitely many } \ell \geq 0) = 0$$

and hence the walk does not visit all the points of Π almost surely. $\square$

A homogeneous Poisson process Π on the real line and its mirrored process $-\Pi$ have the same law. Moreover, the greedy walk exits any finite interval in a finite time and jumps over 0 finitely many times. Thus, an immediate observation is that

$$\mathbb{P}(\lim_{n\to\infty} S_n = +\infty) = \mathbb{P}(\lim_{n\to\infty} S_n = -\infty) = \frac{1}{2}.$$

Since the greedy walk visits finitely many points on one side of 0, in Paper III we find the exact distribution of the number of visited points on the negative half-line, as well as the distribution of the last time when the point on the negative half-line is visited.

Foss *et al.* [18] and Rolla *et al.* [22] study two modifications of the greedy walk on a homogeneous Poisson process on the line. They introduce additional points on the line, which they call “rain” and “dust”, respectively. Foss *et al.* [18] consider a space-time model, starting with a Poisson process at time 0. The positions and times of arrival of new points are given by a Poisson process on the half-plane. Moreover, the expected time that the walk spends at a point is 1. In this case the walk, almost surely, jumps over the starting point finitely many times and the position of the walk diverges logarithmically in time. Rolla *et al.* [22] assign to the points of a Poisson process one mark with probability p or two marks with probability $1-p$. The points with two marks can be visited twice by the greedy walk. The authors show that the points with two marks force the walk to jump over the starting point infinitely many times. Thus, unlike the walk on a Poisson process with only single marks, the walk here almost surely visits all points of the point process.

In Paper V we introduce a third modification of the greedy walk on a real line. We study the greedy walk on an inhomogeneous Poisson process. If the mean measure of the process enforces many long empty intervals, then the greedy walk might visit all points. We find necessary and sufficient conditions which the mean measure of the Poisson process should satisfy so that the greedy walk almost surely visits all points of the point process.

There are only a few results about the behaviour of the greedy walk on a homogeneous Poisson process in higher dimensions. Boyer [12], in a simulation study of the greedy walk on a strip $\mathbb{R}\times[0,\varepsilon]$, observes that the greedy walk bypasses some of the points of the process when the walk visits points around them and those bypassed points cause the walk to change direction later and to return towards the starting point. Assigning randomly two or more marks to the points of the Poisson process on the line mimics the greedy walk on a strip; bypassed points correspond to the points with multiple marks on the line. An explanation is that the greedy walk needs to pass several times over the point with multiple marks to delete all marks, similarly as the greedy walk on the strip might pass several times around the point before it is finally visited. Rolla *et al.* [22] conjecture that whenever the points of a Poisson process are assigned two or more marks with positive probability, the greedy walk visits all points of the point process. In Paper IV we study the greedy walk on two homogeneous Poisson processes placed on two parallel lines at distance r. This can be compared to a strip or the line with “double” marks, because the greedy walk might pass on one line, leaving some of the points on the other line unvisited. Those unvisited points cause the walk to return towards the starting point.

Rolla *et al.* [22] also discuss the behaviour of the greedy walk on a homogeneous point process in $\mathbb{R}^d$, $d\geq 2$. They compare the steps of the greedy walk with the Brownian motion in $\mathbb{R}^d$. The Brownian motion in $\mathbb{R}^2$ is recurrent,

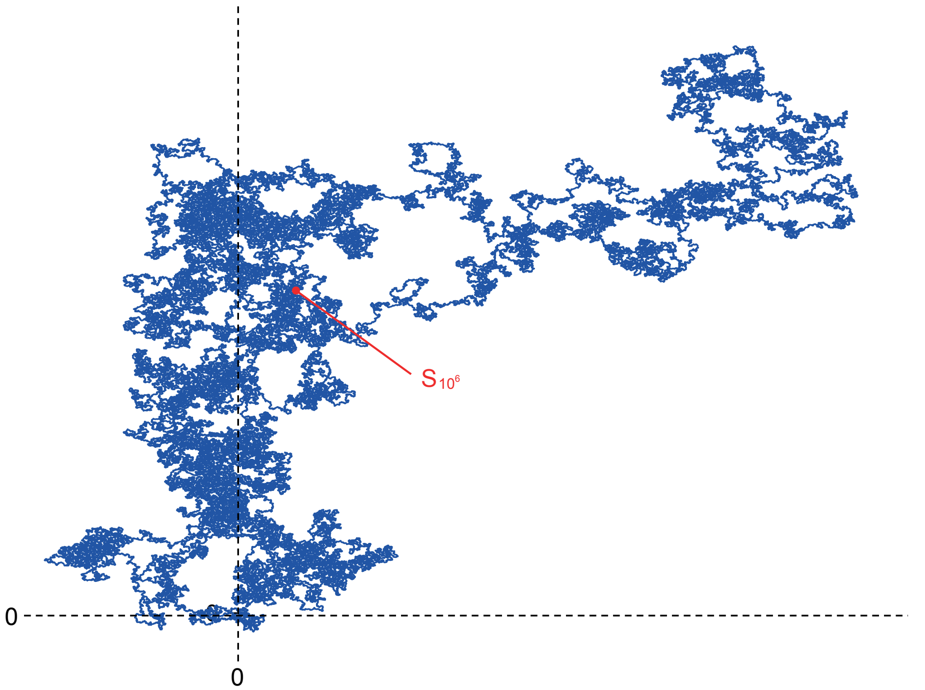

Figure 1.2. The first 10^6 steps of the greedy walk in $\mathbb{R}^2$ starting from $(0,0)$.

but the greedy walk has some local self-repulsion and it is uncertain how this affects the walk in the long run. (See Figure 1.2 for a simulation of the greedy walk in $\mathbb{R}^2$.) The Brownian motion in $\mathbb{R}^d$, $d \geq 3$, is transient and, thus, it is expected that also the greedy walk in $\mathbb{R}^d$, $d \geq 3$, never visits some points.

2. Summary of Papers

2.1 Paper I

In Paper I we consider a directed random graph on $\mathbb{Z}^2$ with vertices ordered according to the product order $\prec$ and with edge probability p. We say that $((i_0, j_0), (i_1, j_1), \ldots, (i_\ell, j_\ell))$, with $(i_0, j_0) \prec (i_1, j_1) \prec \cdots \prec (i_\ell, j_\ell)$, is a path of length ℓ if all pairs of consecutive vertices in the sequence are connected via an edge. We are interested in the random variable $L_{n,m}$, that is defined as the maximum length of all paths between vertices $(0,0)$ and (n,m). Denisov *et al.* [16] found a version of a limit theorem for $L_{n,m}$ when n tends to infinity, but m is constant. In Paper I we prove a limit theorem for $L_{n,m}$ when both n and m tend to infinity, so that $m = \lfloor n^a \rfloor$.

Using a slightly modified definition of the skeleton points from Section 1.3, we can find upper and lower bounds for $L_{n,m}$ defined in terms of the skeleton points. Further, we make a sequence of transformations in order to establish an estimate $S_{n,m}$ of $L_{n,m} - Cn$ which resembles a last-passage percolation model. Similarly as Bodineau and Martin [9], we couple $S_{n,m}$ with a last-passage time $Z_{n,m}$ of the Brownian directed percolation and show that, if properly rescaled, they have the same limit distribution. Thus, the asymptotic distribution of $S_{n,m}$, as well as the the asymptotic distribution of $L_{n,m}$, appropriately rescaled, is the Tracy-Widom distribution as $n \to \infty$, $m = \lfloor n^a \rfloor$ and $a < 3/14$.

2.2 Paper II

In Paper II we use the framework of rooted geometric graphs defined in Subsection 1.2.2. We look at the closure of vertex 0 of a directed random graph on the vertex set $\{0, 1, 2, \ldots\}$, with edge probability n^{-1} and weight function $w_n((i,j)) = n^{-1}|i-j|$.

We prove that the closure of vertex 0 converges in distribution to the Poisson-weighted infinite tree (PWIT) when $n \to \infty$. We do this in three steps: First we show that the probability that the closure of vertex 0 in a finite radius is a tree converges to 1. Whenever the closure of vertex 0 is a tree in a finite radius, it has the same law as the Bernoulli-weighted infinite tree with parameter n^{-1} (BWIT_n) in that radius. Since the BWIT_n converges in distribution to the PWIT, it follows that also the closure of vertex 0 converges in distribution to the PWIT.

Moreover, let L_x be the length of the longest path of the PWIT, between all paths from the root to a vertex at a distance of at most x from the root. Using the related results from Addario-Berry and Ford [1], we prove that the median of L_x is

$$\text{median}(L_x) = x\mathrm{e} - \frac{3}{2}\log x + O(1),$$

where $f(x) = O(g(x))$ means that there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all large x. We also show that the tails of the distribution of $L_x - \text{median}(L_x)$ are exponentially bounded. This implies that the expected value of L_x is $\text{median}(L_x) + O(1)$ and that the variance of L_x is $O(1)$.

2.3 Paper III

In Paper III we study the greedy walk on a homogeneous Poisson process with rate 1 on the real line starting from $S_0 = 0$. Using properties of exponential distribution, we are able to show that the probability that the walk jumps over 0 after visiting S_n is 2^{-n} and that this probability is independent of the steps of the greedy walk before time n. An immediate consequence of this observation is that the expected number of times the walk jumps over 0 is $1/2$.

Furthermore, let N be the number of visited points on the negative half-line and let L be the index of the last step of the greedy walk which is less than or equal to 0. We derive the exact distributions of these two random variables. Moreover, we show that the quantities $\mathbb{P}(N = k)$ and $\mathbb{P}(L = k)$ decay geometrically with factor $1/2$.

2.4 Paper IV

In Paper IV we study the greedy walk on various point processes defined on the union of two lines $E \subset \mathbb{R}^2$, where the distance function d on E is the Euclidean distance, $d\left((x_1,y_1),(x_2,y_2)\right) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$. For each point process, we answer the question whether the greedy walk visits all points of the point process.

We study first a point process Π on two lines intersecting at $(0,0)$ with independent homogeneous Poisson processes on each line. The greedy walk starts from $(0,0)$. When the walk visits a point that is far away from $(0,0)$, then the distance to $(0,0)$ and to any point on the other line is large and the probability of changing lines or jumping over $(0,0)$ is small. Using the Borel-Cantelli lemma we show that almost surely the walk jumps over $(0,0)$ or changes lines only finitely many times, which implies that almost surely the walk does not visit all points of Π.

Thereafter, we look at the greedy walk on point processes placed on two parallel lines at a fixed distance r, $\mathbb{R} \times \{0, r\}$. The behaviour here depends on the definition of the process. The first case we study is a process Π consisting of two identical copies of a homogeneous Poisson process on $\mathbb{R}$, that are placed on the parallel lines. We observe that the greedy walk visits the points of Π in clusters: it visits successively all points on one line which are at the distance less than r. When the next point on the line is at a distance greater than r, the walk changes the line and it visits the copies of the visited points on the other line in the reversed order. The last point the greedy walk visits in a cluster is a copy of the first visited point of the cluster. Therefore, it is enough to have information about the position of the first points of the clusters, to know if the walk jumps over the vertical line $\{0\} \times \mathbb{R}$ infinitely often. Thus, we can just look at the closest points to $(0,0)$ on line 0 of all clusters. Since the distances between two first points of the cluster are identically distributed and independent, we can use a similar argument as in the proof of Theorem 1 in Section 1.5 to show that the greedy walk moving just on the first points of the clusters does not visit all points. Therefore, also the greedy walk on two lines does not visit all the points of Π, but it visits all the points on one side of the vertical line $\{0\} \times \mathbb{R}$ and just finitely many points on the other side.

In the second case, we modify the definition of the process above by deleting exactly one of the copies of each point with probability $p > 0$, independently from the other points, and the line on which the point will be deleted is chosen with probability $1/2$. In particular, if $p = 1$ we have two independent Poisson processes on these lines. For any $p > 0$, the greedy walk almost surely visits all points. The reason is that the greedy walk skips some of the points when it goes away from the vertical line $\{0\} \times \mathbb{R}$ and those points will force the walk to return and jump over the vertical line $\{0\} \times \mathbb{R}$ infinitely many times. We prove this using arguments from [22]. The idea is to show that the walk starting from $(0,0)$ almost surely returns and jumps over the vertical line $\{0\} \times \mathbb{R}$ in a finite time. Then we can repeatedly use that argument to show that the walk jumps over the vertical line $\{0\} \times \mathbb{R}$ infinitely often. In order to prove that the walk jumps over the vertical line $\{0\} \times \mathbb{R}$ in almost surely finite time, we define a stationary and ergodic sequence of points Ξ that will never be visited by the greedy walk. If Ξ is almost surely empty, then the probability that the walk stays on one side of the starting point is 0 and the claim follows. If Ξ is almost surely non-empty, then the distance between two points of Π is infinitely often big enough so that the greedy walk prefers to return, visit the bypassed points of Ξ and to jump over the vertical line $\{0\} \times \mathbb{R}$, instead of moving further from the $\{0\} \times \mathbb{R}$.

In the third case, we place two identical copies of a homogeneous Poisson process on the parallel lines, but this time we shift one of the copies by $|s| < r/\sqrt{3}$. A first guess is that we can divide the points into clusters, as in the first case, so that all points of a cluster are visited successively. However, one can find examples when the walk moves to another cluster without visiting all points of the cluster. Those points that are not visited, will cause the walk to return later and jump over the vertical line $\{0\} \times \mathbb{R}$. Hence, the greedy walk eventually visits all the points. The proof here follows similarly as the proof in the second case. Note that the result in all three cases is independent of the choice of r.

2.5 Paper V

In Paper V we consider the greedy walk on an inhomogeneous Poisson process on the real line. We assume that the greedy walk starts from 0 and we assume that the Poisson process with mean measure μ has almost surely infinitely many points on each half-line. Moreover, we assume that there are no accumulation points. We prove that if the mean measure μ of the Poisson process satisfies

$$\int_0^\infty \exp(-\mu(x,2x+R))\mu(\mathrm{d}x) = \infty \text{ and } \int_{-\infty}^0 \exp(-\mu(2x-R,x))\mu(\mathrm{d}x) = \infty,$$

for all $R \geq 0$ then the greedy walk almost surely visits all points. If either integral is finite for some $R \geq 0$, then the greedy walk almost surely does not visit all points. The proof follows from the observation that the greedy walk crosses 0 infinitely many times if and only if for all $R \geq 0$ there are infinitely many points $X > 0$ such that the interval $(X,2X+R)$ is empty and infinitely many points $Y < 0$ such that $(2Y-R,Y)$ is empty. Given X and Y, the number of points in those intervals have exponential distributions with parameters $\mu(X,2X+R)$ and $\mu(2Y-R,Y)$, respectively. We use Campbell's theorem to determine if the sums over the points of the Poisson process of the probabilities that the intervals are empty is convergent or divergent. The conclusion that the intervals are empty infinitely often if and only if the sum is divergent follows from the extended Borel-Cantelli lemma.

Using the criterion above, we are able to find a threshold function for the property of visiting all points. That is, we can compare the tails of any density function with the threshold function: If the tails of another function are below the threshold function then the greedy walk visits all points and if the tails are significantly above the threshold function, then the greedy walk does not visit all points. We also discuss some cases when the tails are not comparable.

3. Summary in Swedish

Denna avhandling består av en inledning och fem artiklar, av vilka två behandlar riktade slumpgrafer och tre behandlar giriga vandringar på punktprocesser.

I artikel I beaktar vi en riktad slumpgraf med nodmängd $\mathbb{Z}^2$. Dess kantmängd konstrueras genom att för varje par $\{(i_1,i_2),(j_1,j_2)\}$ som uppfyller $i_1 \leq j_1$ och $i_2 \leq j_2$ inkludera en kant från (i_1,i_2) till (j_1,j_2) med sannolikhet p, oberoende av alla andra par. Låt $L_{n,m}$ beteckna den maximala längden av alla vägar som ligger i en $n \times m$–rektangel. Vi visar att för $a < 3/14$, $L_{n,\lfloor n^a \rfloor}$ konvergerar (efter centrering och lämplig omskalning) i fördelning till Tracy-Widomfördelningen. Denna fördelning tros vara den universella gränsvärdesfördelningen för en stor klass av tvådimensionella processer i matematisk fysik och interagerande partikelsystem.

I artikel II koncentrerar vi oss på en riktad slumpgraf med nodmängd $\mathbb{Z}$. Kantmängden konstrueras genom att, för alla $i < j$, inkludera en kant från i till j med sannolikhet p, oberoende av alla andra par av noder. Det slutna höljet av noden 0 är den delgraf som induceras av alla noder i så att det finns en riktad väg från 0 till i. Då $p = n^{-1}$ visar vi att det slutna höljet av 0 (efter lämplig omskalning) konvergerar i fördelning till det Poisson-viktade oändliga trädet. Dessutom härleder vi gränsvärdesresultat för längden av den längsta vägen i det Poisson-viktade oändliga trädet.

En girig vandring $(S_n)_{n\geq 0}$ på en enkel punktprocess Π i ett metriskt rum (E,d) definieras som följer. Vandringen startar i någon punkt $S_0 \in E$ och följer sedan regeln

$$S_{n+1} = \arg\min\big\{d(X,S_n) : X \in \Pi,\ X \notin \{S_0,S_1,\ldots,S_n\}\big\}.$$

Detta innebär att den giriga vandringen hela tiden går till den närmaste ännu icke besökta punkten.

Den giriga vandringen på en homogen Poissonprocess på den reella tallinjen besöker nästan säkert inte alla punkter. Med sannolikhet $1/2$ divergerar den till $+\infty$ och besöker då som mest ändligt många punkter på den negativa delen av tallinjen. I artikel III bestämmer vi fördelningen för antalet punkter som besöks på den negativa delen av tallinjen, och fördelningen för det index för vilket vandringen når sitt minimum.

I artikel IV undersöker vi den giriga vandringen på några olika punktprocesser som definieras på unionen av två linjer i $\mathbb{R}^2$. Vi tittar först på fallet då linjerna skär varandra. Om man placerar två oberoende endimensionella Poissonprocesser på linjerna visar det sig att den giriga vandringen nästan säkert inte besöker alla punkter. Vi tittar sedan på fallet med två parallella linjer. Resultaten här beror på hur processerna definieras. Om vardera linje har en kopia av samma realisering av en homogen Poissonprocess, besöker den giriga vandringen nästan säkert inte alla punkter. Om man för varje punkt i denna process tar bort den (från en av de två linjerna, vald slumpmässigt) med sannolikhet p, oberoende av alla andra punkter, besöker den giriga vandringen nästan säkert alla punkter. Slutligen studeras fallet då varje linje har en kopia av samma realisering av samma Poissonprocess, men där en kopia har förskjutits en sträcka s i sidled (där s är litet). I detta fall besöker den giriga vandringen nästan säkert alla punkter.

I artikel V tittar vi på den giriga vandringen på en ickehomogen Poissonprocess på den reella tallinjen. Vi hittar tillräckliga och nödvändiga villkor på intensiteten för att den giriga vandringen ska besöka alla punkter. Dessutom undersöker vi tröskelbeteendet.

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Paper I

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Convergence to the Tracy-Widom distribution for longest paths in a directed random graph

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Abstract. We consider a directed graph on the 2-dimensional integer lattice, placing a directed edge from vertex (i_1, i_2) to (j_1, j_2), whenever $i_1 \le j_1$, $i_2 \le j_2$, with probability p, independently for each such pair of vertices. Let $L_{n,m}$ denote the maximum length of all paths contained in an $n \times m$ rectangle. We show that there is a positive exponent a, such that, if $m/n^a \to 1$, as $n \to \infty$, then a properly centered/rescaled version of $L_{n,m}$ converges weakly to the Tracy-Widom distribution. A generalization to graphs with non-constant probabilities is also discussed.

1. Introduction

Random directed graphs form a class of stochastic models with applications in computer science (Isopi and Newman, 1994), biology (Cohen and Newman, 1991; Newman, 1992; Newman and Cohen, 1986) and physics (Itoh and Krapivsky, 2012). Perhaps the simplest of all such graphs is a directed version of the standard Erdős-Rényi random graph (Barak and Erdős, 1984) on n vertices, defined as follows: For each pair $\{i, j\}$ of distinct positive integers less than n, toss a coin with probability of heads equal to p, $0 < p < 1$, independently from pair to pair; if heads shows up then introduce an edge directed from $\min(i, j)$ to $\max(i, j)$. There is a natural extension of this graph to the whole of $\mathbb{Z}$ studied in detail in Foss and Konstantopoulos (2003). In particular, if we define the asymptotic growth rate $C = C(p)$, as the a.s. limit of the maximum length of all paths between 1 and n divided by n, Foss and Konstantopoulos (2003) provide sharp bounds on $C(p)$ for all values of $p \in (0, 1)$.

A natural generalization arises when we replace the total order of the vertex set by a partial order, usually implied by the structure of the vertex set. In such

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a model, coins are tossed only for pairs of vertices which are comparable in this partial order. The canonical case is to consider, as a vertex set, the 2-dimensional integer lattice $\mathbb{Z} \times \mathbb{Z}$, equipped with the standard component-wise partial order: $(i_1, i_2) \prec (j_1, j_2)$ if the two pairs are distinct and $i_1 \le i_2$, $j_1 \le j_2$. Such a graph was considered in Denisov et al. (2012). In that paper, it was shown that if $L_{n,m}$ denotes the maximum length of all paths of the graph, restricted to $\{0, \ldots, n\} \times \{1, \ldots, m\}$, then there is a positive κ (depending on p and the fixed integer m), such that

$$\left(\frac{L_{[nt],m} - Cnt}{\kappa\sqrt{n}},\ t \ge 0 \right) \xrightarrow[n\to\infty]{\text{(d)}} (Z_{t,m},\ t \ge 0), \tag{1.1}$$

where $Z_{\bullet,m}$ is the stochastic process defined in terms of m independent standard Brownian motions, $B^{(1)}, \ldots, B^{(m)}$, via the formula

$$Z_{t,m} := \sup_{0=t_0<t_1\cdots<t_{m-1}<t_m=t} \sum_{j=1}^{m} [B^{(j)}_{t_j} - B^{(j)}_{t_{j-1}}], \quad t \ge 0.$$

One can speak of Z as a *Brownian directed percolation model*, the terminology stemming from the picture of a "weighted graph" on $\mathbb{R} \times \{1, \ldots, m\}$ where the weight of a segment $[s,t] \times \{j\}$ equals the change $B^{(j)}_t - B^{(j)}_s$ of a Brownian motion. If a path from $(0,0)$ to (t,m) is defined as a union $\bigcup_{j=1}^m [t_{j-1}, t_j] \times \{j\}$ of such segments, then Z represents the maximum weight of all such paths.

Baryshnikov (2001), answering an open question by Glynn and Whitt (1991), showed that

$$Z_{1,m} \overset{\text{(d)}}{=} \lambda_m,$$

where λ_m is the largest eigenvalue of a GUE matrix of dimension m. Since $Z_{\bullet,m}$ is 1/2-self-similar, we see that

$$Z_{t,m} \overset{\text{(d)}}{=} \sqrt{t}\lambda_m.$$

Now, fluctuations of λ_m around the centering sequence $2\sqrt{m}$ have been quantified by Tracy and Widom (1994) who showed the existence of a limiting law, denoted by F_{TW}:

$$m^{1/6}(\lambda_m - 2\sqrt{m}) \xrightarrow[m\to\infty]{\text{(d)}} F_{\mathrm{TW}}.$$

A natural question then, raised in Denisov et al. (2012), is whether one can obtain F_{TW} as a weak limit of $L_{n,m}$ when n and m tend to infinity simultaneously. Our paper is concerned with resolving this question. To see what scaling we can expect, rewrite the last display, for arbitrary $t > 0$, as

$$m^{1/6}\Big(\frac{Z_{t,m}}{\sqrt{t}} - 2\sqrt{m}\Big) \xrightarrow[m\to\infty]{\text{(d)}} F_{\mathrm{TW}}.$$

A statement of the form $X(t,m) \xrightarrow[m\to\infty]{\text{(d)}} X$, where the distribution of $X(t,m)$ does not depend on the choice of $t > 0$, implies the statement $X(t, m(t)) \xrightarrow[t\to\infty]{\text{(d)}} X$, for any function $m(t)$ such that $m(t) \xrightarrow[t\to\infty]{} \infty$. Hence, upon setting $m = [t^a]$, we have

$$t^{a/6}\left(\frac{Z_{t,[t^a]}}{\sqrt{t}} - 2\sqrt{t^a} \right) \xrightarrow[t\to\infty]{\text{(d)}} F_{\mathrm{TW}}. \tag{1.2}$$

Therefore, it is reasonable to guess that, when a is small enough, an analogous limit theorem holds for a centered scaled version of the largest length $L_{n,[n^a]}$, namely that

$$n^{a/6}\left(\frac{L_{n,[n^a]} - c_1 n}{c_2\sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow[n\to\infty]{(\mathrm{d})} F_{\mathrm{TW}}, \tag{1.3}$$

where c_1, c_2 are appropriate constants.

A stochastic model, bearing some resemblance to ours, is the so-called *directed last passage percolation model* on $\mathbb{Z}^d$ (the case $d = 2$ being of interest here). We are given a collection of i.i.d. random variables indexed by elements of $\mathbb{Z}_+^d$. A path from the origin to the point $n \in \mathbb{Z}_+^d$ is a sequence of elements of $\mathbb{Z}_+^d$, starting from the origin and ending at n, such that the difference of successive members of the sequence is equal to the unit vector in the ith direction, for some $1 \le i \le d$. The weight of a path is the sum of the random variables associated with its members. Specializing to $d = 2$, let $L_{n,m}$ be the largest weight of all paths from $(0,0)$ to (n,m). Assuming that the random variables have a finite moment of order larger than 2, Bodineau and Martin (2005) showed that (1.3) holds for all sufficiently small positive a (the threshold depending on the order of the finite moment). Independently, Baik and Suidan (2005) obtained the same result for random variables with a finite 4th moment and for $a < 3/14$. In both papers, partial sums of i.i.d. random variables were approximated with Brownian motions, in the first case using the Komlós-Major-Tusnády (KMT) construction, while in the second using Skorokhod embedding.

To show that (1.3) holds for our model, we adopt the technique introduced in Denisov et al. (2012), which involves the existence of *skeleton points* on each line $\mathbb{Z} \times \{j\}$. Skeleton points are, by definition, random points which are connected with all the other points on the same line. In Denisov et al. (2012) Denisov, Foss and Konstantopoulos used this fact, together with the fact that, for finite m, one can pick skeleton points common to all m lines, in order to prove (1.1). However, when m tends to infinity simultaneously with n, it is not possible to pick skeleton points common to all lines. Modifying the definition of skeleton points enables us to give a new proof of (1.1), as well as to prove (1.3). To achieve the latter, we borrow the idea of KMT coupling from Bodineau and Martin (2005). However, we need to do some work in order to express the random variable $L_{n,m}$ in a way that resembles a maximum of partial sums.

Although we focus on the case where the edge probability p is constant, it is possible to consider a more general case, where the probability that a vertex $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$ connects to a vertex (j_1, j_2) depends on the distances $|j_1 - i_1|$ and $|j_2 - i_2|$ of the two vertices. This generalization is discussed in the last section of the article.

2. The one-dimensional directed random graph

We summarize below some properties of the directed Erdős-Rényi graph on $\mathbb{Z}$ with connectivity probability p taken from Foss and Konstantopoulos (2003). For $i < j$, let $L[i,j]$ be the maximum length of all paths with start and end points in the interval $[i,j]$. Then, for $i < j < k$, we have $L[i,k] \le L[i,j] + L[j,k] + 1$. Since the distribution of the random graph is invariant under translations, and is also ergodic (the natural invariant σ-field is trivial), it follows, from Kingman's subadditive ergodic theorem, that there is a deterministic constant $C = C(p)$ such that

$$\lim_{n\to\infty} L[1,n]/n = C, \text{ a.s.} \tag{2.1}$$

In fact, $C = \inf_{n\geq 1} EL[1,n]/n$. The function $C(p)$ is not known explicitly; only bounds are known (Foss and Konstantopoulos, 2003, Thm. 10.1). For example, $0.5679 \leq C(1/2) \leq 0.5961$. We also know that there exists, almost surely, a random integer sequence $\{\Gamma_r, r \in \mathbb{Z}\}$ with the property that for all r, all $i < \Gamma_r$, and all $j > \Gamma_r$, there is a path from i to Γ_r and a path from Γ_r to j. The existence of such points, referred to as *skeleton points*, is not hard to establish (Denisov et al., 2012). Since the directed Erdős-Rényi graph is invariant under translations, so is the sequence of skeleton points, i.e., $\{\Gamma_r, r \in \mathbb{Z}\}$ has the same law as $\{n+\Gamma_r, r \in \mathbb{Z}\}$, for all $n \in \mathbb{Z}$. Moreover, it turns out that the sequence forms a stationary renewal process. If we enumerate the skeleton points according to $\cdots < \Gamma_{-1} < \Gamma_0 \leq 0 < \Gamma_1 < \cdots$, we have that $\{\Gamma_{r+1} - \Gamma_r, r \in \mathbb{Z}\}$ are independent random variables, whereas $\{\Gamma_{r+1}-\Gamma_r, r \neq 0\}$ are i.i.d. Stationarity implies that the law of the omitted difference $\Gamma_1 - \Gamma_0$ has a density which is proportional to the tail of the distribution of $\Gamma_2-\Gamma_1$. In Denisov et al. (2012) it is shown that the distance $\Gamma_2-\Gamma_1$ between two successive skeleton points has a finite 2nd moment. One can follow the same steps of the proof, to show that in our case, with constant probability p, this random variable has moments of all orders. Moreover, one can show that for some $\alpha > 0$ (the maximal such α depends on p) it holds that $Ee^{\alpha(\Gamma_2-\Gamma_1)} < \infty$.

The rate λ_0 of the sequence of skeleton points can be expressed as an infinite product:

$$\lambda_0 := \frac{1}{E(\Gamma_2 - \Gamma_1)} = \prod_{k=1}^{\infty} (1-(1-p)^k)^2. \tag{2.2}$$

For example, for $p = 1/2$, $\lambda_0 \approx 1/12$.

A central limit theorem for $L[1,n]$ is also available (Denisov et al., 2012, Thm. 2). If we let

$$\sigma_0^2 := \operatorname{var}(L[\Gamma_1,\Gamma_2] - C(\Gamma_2 - \Gamma_1)), \tag{2.3}$$

then

$$\frac{L[1,n] - Cn}{\sqrt{\lambda_0 \sigma_0^2 n}} \xrightarrow[n\to\infty]{\text{(d)}} N(0,1), \tag{2.4}$$

where $N(0,1)$ is a standard normal random variable. Note that $\sigma_0^2 \neq \operatorname{var}(L[\Gamma_1,\Gamma_2])$. Unfortunately, we have no estimates for σ_0^2, but, interestingly, there is a technique for estimating it, based on perfect simulation. This was briefly explained in Foss and Konstantopoulos (2003) in connection with an infinite-dimensional Markov chain which carries most of the information about the law of the directed Erdős-Rényi random graph.

In addition, it is shown in Foss and Konstantopoulos (2003) that C can also be expressed as

$$C = \frac{EL[\Gamma_1,\Gamma_2]}{E(\Gamma_2 - \Gamma_1)}. \tag{2.5}$$

In fact, if $\{\nu_r, r \in \mathbb{Z}\}$ is a random sequence of integers, defined on the same probability space as the one supporting the random graph, such that $\{\Gamma_{\nu_r}, r \in \mathbb{Z}\}$ is a stationary point process then

$$C = \frac{EL[\Gamma_{\nu_r}, \Gamma_{\nu_{r+1}}]}{E(\Gamma_{\nu_{r+1}} - \Gamma_{\nu_r})}.$$

The most important property of the skeleton points is that if γ is a skeleton point, and if $i \le \gamma \le j$, then a path with length $L[i, j]$ (a maximum length path) must necessarily contain γ. This crucial property will be used several times below, especially since, for every $i < j$, the following equality holds

$$L[\Gamma_i, \Gamma_j] = L[\Gamma_i, \Gamma_{i+1}] + L[\Gamma_{i+1}, \Gamma_{i+2}] + \cdots + L[\Gamma_{j-1}, \Gamma_j].$$

Furthermore, the restriction of the graph on the interval between two successive skeleton points is independent of the restriction on the complement of the interval; hence the summands in the right-hand side of the last display are independent random variables.

3. Statement of the main result

It is clear from (2.4) that the constants c_1, c_2 in (1.3) should be as follows: $c_1 = C$, $c_2 = \sqrt{\lambda_0 \sigma_0^2}$. Now we can formulate the main result.

Theorem 3.1. *Let C, λ_0, σ_0^2 be the quantities associated with the directed random graph on $\mathbb{Z}$ with connectivity probability p, defined by* (2.1) *(equivalently,* (2.5)*),* (2.2)*,* (2.3)*, respectively. Consider the directed random graph on $\mathbb{Z} \times \mathbb{Z}$ and let $L_{n,m}$ be the maximum length of all paths between two vertices in $[0, n] \times [1, m]$. Then, for all $0 < a < 3/14$,*

$$n^{a/6} \left(\frac{L_{n,[n^a]} - Cn}{\sqrt{\lambda_0 \sigma_0^2} \sqrt{n}} - 2\sqrt{n^a} \right) \xrightarrow[n\to\infty]{\text{(d)}} F_{TW}, \tag{3.1}$$

where F_{TW} is the Tracy-Widom distribution.

To prove this theorem, we will first define the notion of skeleton points for the graph on $\mathbb{Z} \times \mathbb{Z}$ and then prove pathwise upper and lower bounds for $L_{n,m}$ which depend on paths going through these skeleton points. This will be done in Section 4. In Section 5.1 we show that the difference between these bounds is of the order $o(n^b)$, where $b = (1/2) - (a/6)$ is the net exponent in the denominator of (3.1). We will then (Section 5.2) introduce a quantity $S_{n,m}$ which resembles a last passage percolation problem and show that it differs from $L_{n,m}$ by a quantity which is of the order $o(n^b)$, when $m = [n^a]$. The problem will then be translated to a last passage percolation problem (with the exception of random indices). This will finally, in Section 5.3 be compared to the Brownian directed percolation problem by means of strong coupling.

4. Skeleton points and pathwise bounds

Our model is a directed random graph G with vertices $\mathbb{Z} \times \mathbb{Z}$. For each pair of vertices $\boldsymbol{i}$, $\boldsymbol{j}$, such that $\boldsymbol{i} \prec \boldsymbol{j}$, toss an independent coin with probability of heads equal to p; if heads shows up introduce an edge directed from $\boldsymbol{i}$ to $\boldsymbol{j}$.

A path of length ℓ in the graph is a sequence $(\boldsymbol{i}_0, \boldsymbol{i}_1, \ldots, \boldsymbol{i}_\ell)$ of vertices $\boldsymbol{i}_0 \prec \boldsymbol{i}_1 \prec \ldots \prec \boldsymbol{i}_\ell$ such that there is an edge between any consecutive vertices.

We denote by $G_{n,m}$ the restriction of G on the set of vertices $\{0, 1, \ldots, n\} \times \{1, \ldots, m\}$. The random variable of interest is

$$L_{n,m} := \text{ the maximum length of all paths in } G_{n,m}.$$

We refer to the set $\mathbb{Z} \times \{j\}$ as "line j" or "jth line", and note that the restriction of G onto $\mathbb{Z} \times \{j\}$ is a directed Erdős-Rényi random graph. We denote this restriction by $G^{(j)}$. Typically, a superscript (j) will refer to a quantity associated with this restriction. For example, for $a \le b$,

$$L^{(j)}[a,b] := \text{the maximum length of all paths in } G^{(j)} \\ \text{with vertices between } (a,j) \text{ and } (b,j)$$

and we agree that $L^{(j)}[a,b] = 0$ if $a \ge b$.

Clearly, the $\{G^{(j)}, j \in \mathbb{Z}\}$ are i.i.d. random graphs, identical in distribution to the directed Erdős-Rényi random graph. Therefore, for each $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} L^{(j)}[1,n]/n = C, \text{ a.s.}$$

To establish upper and lower bounds for $L_{n,m}$, we need to slightly change the definition of a skeleton point in G.

Definition 4.1 (Skeleton points in G)**.** A vertex (i,j) of the directed random graph G is called skeleton point if it is a skeleton point for $G^{(j)}$ (for any $i' < i < i''$, there is a path from (i',j) to (i,j) and a path from (i,j) to (i'',j)) and if there is an edge from (i,j) to $(i,j+1)$.

Therefore, the skeleton points on line j are obtained from the skeleton point sequence of the directed Erdős-Rényi random graph $G^{(j)}$ by independent thinning with probability p. When we refer to skeleton points on line j, we shall be speaking of this thinned sequence. The elements of this sequence are denoted by

$$\cdots < \Gamma_{-1}^{(j)} < \Gamma_0^{(j)} \le 0 < \Gamma_1^{(j)} < \Gamma_2^{(j)} < \cdots$$

and have rate

$$\lambda = \frac{1}{E(\Gamma_2^{(j)} - \Gamma_1^{(j)})} = p\lambda_0 = p\prod_{k=1}^{\infty}(1-(1-p)^k)^2.$$

The associated counting process of skeleton points on line j is defined by

$$\Phi^{(j)}(t) - \Phi^{(j)}(s) = \sum_{r\in\mathbb{Z}} \mathbf{1}(s < \Gamma_r^{(j)} \le t), \quad s,t \in \mathbb{R}, \quad s \le t,$$

together with the agreement that

$$\Phi^{(j)}(0) = 0.$$

Note that we insist on having the parameter t in $\Phi^{(j)}(t)$ as an element of $\mathbb{R}$ (and not just $\mathbb{Z}$). We also let

$$X^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)},$$

$$Y^{(j)}(t) := \Gamma^{(j)}_{\Phi^{(j)}(t)+1},$$

be the skeleton points on line j straddling t:

$$X^{(j)}(t) \le t < Y^{(j)}(t). \tag{4.1}$$

Next we prove upper and lower bounds for $L_{n,m}$. The set of dissections of the interval $[0,n] \subset \mathbb{R}$ in m non-overlapping, possibly empty intervals is denoted by

$$\mathcal{T}_{n,m} := \{\boldsymbol{t} = (t_0, t_1, \ldots, t_m) \in \mathbb{R}^{m+1} : \ 0 = t_0 \le t_1 \le \cdots \le t_{m-1} \le t_m = n\}.$$

Lemma 4.2. (Upper bound) *Define*

$$\overline{L}_{n,m} := \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}),\ Y^{(j)}(t_j)] + m. \tag{4.2}$$

Then $L_{n,m} \le \overline{L}_{n,m}$.

Proof: Let π be a path in $G_{n,m}$. Consider the lines visited by π, denoting their indices by $1 \le \nu_1 < \nu_2 < \cdots < \nu_J \le m$. Let (a_j, ν_j) and (b_j, ν_j) be the first and the last vertex of line ν_j in the path π. Then the length of π satisfies

$$|\pi| \le \sum_{j=1}^{J} L^{(\nu_j)}[a_j, b_j] + J - 1.$$

Since successive vertices in the path should be increasing in the order $\prec$, we have $b_{j-1} \le a_j$, $2 \le j \le J$. Hence, with $b_0 := 0$,

$$|\pi| \le \sum_{j=1}^{J} L^{(\nu_j)}[b_{j-1}, b_j] + J - 1 \le \sum_{j=1}^{J} L^{(\nu_j)}[X^{(\nu_j)}(b_{j-1}),\ Y^{(\nu_j)}(b_j)] + J - 1,$$

where we used (4.1). Since $J \le m$, we can extend $0 = b_0 \le b_1 \le \cdots \le b_J \le n$ to a dissection of $[0, n]$ into m non-overlapping intervals, showing that the right-hand side of the last display is bounded above by $\overline{L}_{n,m}$. Taking the maximum over all π in $G_{n,m}$, we obtain $L_{n,m} \le \overline{L}_{n,m}$, as required. □

Note that the existence and properties of skeleton points were not used in the proof of the upper bound, other than to ensure that the upper bound is a.s. finite.

Lemma 4.3. (Lower bound) *Define*

$$\Delta_n^{(j)} := \max_{0\le i\le \Phi^{(j)}(n)} (\Gamma_{i+1}^{(j)} - \Gamma_i^{(j)}),$$

and

$$\underline{L}_{n,m} := \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] - \sum_{j=1}^{m} \Delta_n^{(j)}.$$

Then $L_{n,m} \ge \underline{L}_{n,m}$.

Proof: We will show that, for all $\boldsymbol{t} = (t_0, \ldots, t_n) \in \mathcal{T}_{n,m}$, there is a path π in $G_{n,m}$ with length $|\pi|$ satisfying

$$\sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] \le |\pi| + \sum_{j=1}^{m} \Delta_n^{(j)}. \tag{4.3}$$

Fix $\boldsymbol{t} \in \mathcal{T}_{n,m}$ and use the notation

$$I_j = [Y^{(j)}(t_{j-1}),\ X^{(j)}(t_j)] = [a_j, b_j], \quad j = 1, \ldots, m.$$

Note that $a_j \ge b_j$ if there is one or no skeleton points on the segment $(t_{j-1}, t_j] \times \{j\}$ and then $L^{(j)}(I_j) = 0$.

Given two skeleton points (x, i), (y, j) we say that there is a *staircase path* from (x, i) to (y, j) if there is a sequence of skeleton points

$$(x, i) = (x_0, i),\ (x_1, i + 1), \ldots,\ (x_{j-i}, j) = (y, j),$$

such that $x = x_0 \le x_1 \le \cdots \le x_{j-i} = y$. See Figure 4.1. Clearly then, there is a path from (x, i) to (y, j) which jumps upwards by one step each time it meets a new skeleton point from the sequence. We denote this by

$$(x, i) \overset{s}{\rightsquigarrow} (y, j).$$

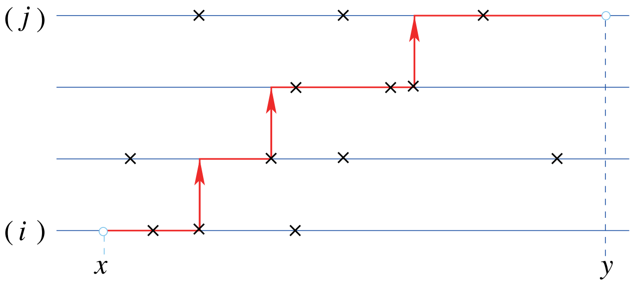

FIGURE 4.1. A staircase path from (x, i) to (y, j) jumps upwards at skeleton points (denoted by $\times$) but may skip several of them before deciding to make a jump

Among all the staircase paths from (x, i) to (y, j), we will consider the *best one*, defined by two properties:

- Property 1: A best path from (x, i) to (y, j) jumps from line k to line $k+1$, $k = i, i+1, \ldots, j-1$, at the first next skeleton point on line k, i.e. at the points x_0 and $x_{k-i+1} = Y^{(k+1)}(x_{k-i})$, $k = i, \ldots, j-2$.
- Property 2: Every horizontal segment of a best path is a path of maximal length.

If all the intervals $I_1, \ldots, I_m$ are empty, the left-hand side of (4.3) is zero and the inequality is trivially satisfied for any path π.

Otherwise, for a fixed $\boldsymbol{t} \in G_{n,m}$ we will construct a path π in $G_{n,m}$ for which (4.3) holds. Define a subsequence $\nu_1 < \nu_2 < \cdots$ of $1, \ldots, m$, inductively, as follows:

$$\nu_1 := \inf\{1 \le j \le m : \ I_j \ne \varnothing\}, \tag{4.4}$$

$$\nu_r := \inf\{j > \nu_{r-1} : \ (b_{\nu_{r-1}}, \nu_{r-1}) \overset{s}{\rightsquigarrow} (b_j, j)\}, \quad r \ge 2. \tag{4.5}$$

See Figure 4.2 for an illustration. The procedure stops if one of the elements of the subsequence exceeds m or if the condition inside the infimum is not satisfied by a path in $G_{n,m}$.

Let J be the last index in the above defined sequence. Let π_1 be a path of maximum length from (a_{ν_1}, ν_1) to (b_{ν_1}, ν_1) and define, for $r = 2, 3, \ldots, J$, a path π_r as a best staircase path from $(b_{\nu_{r-1}}, \nu_{r-1})$ to (b_{ν_r}, ν_r). Note that, for each $r = 2, 3, \ldots, J$, $J \ge 2$, the end vertex of π_{r-1} is the start vertex of π_r. Therefore we can concatenate the paths $\pi_1, \ldots, \pi_J$ to obtain a path π. This path starts from (a_{ν_1}, ν_1) and ends at (b_{ν_J}, ν_J).

Let

$$\pi^{(j)} := \text{ the restriction of path } \pi \text{ on line } j$$

and $|\pi^{(j)}|$ its length on line j. Also, for $j \ge \nu_1$ denote

$$(c_j, j) := \text{the first vertex on line } j \text{ of path } \pi.$$

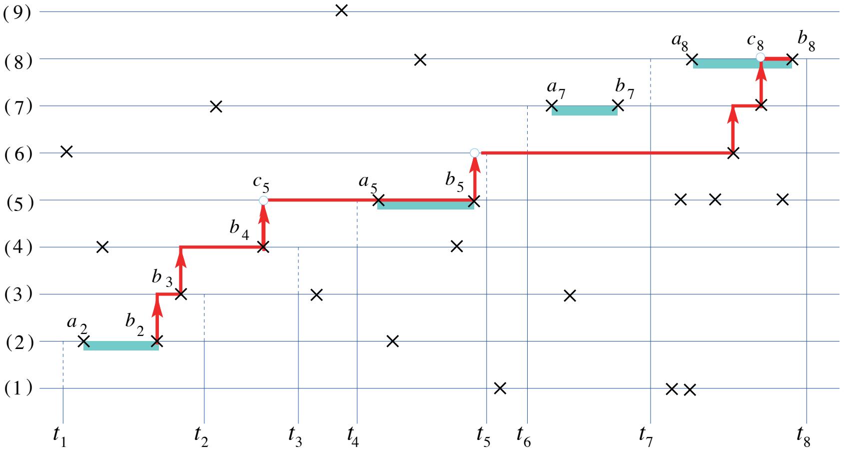

FIGURE 4.2. Illustration of the procedure defined by (4.4)-(4.5). There are four best staircase paths: the path from $(b_2, 2)$ to $(b_3, 3)$, the path from $(b_3, 3)$ to $(b_4, 4)$, the path from $(b_4, 4)$ to $(b_5, 5)$, and the path from $(b_5, 5)$ to $(b_8, 8)$. Observe that $I_j = (a_j, b_j)$, in the figure, are nonempty only for $j = 2, 5, 7$ and 8 (these are the highlighted intervals), but I_7 is not visited by the constructed path. Moreover, I_8 is only partly visited and the path enters I_8 at a point c_8 between b_8.

Split the sum in the left-hand side of (4.3) along the elements of the subsequence $\{\nu_1, \dots, \nu_J\}$:

$$\sum_{j=1}^{m} L^{(j)}(I_j) = \sum_{r=1}^{J+1} \sum_{j=\nu_{r-1}+1}^{\nu_r} L^{(j)}(I_j) =: \sum_{r=1}^{J+1} G_r,$$

where we have conveniently set

$$\nu_0 := 0,\ \nu_{J+1} := m,$$

in order to take care of the first and last terms. By the defintion of ν_1, the intervals $I_1, I_2, \dots, I_{\nu_1 - 1}$ are empty and

$$G_1 = L^{(\nu_1)}(I_{\nu_1}) = |\pi^{(\nu_1)}|.$$

Assume now that $2 \le r \le J$, and write

$$G_r = \sum_{j=\nu_{r-1}+1}^{\nu_r} L^{(j)}(I_j) = \sum_{j=\nu_{r-1}+1}^{\nu_r - 1} L^{(j)}(I_j) + L^{(\nu_r)}(I_{\nu_r}).$$

Since π_{ν_r} is the path of maximal length from (c_{ν_r}, ν_r) to its end-vertex (b_{ν_r}, ν_r) (Property 2), if $c_{\nu_r} < a_{\nu_r}$ then $L^{(\nu_r)}(I_{\nu_r}) \le |\pi^{(\nu_r)}|$. Define in this case $I'_{\nu_r} = \emptyset$. Otherwise, we can write

$$L^{(\nu_r)}(I_{\nu_r}) \le L^{(\nu_r)}[a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})] + L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}].$$

Then, again because of Property 2, $L^{(\nu_r)}[Y^{(\nu_r)}(c_{\nu_r}), b_{\nu_r}] < |\pi^{(\nu_r)}|$ and it is left to find a bound on the interval $I'_{\nu_r} = [a_{\nu_r}, Y^{(\nu_r)}(c_{\nu_r})]$. Recall that by Property

1, depending whether j is a member of the sequence $\{\nu_r, r = 1, \dots, J\}$ or not, $c_{j+1} = b_j$ or $c_{j+1} = Y^{(j)}(c_j)$, respectively. Also, because of $I_j \subseteq [t_{j-1}, t_j]$, we know that $L^{(j)}(I_j) \le t_j - t_{j-1}$. Hence, if $\nu_r - \nu_{r-1} > 1$ it holds

$$\begin{aligned}\sum_{j=\nu_{r-1}+1}^{\nu_r-1} L^{(j)}(I_j) + L^{(j)}(I'_{\nu_r}) &\le \sum_{j=\nu_{r-1}+1}^{\nu_r-1} (t_j - t_{j-1}) + (Y^{(\nu_r)}(c_{\nu_r}) - t_{\nu_r-1}) \\ &\le Y^{(\nu_r)}(c_{\nu_r}) - b_{\nu_{r-1}} = \sum_{j=\nu_{r-1}+1}^{\nu_r} (Y^{(j)}(c_j) - c_j) \le \sum_{j=\nu_{r-1}+1}^{\nu_r} \Delta_n^{(j)}.\end{aligned}$$

Combining the above, we obtain

$$G_r \le \sum_{j=\nu_r+1}^{\nu_r-1} \Delta_n^{(j)} + |\pi^{(\nu_r)}|.$$

If $\nu_J = m$, then $G_{J+1} = 0$. Otherwise, we can extend the sequence $\{c_j, j = \nu_1, \nu_1+1, \dots, \nu_J\}$ defining iteratively $c_{\nu_J+1} := b_{\nu_J}$ and $c_{j+1} := Y^{(j)}(c_j)$ until $c_j > n$ for some j. Let K be the last index such that $c_K \le n$. As there was not possible to construct the best staircase path after the line ν_J, K is at most m. Similarly as above, for G_{J+1} it holds

$$\begin{aligned}G_{J+1} &= \sum_{j=\nu_J+1}^{m} L^{(j)}(I_j) \le \sum_{j=\nu_J+1}^{m} (t_j - t_{j-1}) \le n - b_{\nu_J} \\ &= \sum_{j=\nu_J+1}^{K} (Y^{(j)}(c_j) - c_j) \le \sum_{j=\nu_J+1}^{K} \Delta_n^{(j)}.\end{aligned}$$

Finally, we obtain

$$\sum_{j=1}^{m} L^{(j)}(I_j) \le \sum_{j=1}^{m} \Delta_n^{(j)} + \sum_{r=1}^{J} |\pi^{(\nu_r)}| \le \sum_{j=1}^{m} \Delta_n^{(j)} + |\pi|,$$

as required. □

5. Further estimates in probability and Brownian directed percolation

In the present section we prove Theorem 3.1 as a sequence of lemmas.

5.1. *Asymptotic coincidence of the two bounds.* Looking at (1.3), we can see that the correct scaling requires exponent

$$b := \frac{1}{2} - \frac{a}{6}$$

in the denominator and condition $a < 3/7$, which is equivalent to $a < b$.

In the following two lemmas we will not specifically use the definiton of b and condition on a. Both lemmas hold for more general $a, b > 0$, $0 < b - a < 1$.

Lemma 5.1. *With $b = (1/2) - (a/6)$ and $a < 3/7$,*

$$\frac{\overline{L}_{n,[n^a]} - \underline{L}_{n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

Proof: Let $\boldsymbol{t}$ be such that the maximum in the right-hand side of (4.2) is achieved. Then

$$\begin{aligned}
\overline{L}_{n,m} - \underline{L}_{n,m} &\le m + \sum_{j=1}^{m} \Delta_n^{(j)} \\
&\quad + \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] - \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)] \\
&\le m + \sum_{j=1}^{m} \Delta_n^{(j)} \\
&\quad + \sum_{j=1}^{m} \Big\{ L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_{j-1})] + L^{(j)}[X^{(j)}(t_j), Y^{(j)}(t_j)] \Big\} \\
&\le m + \sum_{j=1}^{m} \Delta_n^{(j)} + 2 \sum_{j=1}^{m} \max_{0 \le i \le \Phi^{(j)}(n)} L^{(j)}[\Gamma_i^{(j)}, \Gamma_{i+1}^{(j)}] \\
&\le m + 3 \sum_{j=1}^{m} \Delta_n^{(j)}.
\end{aligned}$$

Hence

$$\frac{1}{n^b} E[\overline{L}_{n,[n^a]} - \underline{L}_{n,[n^a]}] \le \frac{n^a}{n^b} + 3 \frac{n^a}{n^b} E[\Delta_n^{(1)}].$$

Since $b > a$ and the random variables $\{\Gamma_{i+1}^{(1)} - \Gamma_i^{(1)}, i \ge 1\}$ have a finite $1/(b-a)$-th moment, the converegence to 0 for the second term above follows by Lemma A.1. $\square$

5.2. *Centering.* We introduce the quantity

$$S_{n,m} := \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \sum_{j=1}^{m} \Big\{ L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - C[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \Big\}.$$

This should be "comparable" to $L_{m,n} - Cn$ when $m = [n^a]$. Indeed, we have:

Lemma 5.2. *With $b = (1/2) - (a/6)$, and $a < 3/7$,*

$$\frac{S_{n,[n^a]} - (L_{n,[n^a]} - Cn)}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

Proof: We begin by rewriting the numerator above as

$$\begin{aligned}
&S_{n,m} - (L_{n,m} - Cn) \\
&= \sup_{\boldsymbol{t} \in \mathcal{T}_{n,m}} \Big\{ \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] + Cn - C \sum_{j=1}^{m} [X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \Big\} \\
&\qquad - L_{n,m}.
\end{aligned}$$

Upon writing $n = \sum_{j=1}^{m}(t_j - t_{j-1})$, for any $\boldsymbol{t} \in \mathcal{T}_{n,m}$, we have

$$\begin{aligned}\left| n - \sum_{j=1}^{m}[X^{(j)}(t_j) - X^{(j)}(t_{j-1})] \right| &= \left| \sum_{j=1}^{m}[t_j - X^{(j)}(t_j)] - \sum_{j=1}^{m}[t_{j-1} - X^{(j)}(t_{j-1})] \right| \\ &\le 2\sum_{j=1}^{m}\Delta_n^{(j)}.\end{aligned}$$

Hence, on the one hand we have

$$\begin{aligned}S_{n,m} - (L_{n,m} - Cn) &\le \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} + 2C\sum_{j=1}^{m}\Delta_n^{(j)} \\ &\le \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), Y^{(j)}(t_j)] - L_{n,m} + 2C\sum_{j=1}^{m}\Delta_n^{(j)} \\ &\le \overline{L}_{n,m} - \underline{L}_{n,m} + 2C\sum_{j=1}^{m}\Delta_n^{(j)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}S_{n,m} - (L_{n,m} - Cn) &\ge \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[X^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} - 2C\sum_{j=1}^{m}\Delta_n^{(j)} \\ &\ge \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} L^{(j)}[Y^{(j)}(t_{j-1}), X^{(j)}(t_j)] - L_{n,m} - 2C\sum_{j=1}^{m}\Delta_n^{(j)} \\ &\ge \underline{L}_{n,m} - \overline{L}_{n,m} - 2C\sum_{j=1}^{m}\Delta_n^{(j)} \\ &= -(\overline{L}_{n,m} - \underline{L}_{n,m}) - 2C\sum_{j=1}^{m}\Delta_n^{(j)}.\end{aligned}$$

Therefore,

$$|S_{n,m} - (L_{n,m} - Cn)| \le \overline{L}_{n,m} - \underline{L}_{n,m} + 2C\sum_{j=1}^{m}\Delta_n^{(j)}.$$

Thus, for $m = [n^a]$, the result follows by applying Lemma A.1 and Lemma 5.1. □

Define now variance σ^2 as

$$\sigma^2 := \mathrm{var}(L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)}))$$

and observe that $\sigma^2 = \sigma_0^2/p$. We work with the quantity $\frac{1}{\sigma}S_{n,m}$, which can be rewritten as

$$\frac{1}{\sigma}S_{n,m} = \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} \sum_{k=\Phi^{(j)}(t_{j-1})+1}^{\Phi^{(j)}(t_j)} \chi_k^{(j)},$$

where

$$\chi_k^{(j)} := \frac{1}{\sigma}\big\{L^{(j)}[\Gamma_{k-1}^{(j)}, \Gamma_k^{(j)}] - C(\Gamma_k^{(j)} - \Gamma_{k-1}^{(j)})\big\}.$$

Note that the random variables $\{\chi_k^{(j)}\}_{k\geq 1, j\geq 1}$, indexed by both k and j, are independent and that $\{\chi_k^{(j)}\}_{k\geq 2, j\geq 1}$ are identically distributed with zero mean and unit variance. The fact that the $\{\chi_1^{(j)}\}_{j\geq 1}$ do not have the same distribution will not affect the result, so we will not separately take care of it.

5.3. *Coupling with Brownian motion.* The term $\frac{1}{\sigma}S_{n,m}$ resembles a centered last passage percolation path weight, except that random indices are involved. Therefore, we start using the idea of strong coupling with Brownian motions, analogously to the proof in Bodineau and Martin (2005). Let $B^{(1)}, B^{(2)}, \ldots$ be i.i.d. standard Brownian motions, and recall that

$$Z_{n,m} := \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} [B_{t_j}^{(j)} - B_{t_{j-1}}^{(j)}]. \tag{5.1}$$

Define the random walks $R^{(1)}, R^{(2)}, \ldots$ by

$$R_i^{(j)} = \sum_{k=1}^{i} \chi_k^{(j)}, \quad i = 0, 1, 2, \ldots, \tag{5.2}$$

with $R_0^{(j)} = 0$. With this notation, we have

$$\frac{1}{\sigma} S_{n,m} = \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} [R_{\Phi^{(j)}(t_j)}^{(j)} - R_{\Phi^{(j)}(t_{j-1})}^{(j)}]. \tag{5.3}$$

Taking into account (1.2) and Lemma 5.2, it is evident that to prove Theorem 3.1 it remains to show that

$$\frac{\sigma^{-1}S_{n,[n^a]} - \sqrt{\lambda} Z_{n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0,$$

or, using the scaling property of Brownian motion, that:

Lemma 5.3. *For all $a < 3/14$,*

$$\frac{\sigma^{-1}S_{n,[n^a]} - Z_{\lambda n,[n^a]}}{n^b} \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

To show that the random walks are close enough to the Brownian motion we use the following version of the Komlós-Major-Tusnády strong approximation result (Komlós et al., 1976, Thm. 4):

Theorem 5.4. *For any $0 < r < 1$, $n \in \mathbb{Z}_+$ and $x \in \left[c_1(\log n)^{1/r}, c_2(n\log n)^{1/2}\right]$, starting with a probability space supporting independent Brownian motions $B^{(j)}$, $j = 1, 2, \ldots$, we can jointly construct i.i.d. sequences $\chi^{(j)} = (\chi_1^{(j)}, \chi_2^{(j)}, \ldots)$, $j = 1, 2, \ldots$, with the correct distributions and, moreover, such that, with $R_i^{(j)}$ as in* (5.2) *above,*

$$P(\max_{1\leq i\leq n} |B_i^{(j)} - R_i^{(j)}| > x) \leq Cn\exp\{-\alpha x^r\} \text{ for all } j = 1, 2, \ldots,$$

where the constants C, c_1, c_2 are depending on α, r and distributions of $\chi_1^{(1)}$ and $\chi_2^{(1)}$.

In addition to this, in order to take care of the random indices appearing in (5.3), we need a convergence rate result for the counting processes $\{\Phi^{(j)}, j \geq 1\}$ which is proven in the appendix.

Proof of Lemma 5.3: From (5.3) and (5.1) we have

$$\begin{aligned}
|\sigma^{-1}S_{n,m} - Z_{\lambda n,m}| &\le \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} \Big\{ \big|(R^{(j)}_{\Phi^{(j)}(t_j)} - R^{(j)}_{\Phi^{(j)}(t_{j-1})}) - (B^{(j)}_{\lambda t_j} - B^{(j)}_{\lambda t_{j-1}})\big| \Big\} \\
&= \sup_{\boldsymbol{t}\in\mathcal{T}_{n,m}} \sum_{j=1}^{m} \Big\{ \big|R^{(j)}_{\Phi^{(j)}(t_j)} - B^{(j)}_{\Phi^{(j)}(t_j)}\big| + \big|R^{(j)}_{\Phi^{(j)}(t_{j-1})} - B^{(j)}_{\Phi^{(j)}(t_{j-1})}\big| \\
&\qquad + \big|B^{(j)}_{\Phi^{(j)}(t_j)} - B^{(j)}_{\lambda t_j}\big| + \big|B^{(j)}_{\Phi^{(j)}(t_{j-1})} - B^{(j)}_{\lambda t_{j-1}}\big| \Big\} \\
&\le 2\sum_{j=1}^{m} \Big\{ \max_{0\le i\le n} \big|R^{(j)}_i - B^{(j)}_i\big| + \sup_{0\le s\le n} \big|B^{(j)}_{\Phi^{(j)}(s)} - B^{(j)}_{\lambda s}\big| \Big\} \\
&=: 2\sum_{j=1}^{m} U^{(j)}_n + 2\sum_{j=1}^{m} V^{(j)}_n,
\end{aligned}$$

where

$$U^{(j)}_n := \max_{0\le i\le n} \big|R^{(j)}_i - B^{(j)}_i\big|, \qquad V^{(j)}_n := \sup_{0\le s\le n} \big|B^{(j)}_{\Phi^{(j)}(s)} - B^{(j)}_{\lambda s}\big|.$$

Therefore, it is enough to show that,

$$\frac{1}{n^b}\sum_{j=1}^{[n^a]} U^{(j)}_n \xrightarrow[n\to\infty]{(\mathrm{p})} 0 \quad \text{and} \quad \frac{1}{n^b}\sum_{j=1}^{[n^a]} V^{(j)}_n \xrightarrow[n\to\infty]{(\mathrm{p})} 0.$$

For the first convergence we will take into account the coupling estimate as in Theorem 5.4. That is, we will throughout assume that the random walks and Brownian motions have been constructed *jointly*. The second convergence will be established without this estimate, i.e., we will show that it is true, *regardless* of the joint construction of the Brownian motions and the random walks. This is because *the coupling we use is not detailed enough to give us information about the joint distribution of* $B^{(j)}$ *and the counting process* $\Phi^{(j)}$ (the latter is not a function of the random walks used in the coupling).

Proof of the first convergence. Let $\delta > 0$. We need to show that

$$P\big(\sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\big) \to 0 \quad \text{as } n\to\infty.$$

Let $\varepsilon < b - a$. Then

$$\begin{aligned}
P\big(\sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\big) \le P\big(&\max_{1\le j\le [n^a]} U^{(j)}_n \le n^\varepsilon, \sum_{j=1}^{[n^a]} U^{(j)}_n > \delta n^b\big) \\
&+ P\big(\max_{1\le j\le [n^a]} U^{(j)}_n > n^\varepsilon\big).
\end{aligned} \tag{5.4}$$

The first term from right-hand side is zero for large n. We are allowed, for n large enough, to estimate the second term using Theorem 5.4 for an arbitrary $r \in (0,1)$ and $x = n^\varepsilon$ as

$$P\big(\max_{1\le j\le [n^a]} U^{(j)}_n > n^\varepsilon\big) \le n^a P\big(\max_{1\le i\le n} |B^{(1)}_i - R^{(1)}_i| > n^\varepsilon\big) \le n^a C n \exp\{-\alpha n^{\varepsilon r}\} \to 0,$$

as $n \to \infty$.

Proof of the second convergence. Let $1/4 < \varepsilon < b - a$. Replacing $U_n^{(j)}$ by $V_n^{(j)}$ in (5.4), we see that the first term is again zero for large n and it remains to show that second term converge to 0, i.e., it is enough to show the convergence for its upper-bound

$$n^a P\big(V_n^{(1)} > n^\varepsilon\big) \to 0,$$

as $n \to \infty$. Let $\gamma > 1$ and $1/2 < q < 2\varepsilon$. Then we can write

$$n^a P\big(V_n^{(1)} > n^\varepsilon\big) \le n^a P\big(\sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q\big) \\ + n^a P\big(\sup_{0\le s\le n} |B^{(1)}_{\Phi^{(1)}(s)} - B^{(1)}_{\lambda s}| > n^\varepsilon, \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big).$$

Corollary A.1 implies the convergence of the first term above

$$n^a P\big(\sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| > \gamma n^q\big) \le n^a \exp\{-\alpha(2n)^{qr}\} \to 0,$$

as $n \to \infty$, where $0 < r < 2 - 1/q$. Set $\varphi = \gamma n^q/\lambda$. For the second term using the fact that, under our condition, $\lambda s - \gamma n^q \le \Phi^{(1)}(s) \le \lambda s + \gamma n^q$ for $s \in [0, 2n]$, we have

$$\begin{aligned}
&n^a P\big(\sup_{0\le s\le n} |B^{(1)}_{\Phi^{(1)}(s)} - B^{(1)}_{\lambda s}| > n^\varepsilon, \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big)\\
&\le n^a P\big(\max_{0\le k\le \lfloor n/\varphi\rfloor} \sup_{0\le s\le\varphi} |B^{(1)}_{\Phi^{(1)}(k\varphi+s)} - B^{(1)}_{\lambda(k\varphi+s)}| > n^\varepsilon,\\
&\qquad\qquad \sup_{0\le s\le 2n} |\Phi^{(1)}(s) - \lambda s| \le \gamma n^q\big)\\
&\le n^a P\big(\max_{0\le k\le \lfloor n/\varphi\rfloor} \sup_{\substack{0\le s\le\varphi\\ -\gamma n^q\le t\le \gamma n^q}} |B^{(1)}_{\lambda(k\varphi+s)+t} - B^{(1)}_{\lambda(k\varphi+s)}| > n^\varepsilon\big)\\
&\le n^a(n/\varphi + 1) P\big(\sup_{0\le s\le t\le 3\gamma n^q} |B_t - B_s| > n^\varepsilon\big)\\
&\le n^a(n/\varphi + 1) P\big(\sup_{0\le s\le 3\gamma n^q} |B_s| > n^\varepsilon/2\big)\\
&\le 4n^a(n/\varphi + 1) P(B_{3\gamma n^q} > n^\varepsilon/2) \qquad \text{(a)}\\
&\le 4n^a(n/\varphi + 1) \exp\{-n^{2\varepsilon}/(24\gamma n^q)\} \to 0, \qquad \text{(b)}
\end{aligned}$$

as $n \to \infty$. The inequality (a) is the consequence of $P(\sup_{0<s<t} B_s > x) = 2P(B_t > x)$ and the inequality (b) is an estimate for the tail of the normal distribution. □

Remark 5.5. The condition $a < 3/14$ is equivalent to $b - a > 1/4$ and, thus, it ensures existence of an $\varepsilon > 1/4$ and later existance of $1/2 < q < 2\varepsilon$. Therefore, it is necessary for application of Corollary A.1, as well as for the convergence of the last estimate above.

6. Graph with non-constant edge probabilities

In Denisov et al. (2012), the authors consider a one-dimensional model with the connectivity probability depending on the distance between points, i.e. there is an edge between vertices i and j with probability $p_{|i-j|}$. To prove a central limit theorem for the maximal path length in that graph, two conditions are introduced:

- $0 < p_1 < 1$;
- $\sum_{k=1}^\infty k(1-p_1)\cdots(1-p_k) < \infty$.

The conditions guarantee the existence of skeleton points and finite variance, defined as in (2.3).

We can extend the idea of the graph on $\mathbb{Z} \times \mathbb{Z}$ keeping, whenever possible, the notation from the $\mathbb{Z} \times \mathbb{Z}$ graph with constant probability p.

Let $\{p_{i,j}, i, j \le 0\}$ be a sequence of probabilities that satisfies the following:

- $0 < p_{1,0} < 1$;
- $0 < p_{0,1} < 1$;
- $\sum_{k=1}^{\infty} k^{r-1}(1-p_{1,0})(1-p_{2,0}) \cdots (1-p_{k,0}) < \infty$ for some $r > 2$.

Then the vertices (i_1, i_2) and (j_1, j_2), $(i_1, i_2) \prec (j_1, j_2)$, are connected with probability $p_{j_1-i_1, j_2-i_2}$.

The conditions above are needed to ensure the existence of skeleton points, as in Definition 4.1, and a finite rth moment of the random variables $\chi_2, \chi_3, \ldots$ for some $r > 2$.

From now on, let r denote the order of the highest finite moment, i.e.

$$r = \sup\{q > 0 : \sum_{k=1}^{\infty} k^{q-1}(1-p_{1,0})(1-p_{2,0}) \cdots (1-p_{k,0}) < \infty\}.$$

Now we can state the analogue of Theorem 3.1 for this more general setting:

Theorem 6.1. *For all $a < \min(3/14, (r-2)/(3r/7+1))$,*

$$n^{a/6}\left(\frac{L_{n,[n^a]} - Cn}{\sqrt{\lambda\sigma^2}\sqrt{n}} - 2\sqrt{n^a}\right) \xrightarrow[n\to\infty]{\text{(d)}} F_{TW}.$$

Proof: We follow the lines of the proof of Theorem 3.1, emphasizing the points one should be careful about or which need to be modified.

The construction of the upper and lower bounds for $L_{n,[n^a]}$ is the same as in Section 4. In Lemma 5.1 and Lemma 5.2, the term $n^a E[\Delta_1]/n^b$ converges to 0 if $a + 1/r < b$, which leads to the constraint $a < 6/7(1/2 - 1/r)$.

Next, we rewrite, as in Lemma 5.3,

$$|\sigma^{-1} S_{n,m} - Z_{\lambda n,m}| \le 2\sum_{j=1}^{m} U_n^{(j)} + 2\sum_{j=1}^{m} V_n^{(j)}$$

and prove convergence of each term separately.

Proof of the first convergence. Instead of Theorem 5.4, we use the combination of results from Komlós et al. (1976) and Major (1976), as stated in Proposition 2 of Bodineau and Martin (2005), which allow us to couple Brownian motions $B^{(j)}$, $j = 1, 2, \ldots$ and random walks $R^{(j)}$, $j = 1, 2, \ldots$ so that

$$P\big(\max_{1\le i\le n} |B_i^{(j)} - R_i^{(j)}| > x\big) \le Cnx^{-r}, \text{for all } n \in \mathbb{Z}_+, \text{ and all } x \in [n^{1/r}, n^{1/2}]. \tag{6.1}$$

Let $\varepsilon = 1/2$. We want to establish convergence of the same terms as in (5.4). Applying, on the first term, the Markov inequality and properties of the coupling with Brownian motion (6.1) yield:

$$
\begin{aligned}
&P\Big(\max_{1\le j\le [n^a]} U_n^{(j)} \le n^{1/2}, \sum_{j=1}^{[n^a]} U_n^{(j)} > \delta n^b\Big) \le \frac{n^a}{\delta n^b} E\big[U_n^{(1)}\mathbf{1}(U_n^{(1)} \le n^{1/2})\big] \\
&\le \frac{n^a}{\delta n^b}\left(n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} P(U_n^{(1)} > x)dx\right) \le \frac{n^a}{\delta n^b}\left(n^{1/r} + \int_{n^{1/r}}^{n^{1/2}} Cnx^{-r}dx\right) \\
&= \frac{n^a}{\delta n^b}\left(n^{1/r} - \frac{C}{r-1}n^{3/2-r/2} + \frac{C}{r-1}n^{1/r}\right) \le \left(1+\frac{C}{r-1}\right)\frac{n^a n^{1/r}}{\delta n^b} \to 0
\end{aligned}
$$

as $n \to \infty$. For the second term we use again the coupling properties (6.1):

$$
P\Big(\max_{1\le j\le [n^a]} U_n^{(j)} > n^{1/2}\Big) \le n^a P\Big(\max_{1\le i\le n} |B_i^{(1)} - R_i^{(1)}| > n^{\varepsilon}\Big) \le n^a C n n^{-r/2} \to 0,
$$

as $n \to \infty$ because $a + 1 - r/2 < 0$ due to our new constraint on a.

Proof of the second convergence. Here, the only change is the replacement of Lemma A.2 by part of Theorem 6.12.1 in Gut (2013), in our notation:

If $1/2 \le q < 1$, then for all $\varepsilon > 0$ it holds that $n^{qr-2}P(\max_{0\le k\le n}|\Gamma_k - \lambda k| > \varepsilon n^q) \to 0$ as $n \to \infty$.

One can, in the same fashion as before, prove the following analogue of Corollary A.1,

$$
n^{qr-2}P\Big(\sup_{0\le t\le n}|\Phi(t) - \lambda t| > \varepsilon n^q\Big) \to 0 \quad \text{as} \quad n \to \infty.
$$

Likewise in Remark 5.5, condition $a < 3/14$ is necessary. Another constraint that occurs is $a \le qr - 2$ and it can be shown that this is satisfied if $a \le (r-2)/(3r/7+1)$. □

Appendix A.

Lemma A.1. Let $r \ge 1$ and $\{X_i, i \ge 1\}$ be a sequence of nonnegative i.i.d. random variables such that $E|X_1|^r < \infty$. Then,

$$
\frac{1}{n^{\frac{1}{r}}} E\left[\max_{1\le i\le n} X_i\right] \to 0 \text{ as } n \to \infty.
$$

Proof: It suffices to prove the lemma for $r = 1$, since then the result for $r > 1$ follows easily by Jensen's inequality. Let $M_n = \max_{1\le i\le n} X_i$ and $S_n = X_1 + X_2 + \cdots + X_n$. Borel-Cantelli lemma, together with the assumption $EX < \infty$, give $X_n/n \to 0$, almost surely, as $n \to \infty$. An easy computation shows that then also $M_n/n \to 0$, almost surely, as $n \to \infty$.

On the other hand, we know that $M_n \le S_n$ for all n and that $\{S_n/n, n \ge 1\}$ is uniformly integrable. Therefore, M_n/n is also unformly integrable and $EM_n/n \to 0$ as $n \to \infty$ (Gut, 2013, Thm. 5.4.5, Thm. 5.5.2). □

The following lemma is an analogue of a result by Lanzinger (1998, Prop. 2), with slight improvement in the probability bound. Moreover, we specialize the lemma to our case.

Lemma A.2. Let $1/2 < q \le 1$ and $r < 2 - 1/q$. Suppose that $X, X_1, X_2, \ldots$ are i.i.d. random variables such that $EX = \mu$ and $Ee^{\alpha|X|} < \infty$ for some $\alpha > 0$, and set $S_n = \sum_{i=1}^n X_k$. Then, for all $\varepsilon > 1$,

$$
\exp\{\alpha n^{qr}\}P\Big(\max_{1\le k\le n}|S_k - k\mu| > \varepsilon n^q\Big) \to 0
$$

as $n \to \infty$.

Proof: Without loss of generality we can assume that $\mu = 0$. We first show that, for all $\varepsilon > 1$, $\exp\{\alpha n^{qr}\}P(S_n > \varepsilon n^q) \to 0$ as $n \to \infty$.

For fixed $n \in \mathbb{Z}_+$, define $X'_k = X_k \mathbf{1}(X_k \le \varepsilon n^q)$ and set $S'_n = \sum_{k=1}^n X'_k$. Note that $EX' \le 0$ and that for α', where $\alpha' < \alpha$ and $\alpha'\varepsilon^r > \alpha$, it holds that $EX^2 e^{\alpha'|X|^r} < \infty$. We can write

$$P(S_n > \varepsilon n^q) \le P(S'_n > \varepsilon n^q) + nP(X > \varepsilon n^q). \tag{A.1}$$

We first find a bound for the moment generating function for X' in $\alpha'(\varepsilon n^q)^{r-1}$:

$$\begin{aligned}
Ee^{\alpha'(\varepsilon n^q)^{r-1}X'} &\le 1 + \frac{1}{2}\alpha'^2(\varepsilon n^q)^{2(r-1)}(EX'^2 + EX'^2 e^{\alpha'(\varepsilon n^q)^{r-1}X'}\mathbf{1}\{X' > 0\}) \\
&\le 1 + \frac{1}{2}\alpha'^2(\varepsilon n^q)^{2(r-1)}(EX^2 + EX^2 e^{\alpha'|X|^r}) \\
&\le 1 + Cn^{2q(r-1)} \le \exp\{Cn^{-2q(1-r)}\},
\end{aligned}$$

where for the first inequality we used $e^y \le 1 + \max\{1, e^y\}y^2/2$ and for the last one $1 + y \le e^y$. Now, using Markov's inequality and the bound above for the first term in (A.1) yields

$$\begin{aligned}
P(S'_n > \varepsilon n^q) &= P(e^{\alpha'(\varepsilon n^q)^{r-1}S'_n} > e^{\alpha'(\varepsilon n^q)^r}) \le e^{-\alpha'(\varepsilon n^q)^r}\left(Ee^{\alpha'(\varepsilon n^q)^{r-1}X'}\right)^n \\
&\le \exp\{-\alpha'(\varepsilon n^q)^r + Cnn^{-2q(1-r)}\}.
\end{aligned}$$

For the second term in (A.1), again using Markov's inequality, we have

$$nP(X > \varepsilon n^q) = nP(e^{\alpha X^r} > e^{\alpha(\varepsilon n^q)^r}) \le ne^{-\alpha(\varepsilon n^q)^r}E(e^{\alpha|X|^r}).$$

Combining the two estimates finally establishes that

$$\begin{aligned}
\exp\{\alpha n^{qr}\}P(S_n > \varepsilon n^q) &\le \exp\{-n^{qr}(\alpha'\varepsilon^r - \alpha - Cn^{1-2q+qr})\} \\
&\quad + n\exp\{-n^{qr}\alpha(\varepsilon^r - 1)\}Ee^{\alpha|X|^r} \to 0
\end{aligned}$$

as $n \to \infty$ because of the choice of r, α' and ε.

By symmetry, the same convergence rate holds also for $P(S_n < -\varepsilon n^q)$.

Thus, the statement of the theorem follows using Lévy inequality (Gut, 2013, Thm. 3.7.2) and the fact that for n large enough we can find ε' such that $1 < \varepsilon' < \varepsilon - \sqrt{2\sigma^2}n^{1/2-q}$:

$$\begin{aligned}
P(\max_{1\le k\le n}|S_k| > \varepsilon n^q) &\le 2P(|S_n| > \varepsilon n^q - \sqrt{2n\sigma^2}) = 2P(|S_n| > n^q(\varepsilon - \sqrt{2\sigma^2}n^{1/2-q})) \\
&\le 2P(|S_n| > \varepsilon' n^q).
\end{aligned}$$

□

Corollary A.1. Let $X, X_1, X_2, \dots$ be positive, integer-valued i.i.d. random variables and suppose that the assumptions of Lemma A.2 are satisfied. Then, for the counting process Φ, where $\Phi(t) = \max\{n : S_n \le t\}$, it holds that

$$\exp\{\alpha n^{qr}\}P(\sup_{0\le t\le n}|\mu\Phi(t) - t| > \varepsilon n^q) \to 0 \text{ for all } \varepsilon > 1.$$

Proof: For the counting process we have $t - X_{\Phi(t)+1} \leq S_{\Phi(t)} \leq t$. Therefore, we can write

$$\begin{aligned}
&\{\sup_{0\leq t\leq n} |\mu\Phi(t) - t| > \varepsilon n^q\} = \\
&= \{\sup_{0\leq t\leq n} (\mu\Phi(t) - t) > \varepsilon n^q\} \cup \{\inf_{0\leq t\leq n} (\mu\Phi(t) - t) < -\varepsilon n^q\} \\
&\subset \{\sup_{0\leq t\leq n} (\mu\Phi(t) - S_{\Phi(t)}) > \varepsilon n^q\} \cup \{\inf_{0\leq t\leq n} (\mu\Phi(t) - S_{\Phi(t)} - X_{\Phi(t)+1}) < -\varepsilon n^q\} \\
&\subset \{\sup_{0\leq t\leq n} (\mu\Phi(t) - S_{\Phi(t)}) > \varepsilon n^q\} \cup \{\inf_{0\leq t\leq n} (\mu\Phi(t) - S_{\Phi(t)}) < -\varepsilon' n^q\} \\
&\quad \cup \{\sup_{0\leq t\leq n} X_{\Phi(t)+1} > (\varepsilon - \varepsilon')n^q\} \\
&\subset \{\max_{1\leq k\leq n} |S_k - k\mu| > \varepsilon n^q\} \cup \{\max_{1\leq k\leq n} |S_k - k\mu| > \varepsilon' n^q\} \\
&\quad \cup \{\max_{1\leq k\leq n+1} X_k > (\varepsilon - \varepsilon')n^q\},
\end{aligned}$$

where $1 < \varepsilon' < \varepsilon$. Thus,

$$\begin{aligned}
&\exp\{\alpha n^{qr}\} P(\sup_{0\leq t\leq n} |\mu\Phi(t) - t| > \varepsilon n^q) \leq \exp\{\alpha n^{qr}\} P(\max_{1\leq k\leq n} |S_k - k\mu| > \varepsilon n^q) \\
&+ \exp\{\alpha n^{qr}\} P(\max_{1\leq k\leq n} |S_k - k\mu| > \varepsilon' n^q) + (n+1)\exp\{\alpha n^{qr}\} P(X > (\varepsilon - \varepsilon')n^q).
\end{aligned}$$

The first and the second term above converge to 0 as $n \to 0$ by Lemma A.2. We prove convergence to 0 for the third term using Markov's inequality,

$$(n+1)e^{\alpha n^{qr}} P(X > (\varepsilon - \varepsilon')n^q) \leq (n+1)e^{\alpha(n^{qr} - (\varepsilon-\varepsilon')n^q)} E e^{\alpha X} \to 0 \text{ as } n \to \infty. \qquad \square$$

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CONVERGENCE OF DIRECTED RANDOM GRAPHS TO THE POISSON-WEIGHTED INFINITE TREE

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Abstract

We consider a directed graph on the integers with a directed edge from vertex i to j present with probability n^{-1}, whenever $i < j$, independently of all other edges. Moreover, to each edge (i, j) we assign weight $n^{-1}(j-i)$. We show that the closure of vertex 0 in such a weighted random graph converges in distribution to the Poisson-weighted infinite tree as $n \to \infty$. In addition, we derive limit theorems for the length of the longest path in the subgraph of the Poisson-weighted infinite tree which has all vertices at weighted distance of at most ρ from the root.

Keywords: Directed random graph; Poisson-weighted infinite tree; rooted geometric graph

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1. Introduction and main results

In this paper we consider a directed version of the standard Erdős–Rényi random graph [7] on n vertices, defined as follows. For each pair $\{i, j\}$ of distinct positive integers, $i, j \leq n$, toss a coin with probability of heads equal to p, $0 < p < 1$, independently from pair to pair; if heads shows up then introduce an edge directed from $\min(i, j)$ to $\max(i, j)$. A path in a directed graph is a sequence of increasing vertices which are successively connected and the length of a path is the number of edges in the path. The longest path in the graph is a path with maximal length. The length of the longest path in a directed random graph, sometimes also called the height of a graph, was the subject in a number of papers, see [9] and [12]–[15]. Newman [15] and Itoh and Krapivsky [14] showed that the length of the longest path starting from vertex 1 in a directed random graph on n vertices with edge probability n^{-1}, rescaled by n^{-1}, converges to the constant e. In both papers, the authors also describe the limiting object of the closure of vertex 1. This limiting object can be found in the literature under the name Poisson-weighted infinite tree.

Let $U = \{\varnothing\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k$ and, if $u, v \in U$, denote by uv the concatenated element and $\varnothing u = u \varnothing = u$. The Ulam–Harris tree is the infinite rooted tree with vertex set U, the root in the vertex with label $\varnothing$, and an edge joining vertices u and uj for any $u \in U$ and $j \in \mathbb{N}$. This, in the language of trees, means that if $u \in U$ then the children of u are $u1, u2, u3, \ldots$.

The *Poisson-weighted infinite tree* (PWIT) is the Ulam–Harris tree with weight function w on edges defined as follows. Assign to each vertex $u \in U$ an independent realization of a Poisson process with rate 1. Let the arrival times of the Poisson process assigned to vertex u be $\{\xi_j^u, j \in \mathbb{N}\}$. Then the weight function w of the edges from the vertex u is given by $w(u, uj) = \xi_j^u$, $j \in \mathbb{N}$.

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In their survey, Aldous and Steele [4] defined rooted geometric graphs and their convergence, a framework which we use here in order to prove rigorously that the limit graph of directed random graphs is the PWIT.

Let $G = (V, E)$ be a connected graph with a finite or countably infinite vertex set V, edge set E, and associated function $w\colon E \to (0, \infty]$ for weights of the edges of G. Moreover, for any pair of vertices u and v, the distance between u and v is defined as the infimum over all paths from u to v of the sum of the weights of the edges in the path. The graph G is called a *geometric graph* if the weight function makes G locally finite in the sense that for each vertex v and each $\rho > 0$ the number of vertices within distance ρ from v is finite. If, in addition, there is a distinguished vertex v, we say that G is a *rooted geometric graph* with root v. The set of rooted geometric graphs will be denoted by $\mathcal{G}_*$.

For any $\rho > 0$ and for any rooted geometric graph G denote by $N_\rho(G)$ the graph whose vertex set $V_\rho(G)$ is the set of vertices of G that are at a distance of at most ρ from the root and whose edge set consists of just those edges of G that have both vertices in $V_\rho(G)$. We say that ρ is a *continuity point* of G if no vertex of G is exactly at a distance ρ from the root of G.

A *graph isomorphism* between rooted geometric graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $\Phi\colon V \to V'$ such that $(\Phi(u), \Phi(v)) \in E'$ if and only if $(u, v) \in E$ and Φ maps the root of G to the root of G'. A *geometric isomorphism* between rooted geometric graphs G and G' is a graph isomorphism Φ between G and G' which preserves edge weights, that is, $w'(\Phi(e)) = w(e)$ for all $e \in E$ (where $\Phi(e)$ denotes $(\Phi(u), \Phi(v))$ for $e = (u, v)$).

We say that G_n *converges* (locally) to G in $\mathcal{G}_*$ if for each continuity point ρ of G there is an $n_0 = n_0(\rho, G)$ such that for all $n \geq n_0$ there exists a graph isomorphism $\Phi_{\rho,n}$ from the rooted geometric graph $N_\rho(G)$ to the rooted geometric graph $N_\rho(G_n)$ such that for each edge e of $N_\rho(G)$ the weight of $\Phi_{\rho,n}(e)$ converges to the weight of e.

In Appendix A we propose a distance function d which makes $\mathcal{G}_*$ into a complete separable metric space and for which the notion of convergence in the metric space is equivalent with the definition of convergence above. In this setting we can use the usual definition of convergence in distribution, see, e.g. [8].

Consider now an extension of a directed random graph to the infinite, countable vertex set with labels $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and with edge probability n^{-1}. As before, any two vertices are connected with an edge with probability n^{-1}, independently of the other edges, and an edge is always directed from the vertex with the smaller label to the vertex with the larger label. We define the weight of an edge in this graph as the absolute difference between its labels rescaled by n^{-1}, that is, if $e = (i, j)$ is an edge of the graph then its weight is $w_n(e) = n^{-1}|j - i|$. The *closure of vertex* 0 in a directed random graph is its subgraph consisting of all vertices which are connected with vertex 0 through a path (including vertex 0) and all the edges between these vertices. We call vertex 0 the root of the closure of vertex 0.

Our main result is the following theorem.

Theorem 1. *Let T_n, $n \in \mathbb{N}$, be the closure of vertex* 0 *in a directed random graph on $\mathbb{N}_0$, with edge probability n^{-1} and weight function w_n. Then T_n converges in distribution to a Poisson-weighted infinite tree as $n \to \infty$.*

We show convergence in distribution of the closure of vertex 0 of the directed random graphs on an infinite vertex set to the PWIT, instead for their finite versions. A reason is that directed random graphs on n vertices with weight function w_n have all vertices at the distance at most 1 from the root and their weak limit is a subgraph of a PWIT T, $N_1(T)$. To obtain a PWIT in the limit, one could consider finite directed random graphs but then the number of vertices need to grow faster than linearly.

The PWIT appears also as a weak limit of a randomly rooted complete graph on n vertices with independent and exponentially distributed edge weights with mean n. Aldous [3] proved this limit theorem and used it in a study of a random assignment problem with n jobs, n machines, and the costs of performing a job on each machine being independent and exponentially distributed with mean n.

The PWIT was also studied in the domain of branching random walks [1]. A branching random walk is an extension of the Galton–Watson branching process obtained by associating to each individual of the process a real-valued random variable. In a tree generated by the branching process, the displacement of a particle is defined as the sum of random variables associated to the particle's ancestors, from the root to the particle itself. An object of interest in a branching random walk is the minimal displacement of particles in the nth generation, denoted by M_n.

Let L_x be the length of the longest path in the PWIT, between the root vertex and all the vertices at a distance of at most x from the root. Itoh and Krapivsky [14] gave arguments supporting the assertion that the distribution of L_x attains travelling wave behaviour for large x. It is assumed that the convergence to a travelling wave is a general phenomenon for the minimal displacement in branching random walks, but only a few examples exist where it is explicitly proven [6], [11]. A consequence of the travelling wave behaviour in the limit is tightness of the minimal displacements, that is, the tightness of the family $\{M_n - \text{median}(M_n), n \geq 1\}$, which was also separately studied in [10]. A stronger result than tightness is the existence of an exponential upper bound for the deviation probability for $M_n - \text{median}(M_n)$, which was obtained in [1], [2], and [11].

In this paper we obtain the asymptotic expression for the median of the length of the longest path L_x in the PWIT and show that the distribution of L_x has uniform exponential tails.

Theorem 2. *The median of the length of the longest path of the PWIT, for $x \geq 1$, is*

$$\text{median}(L_x) = x\mathrm{e} - \tfrac{3}{2}\log x + O(1).$$

Theorem 3. *For the length of the longest path of the PWIT, for all $x \geq 1$, $y \geq 0$, and any $c_1 < 1/\mathrm{e}$, there exists a constant C_1, dependent only on c_1, so that*

$$\mathbb{P}(L_x \leq \text{median}(L_x) - y) \leq C_1 \mathrm{e}^{-c_1 y},$$

and for any $c_2 < 1$ there exists a constant C_2, dependent only on c_2, so that

$$\mathbb{P}(L_x \geq \text{median}(L_x) + y) \leq C_2 \mathrm{e}^{-c_2 y}.$$

In particular,

$$\mathbb{E}L_x = x\mathrm{e} - \tfrac{3}{2}\log x + O(1) \quad \textit{and} \quad \text{var}(L_x) = O(1).$$

The paper is organized as follows. In Section 2 we show that the studied random graphs are almost surely rooted geometric graphs. Theorem 1 is proven in Section 3 and the proofs of Theorem 2 and Theorem 3 are given in Section 4. In Section 3 we use a definition of distance between two rooted geometric graphs, which we study in more detail in Appendix A.

2. Rooted geometric graphs

The closure of vertex 0 in a directed random graph on vertex set $\mathbb{N}_0$ with edge probability n^{-1} is a graph with the root at vertex 0 and a countably infinite vertex set. The weight function w_n makes the graph locally finite as, for any $\rho > 0$, the number of vertices within the distance ρ from a vertex v of the graph is at most $2\lfloor \rho n \rfloor + 1$.

The PWIT is also a rooted graph with a countably infinite vertex set. To show that the PWIT is almost surely a rooted geometric graph, it is left to prove that is almost surely locally finite.

We start by a connection between the PWIT and the Yule process. The *Yule process* $\{Y(t),\ t \geq 0\}$ is a branching process which starts at time 0 with one particle, $Y(0) = 1$, and each particle, independently of all others and of the past of the process, after an exponential waiting time with mean 1 splits into two particles. The number of particles present at time t is $Y(t)$.

Lemma 1. *Let $\{Y(t),\ t \geq 0\}$ be the Yule process and let T be the PWIT. Let, for $t > 0$, $|V_t(T)|$ be the number of vertices of the graph T at a distance of at most t from the root. Then*

$$\{Y(t), t \geq 0\} \stackrel{\mathrm{D}}{=} \{|V_t(T)|,\ t \geq 0\},$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution.

Proof. For $t = 0$, $Y(0) = 1$, $V_0(T) = \varnothing$, and $|V_0(T)| = 1$. Because of the memoryless property of the exponential distribution, at any time $t \geq 0$ the time left until the next splitting occurs in the Yule process is the minimum of $Y(t)$ exponential random variables with mean 1, that is, exponential random variable with mean $Y(t)^{-1}$. Similarly for the PWIT, at time $t \geq 0$ there are $|V_t(T)|$ vertices and each vertex is assigned a Poisson process which defines the weights of the edges. The closest vertex to the root that is outside $V_t(T)$ is determined by the position of the first point after t in these $|V_t(T)|$ Poisson processes. Thus, the first vertex with distance from the root greater than t is at the distance $t + X$ from the root, where X is exponentially distributed random variable with mean $|V_t(T)|^{-1}$. $\square$

Proposition 1. *The PWIT is almost surely a locally finite graph.*

Proof. Let T be a PWIT and let $\{Y(t), t \geq 0\}$ be the Yule process. For the Yule process is $\mathbb{E}(Y(t)) = \mathrm{e}^t$ [5, Chapter III.4] and, thus, by Lemma 1, we have $\mathbb{E}(|V_t(T)|) = \mathrm{e}^t$. Hence, the number of vertices within the distance t from the root is, almost surely, finite.

Let v be a vertex of T and denote by r the distance between v and the root. Note that $r < \infty$. Then the set of vertices that are within distance ρ from vertex v is a subset of $V_{r+\rho}(T)$. Therefore, there exists almost surely finitely many such vertices and the PWIT is almost surely a locally finite graph. $\square$

3. Convergence to the PWIT

In this section we prove Theorem 1. The first step is to show that for large enough n, with high probability, the directed graph on the segment $[0, \rho]$ is a tree, which suggests that the limiting object should also be a tree.

Lemma 2. *Let T_n, $n \in \mathbb{N}$, be the closure of vertex 0 in a directed random graph on $\mathbb{N}_0$, with edge probability n^{-1} and weight function w_n. For $\rho > 0$,*

$$\mathbb{P}(N_\rho(T_n) \textit{ is a tree}) \to 1 \quad \textit{as } n \to \infty.$$

Proof. Note first that for $\rho > 0$ it holds that $|V_\rho(T_n)| \le \lfloor n\rho \rfloor + 1$. For vertices i and j, $i < j$, we denote by $i \rightsquigarrow j$ the event that there exists a path from vertex i to vertex j and by $i \overset{2}{\rightsquigarrow} j$ the event that there exist two nonintersecting paths from vertex i to j, where by nonintersecting we mean that the paths do not have any common vertices except i and j. Define now for $n \in \mathbb{N}$ and $\rho > 0$ the events

$$A^{\rho,n} = \{N_\rho(T_n) \text{ is not a tree}\}$$

and for $i, j \in \mathbb{N}_0$, $i < j$,

$$A^{\rho,n}_{i,j} = \{i, j \in V_\rho(T_n),\ i \overset{2}{\rightsquigarrow} j\}.$$

Obviously, when $j = i + 1$ the events $A^{\rho,n}_{i,j}$ are empty. If the graph $N_\rho(T_n)$ is not a tree, there exist vertices $i_1, i_2, j \in V_\rho(T_n)$, $i_1, i_2 < j$, such that (i_1, j) and (i_2, j) are edges of $N_\rho(T_n)$. Then we can find a vertex $i \in V_\rho(T_n)$, $i \le \min\{i_1, i_2\}$, such that i and j are connected with two nonintersecting paths. Therefore, we can write

$$\mathbb{P}(A^{\rho,n}) \le \sum_{0 \le i < j \le \lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{i,j}) = \sum_{j=2}^{\lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{0,j}) + \sum_{i=1}^{\lfloor n\rho \rfloor - 2} \sum_{j=i+2}^{\lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{i,j}). \tag{1}$$

For $i \ge 1,\ i + 2 \le j \le \lfloor n\rho \rfloor$, we have

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}_{i,j}) &= \mathbb{P}(0 \rightsquigarrow i)\mathbb{P}(i \overset{2}{\rightsquigarrow} j) \\
&\le \sum_{k=0}^{i-1} \mathbb{P}((0, v_1, v_2, \dots, v_k, i) \text{ is a path in } N_\rho(T_n)) \\
&\quad \times \sum_{\ell=1}^{j-i-1} \mathbb{P}((i, u_1, u_2, \dots, u_{\ell_1}, j) \text{ and } (i, u'_1, u'_2, \dots, u'_{\ell_2}, j) \text{ are} \\
&\qquad\qquad \text{nonintersecting paths in } N_\rho(T_n),\ \ell_1 + \ell_2 = \ell) \\
&= \sum_{k=0}^{i-1} \binom{i-1}{k} p^{k+1} \sum_{\ell=1}^{j-i-1} \binom{j-i-1}{\ell} 2^{\ell-1} p^{\ell+2} \\
&< \frac{p^3}{2}(1+p)^{i-1}(1+2p)^{j-i-1}.
\end{aligned}$$

Similarly, for $i = 0,\ 2 \le j \le \lfloor n\rho \rfloor$,

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}_{0,j}) &= \mathbb{P}(0 \overset{2}{\rightsquigarrow} j) \\
&\le \sum_{\ell=1}^{j-i-1} \mathbb{P}((0, u_1, u_2, \dots, u_{\ell_1}, j) \text{ and } (0, u'_1, u'_2, \dots, u'_{\ell_2}, j) \text{ are} \\
&\qquad\qquad \text{nonintersecting paths in } N_\rho(T_n),\ \ell_1 + \ell_2 = \ell) \\
&= \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} 2^{\ell-1} p^{\ell+2} \\
&< \frac{p^2}{2}(1+2p)^{j-1}.
\end{aligned}$$

Applying the above inequalities to (1) and replacing p with n^{-1}, we obtain

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}) &< \sum_{j=2}^{\lfloor n\rho\rfloor} \frac{p^2}{2}(1+2p)^{j-1} + \sum_{i=1}^{\lfloor n\rho\rfloor-2} \sum_{j=i+2}^{\lfloor n\rho\rfloor} \frac{p^3}{2}(1+p)^{i-1}(1+2p)^{j-i-1} \\
&< \frac{p^2}{2}\frac{(1+2p)^{\lfloor n\rho\rfloor}-(1+2p)}{2p} + \frac{p^3}{2}\sum_{i=1}^{\lfloor n\rho\rfloor-2}(1+p)^{i-1}\frac{(1+2p)^{\lfloor n\rho\rfloor-i}-(1+2p)}{2p} \\
&< \frac{p}{4}(1+2p)^{\lfloor n\rho\rfloor} \\
&\quad + \frac{p^2}{4}(1+2p)^{\lfloor n\rho\rfloor-1}\left\{\left(1-\left(\frac{1+p}{1+2p}\right)^{\lfloor n\rho\rfloor-2}\right)\left(1-\frac{1+p}{1+2p}\right)^{-1}\right\} \\
&< \frac{p}{2}(1+2p)^{\lfloor n\rho\rfloor} \\
&< \frac{1}{2n}\mathrm{e}^{2\rho},
\end{aligned}$$

which gives

$$\mathbb{P}(N_\rho(T_n) \text{ is a tree}) = 1 - \mathbb{P}(A^{\rho,n}) \geq 1 - \frac{1}{2n}\mathrm{e}^{2\rho} \to 1 \quad \text{as } n \to \infty. \qquad \square$$

In the proofs of Lemma 3 and Theorem 1 we use the following definition for a distance between two geometric graphs $G_1, G_2 \in \mathcal{G}_*$:

$$d(G_1, G_2) = \int_0^\infty \frac{R(N_\rho(G_1), N_\rho(G_2))}{\mathrm{e}^\rho}\,\mathrm{d}\rho,$$

where

$$R(N_\rho(G_1), N_\rho(G_2)) = \min\left\{1, \min_{\substack{\{\Phi\colon \Phi\colon V_\rho(G_1)\to V_\rho(G_2)\\ \text{graph isomorphism}\}}} \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e) - w(\Phi(e))|\right\};$$

see Appendix A for more details.

To show convergence in distribution of rooted geometric graphs $\{G_n, n \geq 1\}$ to G, we show one of the equivalent statements of the Portmanteau theorem [8, Theorem 2.1]:

$$\lim_{n\to\infty} \mathbb{P}(G_n \in A) = \mathbb{P}(G \in A)$$

for all Borel sets A whose boundary ∂A satisfies $\mathbb{P}(G \in \partial A) = 0$.

Furthermore, define the *Bernoulli-weighted infinite tree with parameter n^{-1}* (BWIT_n) to be the Ulam–Harris tree with weight function $\hat{w}_n$ on edges defined as follows. Let $\{X_j^u, u \in U, j \in \mathbb{N}\}$ be a sequence of independent and geometrically distributed random variables with law $\mathbb{P}(X_j^u = k) = n^{-1}(1-n^{-1})^{k-1}, k \in \mathbb{N}$. The weight function $\hat{w}_n$ is $\hat{w}_n(u, uj) = n^{-1}\sum_{k=1}^{j} X_k^u$.

Lemma 3. *Let $\hat{T}_n$, $n \in \mathbb{N}$, be a $BWIT_n$. Then the sequence $\hat{T}_n$ converges in distribution to a PWIT as $n \to \infty$.*

Proof. Let T and $\widetilde{T}_n$, $n \in \mathbb{N}$, be the Ulam–Harris trees with weight functions w and $\widetilde{w}_n$ on edges defined as follows. Let $\{Y_j^u, u \in U, j \in \mathbb{N}\}$ be independent and exponentially distributed random variables, $\mathbb{P}(Y_j^u \leq x) = 1 - \mathrm{e}^{-x}$, $x \geq 0$. For $n \in \mathbb{N}$, let $a_n = 1/\log(n/(n-1))$ and set $X^u_{n,j} = \lceil a_n Y^u_j \rceil$, $u \in U$, $j \in \mathbb{N}$. Then the random variables $\{X^u_{n,j}, u \in U, j \in \mathbb{N}\}$ have geometric distribution with parameter n^{-1}. The weight functions of the trees T and $\widetilde{T}_n$ are

$$w(u, uj) = \sum_{k=1}^{j} Y^u_k \quad \text{and} \quad \widetilde{w}_n(u, uj) = \frac{1}{n} \sum_{k=1}^{j} X^u_{n,k} \quad \text{for } u \in U,\ j \in \mathbb{N}.$$

The tree T has the law of a PWIT and $\widetilde{T}_n$ has, the same as $\hat{T}_n$, the law of a BWIT_n. Moreover, $n^{-1} X^u_{n,j} \to Y^u_j$, almost surely, as $n \to \infty$, so the weights of the edges of trees $\widetilde{T}_n$ converge, almost surely, to the weights of the corresponding edges of the tree T. Therefore, almost surely, for all continuity points ρ of T, there is $n_0 \in \mathbb{N}$, such that $N_\rho(\widetilde{T}_n)$ and $N_\rho(T)$ are graph isomorphic for all $n \geq n_0$ and $R(N_\rho(\widetilde{T}_n), N_\rho(T)) \to 0$ as $n \to \infty$. By the dominated convergence theorem, it follows that $d(\widetilde{T}_n, T) \to 0$, almost surely, as $n \to \infty$. Thus, for any Borel set A with $\mathbb{P}(T \in \partial A) = 0$, we have

$$\lim_{n\to\infty} \mathbb{P}(\hat{T}_n \in A) = \lim_{n\to\infty} \mathbb{P}(\widetilde{T}_n \in A) = \mathbb{P}(T \in A). \qquad \square$$

Proof of Theorem 1. Fix $\rho > 0$ and let $\mathcal{T}_{n,\rho}$ be the set of all rooted geometric graphs G which are trees and for which $\mathbb{P}(N_\rho(T_n) = G) > 0$. Since a vertex i of T_n is connected with any vertex in $\{i+1, i+2, \dots\}$ with probability n^{-1}, the corresponding weights of the edges from i are sums of geometrically distributed random variables with parameter n^{-1} rescaled by n^{-1}. Thus, for $G \in \mathcal{T}_{n,\rho}$ is $\mathbb{P}(N_\rho(T_n) = G) = \mathbb{P}(N_\rho(\hat{T}_n) = G)$, where $\hat{T}_n$ is a BWIT_n. Moreover, whenever the graphs $N_\rho(T_n)$ and $N_\rho(\hat{T}_n)$ are geometric isomorphic, the distance between T_n and $\hat{T}_n$ is

$$d(T_n, \hat{T}_n) = \int_\rho^\infty \frac{R(N_r(T_n), N_r(\hat{T}_n))}{\mathrm{e}^r}\, \mathrm{d}r \leq \int_\rho^\infty \frac{1}{\mathrm{e}^r}\, \mathrm{d}r = \frac{1}{\mathrm{e}^\rho}. \tag{2}$$

Let $A \subset \mathcal{G}_*$ be a Borel set with $\mathbb{P}(T \in \partial A) = 0$. Furthermore, for $\varepsilon > 0$, define $A_{+\varepsilon} = \{G \in \mathcal{G}_* \colon \text{there is } G' \in A \text{ such that } d(G, G') < \varepsilon\}$ and $A_{-\varepsilon} = (A^c_{+\varepsilon})^c$. Using these definitions and the upper bound (2), we can write

$$\mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}, N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) \leq \mathbb{P}(T_n \in A, N_\rho(T_n) \in \mathcal{T}_{n,\rho}) \leq \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}). \tag{3}$$

Using

$$\mathbb{P}(T_n \in A) = \mathbb{P}(T_n \in A, N_\rho(T_n) \in \mathcal{T}_{n,\rho}) + \mathbb{P}(T_n \in A, N_\rho(T_n) \notin \mathcal{T}_{n,\rho})$$

and (3), we obtain

$$\begin{aligned} \mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}, N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) &\leq \mathbb{P}(T_n \in A) \\ &\leq \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}) + \mathbb{P}(N_\rho(T_n) \notin \mathcal{T}_{n,\rho}). \end{aligned}$$

Letting $n \to \infty$ and using the result of Lemma 2 that $\mathbb{P}(N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) = \mathbb{P}(N_\rho(T_n) \in \mathcal{T}_{n,\rho}) \to 1$ as $n \to \infty$ yields

$$\liminf_{n\to\infty} \mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}) \leq \liminf_{n\to\infty} \mathbb{P}(T_n \in A) \leq \limsup_{n\to\infty} \mathbb{P}(T_n \in A) \leq \limsup_{n\to\infty} \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}).$$

Finally, letting $\rho \to \infty$ in the inequalities above and Lemma 3, we obtain

$$\lim_{n\to\infty} \mathbb{P}(T_n \in A) = \lim_{n\to\infty} \mathbb{P}(\hat{T}_n \in A) = \mathbb{P}(T \in A),$$

where T is a PWIT. $\square$

4. The length of the longest path in the PWIT

Denote by $S(v)$ the distance of a vertex v from the root of the PWIT. For $n \in \mathbb{N}$, define the minimal displacement over all individuals in the nth generation to be $M_n = \inf_{v \in \mathbb{N}^n} S(v)$ and let $\widetilde{M}_n = \operatorname{median}(M_n) = \sup\{x\colon \mathbb{P}(M_n < x) < \frac{1}{2}\}$. The minimal displacement of the nth generation of a PWIT was studied by Addario-Berry and Ford [1], who obtained the following results.

Theorem 4. *The median of the minimal displacement of the nth generation of the PWIT is*

$$\widetilde{M}_n = \frac{n}{\mathrm{e}} + \frac{3}{2\mathrm{e}} \log n + O(1).$$

Theorem 5. *For the minimal displacement M_n of the PWIT, for all $n \geq 1$, $x \geq 0$, and any $c_1 < \mathrm{e}$, there exists a constant C_1, dependent only on c_1, such that*

$$\mathbb{P}(M_n \leq \widetilde{M}_n - x) \leq C_1 \mathrm{e}^{-c_1 x},$$

and for any $c_2 < 1$, there exists a constant C_2, dependent only on c_2, such that

$$\mathbb{P}(M_n \geq \widetilde{M}_n + x) \leq C_2 \mathrm{e}^{-c_2 x}.$$

The length of a path is the number of edges in the path. We denote by L_x the length of the longest path in the PWIT between all the paths of a tree starting from the root and ending in a vertex at a distance of at most x from the root. Let $\widetilde{L}_x$ be a median of L_x and let $\hat{L}_x = \max\{n\colon \mathbb{P}(L_x \geq n) \geq \frac{1}{2}\}$. Note that $|\hat{L}_x - \widetilde{L}_x| \leq 1$, so the asymptotic result for $\hat{L}_x$ holds for the median as well. We will show that Theorem 2 and Theorem 3 follow from Theorem 4 and Theorem 5, respectively, and from the observation: $L_x \geq n$ if and only if $M_n \leq x$.

Proof of Theorem 2. Combining the definition of $\hat{L}_x$ with the above observation, we obtain

$$\hat{L}_x = \max\bigl\{n\colon \mathbb{P}(L_x \geq n) \geq \tfrac{1}{2}\bigr\} = \max\bigl\{n\colon \mathbb{P}(M_n \leq x) \geq \tfrac{1}{2}\bigr\}.$$

The maximum in the last term above will be achieved for n such that $\widetilde{M}_n \leq x < \widetilde{M}_{n+1}$. Using Theorem 4 we can assert that the n that satisfies these inequalities is $n = x\mathrm{e} - \frac{3}{2}\log x + O(1)$, which is then the asymptotic value of $\hat{L}_x$ and $\widetilde{L}_x$. □

Proof of Theorem 3. Using the second part of Theorem 5 and definitions of $\widetilde{M}_n$ and $\widetilde{L}_x$ from Theorem 4 and Theorem 2, respectively, for any $x \geq 1$, $0 \leq y \leq \widetilde{L}_x$, and $c_2 < 1$, we have

$$\begin{aligned}
\mathbb{P}(L_x \leq \widetilde{L}_x - y) &\leq \mathbb{P}(M_{\lfloor \widetilde{L}_x - y \rfloor + 1} > x) \\
&\leq C_2 \exp\{-c_2(x - \widetilde{M}_{\lfloor \widetilde{L}_x - y \rfloor + 1})\} \\
&\leq C_2 \exp\biggl\{-c_2\biggl(x - \frac{\lfloor \widetilde{L}_x - y \rfloor + 1}{\mathrm{e}} - \frac{3}{2\mathrm{e}} \log(\lfloor \widetilde{L}_x - y \rfloor + 1) + O(1)\biggr)\biggr\} \\
&\leq C_2 \exp\biggl\{-c_2 \frac{y}{\mathrm{e}} + c_2 \frac{3}{2\mathrm{e}} \log \frac{x\mathrm{e} - (3/2)\log x}{x} + O(1)\biggr\} \\
&\leq C_1' \mathrm{e}^{-c_1' y},
\end{aligned}$$

where $c_1' = c_2/\mathrm{e}$. For $y \geq \widetilde{L}_x$ the same holds because $\mathbb{P}(L_x \leq \widetilde{L}_x - y) = 0$.

The second statement of the theorem can be obtained in an analogous way.

Applying Theorem 3 to $\mathbb{E}L_x = \sum_{n=1}^{\infty}\mathbb{P}(L_x \geq n)$, we obtain $\mathbb{E}L_x = \widetilde{L}_x + O(1)$ and then from Theorem 2, we have $\mathbb{E}L_x = x\mathrm{e} - 3\log x/2 + O(1)$. Moreover, $\widetilde{L}_x$ can be replaced by $\mathbb{E}L_x$ in the first part of Theorem 3 and this can be used to calculate $\operatorname{var}(L_x) = O(1)$. □

Appendix A. A metric on $\mathcal{G}_*$

On the set of simple rooted graphs (rooted geometric graphs with all edges of weight 1), a distance between two graphs G_1 and G_2 is sometimes defined as

$$d(G_1, G_2) = \frac{1}{2^{R(G_1,G_2)}},$$

where $R(G_1, G_2) = \inf\{\rho \in \mathbb{N}_0 \colon N_\rho(G_1) \text{ and } N_\rho(G_2) \text{ are not graph isomorphic}\}$.

We expand this formula to the definition of the distance for rooted geometric graphs in the following way. Let $G_1, G_2 \in \mathcal{G}_*$, then

$$d(G_1, G_2) = \int_0^\infty \frac{R(N_\rho(G_1), N_\rho(G_2))}{\mathrm{e}^\rho}\,\mathrm{d}\rho,$$

where

$$R(N_\rho(G_1), N_\rho(G_2)) = \min\Big\{1, \min_{\substack{\{\Phi\colon \Phi\colon V_\rho(G_1)\to V_\rho(G_2)\\ \text{graph isomorphism}\}}} \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e) - w(\Phi(e))|\Big\}.$$

Note that $R(N_\rho(G_1), N_\rho(G_2)) = 1$ if there is no graph isomorphisms from $N_\rho(G_1)$ to $N_\rho(G_2)$. Whenever there exist an isomorphism then the set of edges will be nonempty with an exception if the graphs contain just the root vertex. In the latter case, we define $R(N_\rho(G_1), N_\rho(G_2)) = 0$.

Proposition 2. *The function d is a metric on $\mathcal{G}_*$ and $(\mathcal{G}_*, d)$ is a complete separable metric space.*

Proof. The function d is symmetric and nonnegative, with $d(G_1, G_2) = 0$ if and only if the graphs G_1 and G_2 are graph isomorphic and all the corresponding edges have the same weights, i.e. if there exists a geometric isomorphism from G_1 to G_2.

Now fix $\rho > 0$ and let $G_1, G_2, G_3 \in \mathcal{G}_*$. We want to show that the triangle inequality is satisfied for the function R, which by integrating over ρ immediately gives the triangle inequality for the function d. If the graphs $N_\rho(G_1), N_\rho(G_2), N_\rho(G_3)$ are not graph isomorphic the triangle inequality holds. Otherwise, let $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ be the graph isomorphisms from $N_\rho(G_1)$ to $N_\rho(G_2)$ and from $N_\rho(G_2)$ to $N_\rho(G_3)$, respectively, for which the functions $R(N_\rho(G_1), N_\rho(G_2))$ and $R(N_\rho(G_2), N_\rho(G_3))$ attain the minima. Then, by the triangle inequality, for any edge e of $N_\rho(G_1)$, we have

$$|w(e) - w(\Phi_{\rho,1}(e))| + |w(\Phi_{\rho,1}(e)) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))| \geq |w(e) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))|.$$

Taking a maximum over edges of $N_\rho(G_1)$ yields

$$\begin{aligned} &R(N_\rho(G_1), N_\rho(G_2)) + R(N_\rho(G_2), N_\rho(G_3)) \\ &\quad\geq \min\Big\{1, \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))|\Big\} \\ &\quad\geq R(N_\rho(G_1), N_\rho(G_3)), \end{aligned}$$

which is precisely the triangle inequality for function R.

We now proceed to show that this metric space is complete. Let $\{G_n,\ n \geq 1\}$ be a Cauchy sequence of graphs in $\mathcal{G}_*$, that is, a sequence for which for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for all $m, n \geq n_0$ the distance $d(G_n, G_m) < \varepsilon$. We want to show that there exists a graph $G \in \mathcal{G}_*$ for which $d(G_n, G) \to 0$ when $n \to \infty$.

We start by choosing a subsequence $\{G_{n_i},\ i \geq 1\}$ of the Cauchy sequence of graphs such that $d(G_{n_i}, G_{n_{i+1}}) < 2^{-i}$. Then, we have

$$1 > \sum_{i=1}^{\infty} d(G_{n_i}, G_{n_{i+1}}) = \int_0^{\infty} \sum_{i=1}^{\infty} \frac{R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}}))}{\mathrm{e}^\rho} \,\mathrm{d}\rho$$

and, thus, $\sum_{i=1}^{\infty} R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}})) < \infty$ for almost all ρ. For all such ρ and for all ε, $0 < \varepsilon < 1$, there exists an $i_0 = i_0(\rho, \varepsilon)$ such that $R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}})) < \varepsilon$ for all $i \geq i_0$. Therefore, for $i \geq i_0$ the pairs of graphs $N_\rho(G_{n_i})$ and $N_\rho(G_{n_{i+1}})$ are graph isomorphic and, hence, all the graphs $\{N_\rho(G_{n_i}), i \geq i_0\}$ are graph isomorphic. Denote now by $\Phi_{\rho,i}$ a graph isomorphism from $N_\rho(G_{n_i})$ to $N_\rho(G_{n_{i+1}})$ for which the maximum of the edge weight differences is minimal. Furthermore, define for $i < j$ graph isomorphism $\Phi_{\rho,(i,j)} = \Phi_{\rho,j-1} \circ \Phi_{\rho,j-2} \cdots \circ \Phi_{\rho,i}$ and let $\Phi_{\rho,(i,i)}$ be the identity graph isomorphism. Then, for all $\varepsilon' > 0$, there exists a $j_0 = j_0(\rho, \varepsilon')$, $j_0 \geq i_0$, such that for $j > i \geq j_0$ and for any edge e of the graph $N_\rho(G_{n_i})$,

$$\begin{aligned}
|w(e) - w(\Phi_{\rho,(i,j)}(e))| &\leq \sum_{k=i}^{j-1} |w(\Phi_{\rho,(i,k)}(e) - w(\Phi_{\rho,(i,k+1)}(e))| \\
&\leq \sum_{k=i}^{j-1} R(N_\rho(G_{n_k}), N_\rho(G_{n_{k+1}})) \\
&\leq \sum_{k=i}^{\infty} R(N_\rho(G_{n_k}), N_\rho(G_{n_{k+1}})) \\
&< \varepsilon'.
\end{aligned}$$

Thus, for each edge e of the graph $N_\rho(G_{n_{i_0}})$ the sequence $\{w(\Phi_{\rho,(i_0,i)}(e)),\ i \geq i_0\}$ is a Cauchy sequence and, hence, it converges.

Construct now the limit graph G in radius ρ in the following way. Set $N_\rho(G)$ to be graph isomorphic to $N_\rho(G_{n_{i_0}})$ and let Φ_ρ be the graph isomorphism from $N_\rho(G)$ to $N_\rho(G_{n_{i_0}})$. Furthermore, set the weight of every edge e of $N_\rho(G)$ to be $w(e) = \lim_{i\to\infty} w(\Phi_{\rho,(i_0,i)} \circ \Phi_\rho(e))$. In the same way, we can construct the limit graph G in any arbitrary large radius. For each continuity point ρ of the limit graph G, the function $R(N_\rho(G_{n_i}), N_\rho(G)) \to 0$ as $i \to \infty$ and then by the dominated convergence theorem $d(G_{n_i}, G) \to 0$ as $i \to \infty$.

Moreover, we need to show that $d(G_n, G) \to 0$. For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for $m, n \geq n_0$ $d(G_n, G_m) \leq \varepsilon/2$ and there exists $i_0 = i_0(\varepsilon)$ such that for $i \geq i_0$ is $n_i \geq n_0$ and $d(G_{n_i}, G) \leq \varepsilon/2$. Then, by the triangle inequality, we have

$$d(G_n, G) \leq d(G_n, G_{n_i}) + d(G_{n_i}, G) \leq \varepsilon$$

and, thus, $d(G_n, G) \to 0$ as $n \to \infty$.

Separability follows from the facts that the geometric graphs are locally finite and rational numbers are a countable dense subset of the real line. Hence, an example of a countable dense subset of the set of the rooted geometric graphs is the set of all rooted geometric graphs with finite number of vertices and rational edge weights. $\square$

Proposition 3. *Rooted geometric graphs G_n, $n \in \mathbb{N}$, converge to rooted geometric graph G if and only if $d(G_n, G) \to 0$ as $n \to \infty$.*

Proof. If rooted geometric graphs G_n converge to G, then $R(N_\rho(G_n), N_\rho(G)) \to 0$ as $n \to \infty$ for all continuity points ρ. Moreover, the function $R(N_\rho(G_n), N_\rho(G))\mathrm{e}^{-\rho}$ is bounded by the integrable function $\mathrm{e}^{-\rho}$. Since there are countably many discontinuity points, by the dominated convergence theorem,

$$d(G_n, G) = \int_0^\infty \frac{R(N_\rho(G_n), N_\rho(G))}{\mathrm{e}^\rho} \,\mathrm{d}\rho \to 0 \quad \text{when } n \to \infty,$$

which proves one direction of the proposition.

Assume now that $d(G_n, G) \to 0$ as $n \to \infty$. Let $\rho > 0$ be a continuity point of graph G and let x be the last discontinuity point of G before ρ and y be the first discontinuity point after ρ. Set $\delta_1 = \rho - x$ and $\delta_2 = y - \rho$. For any ε, $0 < \varepsilon < \min\{\delta_1 \mathrm{e}^{-\rho}, \delta_2 \mathrm{e}^{-y}\}$, there exists $n_0 = n_0(\varepsilon)$ such that $d(G_n, G) < \varepsilon$ for all $n \geq n_0$. Assume now that $N_\rho(G_n)$ and $N_\rho(G)$ are not graph isomorphic for some $n \geq n_0$. Then, we have two cases, either the graphs are not graph isomorphic for all the points of the interval $[x, \rho]$ or there is a discontinuity point x' of G_n in the interval $[x, \rho]$, such that the graphs are graph isomorphic in the interval $[x, x')$, but not graph isomorphic in the interval $[x', \rho]$. In the latter case, the graphs are not graph isomorphic for all the points in the interval $[\rho, y)$. In these two cases, we have

$$d(G_n, G) \geq \int_x^\rho \frac{1}{\mathrm{e}^r} \,\mathrm{d}r > \frac{\delta_1}{\mathrm{e}^\rho} \quad \text{or} \quad d(G_n, G) \geq \int_\rho^y \frac{1}{\mathrm{e}^r} \,\mathrm{d}r > \frac{\delta_2}{\mathrm{e}^y},$$

which contradicts $d(G_n, G) < \varepsilon$. Thus, the graphs $N_\rho(G_n)$ and $N_\rho(G)$ are graph isomorphic for all $n \geq n_0$. Assume now that the weights of the edges do not converge to the weights of the edges of G, i.e. that there exist an $\varepsilon' > 0$ such that for all $n_1 \geq n_0$, there is an $n \geq n_1$ for which

$$\min_{\{\Phi\colon \Phi\colon V_\rho(G) \to V_\rho(G_n) \text{ graph isomorphism}\}} \; \max_{\{e\colon e \text{ edge of } N_\rho(G)\}} |w(e) - w(\Phi(e))| > \varepsilon'.$$

But then the function $R(N_r(G_n), N_r(G))$ is greater than ε' for r between x and ρ and, thus, similarly as above $d(G_n, G) > \varepsilon' \delta_1 \mathrm{e}^{-\rho}$ which is impossible. Hence, for all continuity points ρ, the graphs are graph isomorphic for large enough n and the weights of the edges converge to the weights of the corresponding edges of G, which is precisely the definition of convergence of rooted geometric graphs. □

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Paper III

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DISTRIBUTION OF THE SMALLEST VISITED POINT IN A GREEDY WALK ON THE LINE

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Abstract

We consider a greedy walk on a Poisson process on the real line. It is known that the walk does not visit all points of the process. In this paper we first obtain some useful independence properties associated with this process which enable us to compute the distribution of the sequence of indices of visited points. Given that the walk tends to $+\infty$, we find the distribution of the number of visited points in the negative half-line, as well as the distribution of the time at which the walk achieves its minimum.

Keywords: Poisson point process; greedy walk

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1. Introduction

We consider a stationary Poisson process Π with rate 1 on the real line and define a greedy walk on Π as follows. The walk starts from point $S_0 = 0$ and always moves on the points of Π by picking the point closest to its current position that has not been visited before. In other words, the sequence of the visited points of Π of the walk $(S_n)_{n\geq 0}$ is defined recursively by

$$S_{n+1} = \arg\min\{d(X, S_n) \colon X \in \Pi \setminus \{S_0, S_1, \dots, S_n\}\}, \tag{1}$$

where d is the usual Euclidean distance. Note that the sequence $(S_n)_{n\geq 0}$ is almost surely (a.s.) well defined, i.e. there is a.s. a unique point which is arg min in the definition of the walk (1). Moreover, $0 \notin \Pi$ a.s.

The distance between the current position and the closest nonvisited point on the other side of 0 increases with each step. Using the Borel–Cantelli lemma, one can show that the walk a.s. jumps over 0 finitely many times. Therefore, $\{S_1, S_2, \dots\} \neq \Pi$ a.s. and, moreover,

$$\mathbb{P}\Big(\lim_{n\to\infty} S_n = +\infty\Big) = \mathbb{P}\Big(\lim_{n\to\infty} S_n = -\infty\Big) = \tfrac{1}{2}. \tag{2}$$

In this paper we study the distribution of the number of visited points of Π which are less than 0 and the distribution of the index of the last point in the sequence $(S_n)_{n\geq 0}$ which is less than or equal to 0. We denote these random variables by N and L, respectively.

The greedy walk is a model in queueing systems where the points of the process in our case represent positions of customers and a server (the walker) moving towards customers. Applications of such a system can be found, for example, in telecommunications and computer networks or transportation. As described in [1], the model of a greedy walk on a point process can be defined in various ways or/and on different spaces. For example, Coffman and Gilbert [2]

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and Leskelä and Unger [5] studied a dynamic version of the greedy walk on a circle with new customers arriving to the system according to a Poisson process. Two modifications of the greedy walk model on a homogeneous Poisson process on the real line were studied by Foss *et al.* [3] and Rolla *et al.* [6]. In the first paper, the authors considered a space-time model, starting with a Poisson process at time 0 and the time and position of arrivals of new points are given by a Poisson process on the space-time half-plane. Moreover, the expected time that the walk spends at a point is 1. In this case the walk, a.s., jumps over the starting point finitely many times and the position of the walk diverges logarithmically in time. In the second paper, the points of a Poisson process were assigned one or two marks at random. The walk always moves to the point closest to the current position which still has at least one mark and then removes exactly one mark from the point. The authors showed that introducing points with two marks will force the walk to change sides infinitely many times and, thus, to visit all the points of Π. There is not much known about the behaviour of the greedy walk on a homogeneous Poisson process in higher dimensions. For example, it is an open problem whether the greedy walk on the points of a homogeneous Poisson process on $\mathbb{R}^2$ visits all points [6].

The paper is organized as follows. In Section 2 we calculate the probability that the server visits the points of Π in a particular order. The distributions of random variables N and L are studied in Section 3 and Section 4, respectively.

2. The probability to visit points in a predefined order

The points of $\Pi \cup \{0\}$ can be written in order as

$$\cdots < X_{-3} < X_{-2} < X_{-1} < X_0 = 0 < X_1 < X_2 < X_3 < \cdots .$$

For $n \in \mathbb{Z}$, let

$$Y_n = X_n - X_{n-1}.$$

Since Π is a stationary Poisson process with rate 1, $\{Y_n\}_{n\in\mathbb{Z}}$ are independent and exponentially distributed random variables with parameter 1.

Lemma 1. *Let $\{Y_i\}_{i\geq 1}$ be independent and exponentially distributed random variables with mean* 1 *and let $D_n = \sum_{i=1}^{n} Y_i$. Then*

$$\mathbb{P}(D_n < Y_{n+1}) = \frac{1}{2^n}$$

and the events $\{D_n < Y_{n+1}\}_{n\geq 1}$ are independent.

Proof. For $n = 1$ this is $\mathbb{P}(D_1 < Y_2) = \mathbb{P}(Y_1 < Y_2) = \frac{1}{2}$. Using the memoryless property of the exponential random variable Y_{n+1}, we have

$$\begin{aligned}
\mathbb{P}(D_n < Y_{n+1}) &= \mathbb{P}(D_n < Y_{n+1} \mid D_{n-1} < Y_{n+1})\mathbb{P}(D_{n-1} < Y_{n+1}) \\
&= \mathbb{P}(Y_n < Y_{n+1})\mathbb{P}(D_{n-1} < Y_{n+1}) \\
&= \tfrac{1}{2}\mathbb{P}(D_{n-1} < Y_{n+1}).
\end{aligned}$$

Thus, by induction, we obtain $\mathbb{P}(D_n < Y_{n+1}) = 1/2^n$.

The event $\{D_n > Y_{n+1}\}$ can be written as

$$\{D_n > Y_{n+1}\} = \left\{\frac{Y_{n+1}}{D_{n+1}} < \frac{1}{2}\right\} = \left\{\frac{D_n}{D_{n+1}} > \frac{1}{2}\right\}.$$

Therefore, in order to show the independence of the events $\{D_n > Y_{n+1}\}_{n\geq 1}$, we prove the independence of the random variables $\{D_n/D_{n+1}\}_{n\geq 1}$.

For $i < j$, let $U_{i,j} = D_i/D_j$. For any $\ell > 0$, choose integers $i_1 < i_2 < \cdots < i_\ell$. The random variable $U_{i_1,i_2} = D_{i_1}/D_{i_2}$ is independent of D_{i_2} (see, for example, [4, Theorem 8.4.1]) and it is also independent of $D_{i_3} - D_{i_2}, \ldots, D_{i_\ell} - D_{i_{\ell-1}}$. This implies that the random variable U_{i_1,i_2} is independent of the random variables $\{U_{i_j,i_{j+1}}\}_{j=2}^{\ell-1}$ since

$$U_{i_j,i_{j+1}} = \frac{D_{i_2} + \sum_{k=2}^{j-1}\left(D_{i_{k+1}} - D_{i_k}\right)}{D_{i_2} + \sum_{k=2}^{j}\left(D_{i_{k+1}} - D_{i_k}\right)} \quad \text{for } 2 \le j < \ell.$$

We can use the same argument to show that, for $2 \le k < \ell-1$, the random variable $U_{i_k,i_{k+1}}$ is also independent of $\{U_{i_j,i_{j+1}}\}_{j=k+1}^{\ell-1}$. Therefore, the random variables $\{U_{i_j,i_{j+1}}\}_{j=1}^{\ell-1}$ are independent for any choice of ℓ and for any increasing sequence $i_1, i_2, \ldots, i_\ell$ and, thus, the random variables $\{U_{n,n+1}\}_{n\ge1} = \{D_n/D_{n+1}\}_{n\ge1}$ are independent as well. □

Let

$$\mathcal{S} = \left\{(i_n)_{n\ge0} \in \mathbb{Z}^\infty \colon i_0 = 0 \text{ and } i_n \in \left\{\min_{0\le j\le n-1} i_j - 1, \max_{0\le j\le n-1} i_j + 1\right\} \text{ for all } n \in \mathbb{N}\right\}.$$

If the sequence $(S_n)_{n\ge0}$ satisfies (1) then there exists a sequence $(i_n)_{n\ge0} \in \mathcal{S}$ such that $S_n = X_{i_n}$ for all $n \ge 0$. Furthermore, denote by $\pi(S_n) = i_n$ the index of the nth visited point. In the following lemma we compute the probability that the sequence $(\pi(S_n))_{n\ge0}$ is $(i_n)_{n\ge0} \in \mathcal{S}$.

Lemma 2. *Let $(i_n)_{n\ge0} \in \mathcal{S}$ and let*

$$\delta_1 = \begin{cases} 0 & \text{if } i_1 = 1, \\ 1 & \text{if } i_1 = -1, \end{cases} \qquad \text{and, for } n \ge 2, \qquad \delta_n = \begin{cases} 0 & \text{if } |i_n - i_{n-1}| = 1, \\ 1 & \text{otherwise}. \end{cases}$$

Then

$$\mathbb{P}((\pi(S_n))_{n\ge0} = (i_n)_{n\ge0}) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)^{1-\delta_n}\left(\frac{1}{2^n}\right)^{\delta_n}.$$

In other words, the random variables $\{\delta_n\}_{n\ge1}$ are independent and δ_n has a Bernoulli distribution with parameter 2^{-n}.

Proof. Let $M_n = \max_{0\le j\le n} i_j$ and let $m_n = \min_{0\le j\le n} i_j$. Moreover, define random variables $\{Z_n\}_{n\ge1}$ as

$$Z_1 = Y_0, \qquad Z_2 = Y_1, \qquad Z_n = \begin{cases} Y_{i_{n-2}+1} & \text{if } i_{n-2} > 0, \\ Y_{i_{n-2}} & \text{if } i_{n-2} < 0, \end{cases} \qquad n \ge 3.$$

Assume that the event $\{(\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\}$ occurs and, without loss of generality, assume that $i_n = M_n$. If $i_{n+1} = M_n + 1$ then $\delta_{n+1} = 0$ and

$$\begin{aligned} &\mathbb{P}(\pi(S_{n+1}) = i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\ &\quad = \mathbb{P}(X_{M_n+1} - X_{i_n} < X_{i_n} - X_{m_n-1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\ &\quad = \mathbb{P}\left(Y_{M_n+1} < \sum_{j=m_n}^{M_n} Y_j \;\middle|\; (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\right) \\ &\quad = \mathbb{P}\left(Z_{n+2} < \sum_{j=1}^{n+1} Z_j \;\middle|\; (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\right). \end{aligned} \tag{3}$$

Since $\{Z_n\}_{n\geq 1}$ are independent and identically distributed exponential random variables with parameter 1 and the event $\{(\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\}$ is the intersection of the events $\{Z_{k+2} < \sum_{j=1}^{k+1} Z_j\}_{k\geq 0}^{n-1}$ or their complements, from Lemma 1, it follows that the value of (3) is $1-2^{-(n+1)}$. If $i_{n+1} = m_n - 1$ then $\delta_{n+1} = 1$ and

$$\begin{aligned}\mathbb{P}(\pi(S_{n+1}) &= i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\ &= \mathbb{P}(X_{M_n+1} - X_{i_n} > X_{i_n} - X_{m_n-1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n).\end{aligned}$$

We deduce, again from Lemma 1, that the probability above is $2^{-(n+1)}$. Thus, we can write

$$\mathbb{P}(\pi(S_{n+1}) = i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) = \left(1 - \frac{1}{2^{n+1}}\right)^{1-\delta_{n+1}} \left(\frac{1}{2^{n+1}}\right)^{\delta_{n+1}},$$

and the claim of the lemma follows. □

Corollary 1. *The expected number of times the sequence $(S_n)_{n\geq 1}$ changes sign is $\frac{1}{2}$.*

Proof. The sequence changes sign after visiting point S_n, $n \geq 1$, if $\delta_{n+1} = 1$. Since $\{\delta_n\}_{n\geq 1}$ are independent random variables with $\mathbb{P}(\delta_n = 1) = 2^{-n}$, the expected number of times the sequence changes sign is

$$\mathbb{E}\left(\sum_{n=2}^{\infty} \delta_n\right) = \sum_{n=2}^{\infty} \mathbb{P}(\delta_n = 1) = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}. \qquad \square$$

Remark 1. Corollary 1 implies that the number of times the sequence $(S_n)_{n\geq 0}$ changes sign is a.s. finite and, thus, the walk a.s. does not visit all the points of Π. This is another way to prove the fact that $\{S_1, S_2, \dots\} \neq \Pi$ a.s.

3. Distribution of the random variable N

In this section we study the distribution of the random variable N, the number of visited points of Π which are less than $S_0 = 0$. We can write $N = -\min_{n\geq 0} \pi(S_n)$. From (2), we know that N is a defective random variable with $\mathbb{P}(N = \infty) = \frac{1}{2}$ and the law of N, when N is finite, is given in the following theorem.

Theorem 1. *Let*

$$C(k,\ell) = \sum_{\substack{1\leq i_1<j_1<i_2<j_2<\cdots<i_\ell<j_\ell \\ \sum_{m=1}^{\ell}(j_m-i_m)=k}} \frac{1}{(2^{i_1}-1)(2^{j_1}-1)\cdots(2^{i_\ell}-1)(2^{j_\ell}-1)} \quad \text{for } k,\ell \geq 1.$$

Then,

$$\mathbb{P}(N = 0) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)$$

and, for $k \geq 1$,

$$\mathbb{P}(N = k) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right) \sum_{\ell=1}^{k} C(k,\ell).$$

Moreover, for $k \geq 0$,

$$\mathbb{P}(N = k) = \frac{c}{2^k} + O\left(\frac{1}{2^{2k}}\right),$$

where c is a positive constant that does not depend on k.

Proof. If $N = 0$, the walk visits points $X_0, X_1, X_2, \ldots$ successively. This implies that $(\pi(S_n))_{n\geq 0} = (n)_{n\geq 0}$ and the sequence $(\delta_n)_{n\geq 0}$, defined in Lemma 2, is identically 0. Now, the result follows directly from Lemma 2.

If $N = k \geq 1$, the set of indices of visited points is $\{\pi(S_0), \pi(S_1), \pi(S_2), \ldots\} = \{-k, -k+1, -k+2, \ldots\}$. Then there is ℓ, $1 \leq \ell \leq k$, and sequences $i_1, i_2, \ldots, i_\ell$ and $j_1, j_2, \ldots, j_\ell$ such that $0 < i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell$ and the sequence $(S_n)_{n\geq 0}$ is negative when $i_m \leq n \leq j_m - 1$, $m \in \{1, 2, \ldots, \ell\}$ and nonnegative otherwise. Since the walk visits exactly k points on the left, we have $\sum_{m=1}^{\ell}(j_m - i_m) = k$, $\delta_n = 1$ for $n \in \{i_1, i_2, \ldots, i_\ell, j_1, j_2, \ldots, j_\ell\}$ and $\delta_n = 0$ otherwise.

Therefore, from Lemma 2, we obtain

$$\begin{aligned}
&\mathbb{P}(N = k) \\
&\quad= \sum_{\substack{(i_n)_{n\geq 0}\in\mathcal{S} \\ \{i_0,i_1,i_2,\ldots\}=\{-k,-k+1,-k+2,\ldots\}}} \mathbb{P}((\pi(S_n))_{n\geq 0} = (i_n)_{n\geq 0}) \\
&\quad= \sum_{\ell=1}^{k} \sum_{\substack{1\leq i_1<j_1<i_2<j_2<\cdots<i_\ell<j_\ell \\ \sum_{m=1}^{\ell}(j_m-i_m)=k}} \prod_{i\notin\{i_1,\ldots,i_\ell,j_1,\ldots,j_\ell\}} \left(1 - \frac{1}{2^i}\right) \prod_{i\in\{i_1,\ldots,i_\ell,j_1,\ldots,j_\ell\}} \frac{1}{2^i} \\
&\quad= \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right) \sum_{\ell=1}^{k} \sum_{\substack{1\leq i_1<j_1<i_2<j_2<\cdots<i_\ell<j_\ell \\ \sum_{m=1}^{\ell}(j_m-i_m)=k}} \frac{1}{(2^{i_1}-1)(2^{j_1}-1)\cdots(2^{i_\ell}-1)(2^{j_\ell}-1)} \\
&\quad= \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right) \sum_{\ell=1}^{k} C(k,\ell).
\end{aligned}$$

In order to find the asymptotic value of the expression above, we first obtain an upper bound for $2^k C(k,\ell)$ by using the inequalities $1/(2^k - 1) \leq 2/2^k$ and $1/(4^k - 1) \leq 2/4^k$. Thus,

$$\begin{aligned}
&2^k C(k,\ell) \\
&\quad= \sum_{\substack{1\leq i_1,i_2,\ldots,i_\ell \\ 1\leq k_1,k_2,\ldots,k_{\ell-1} \\ \sum_{i=1}^{\ell-1} k_i<k}} \frac{2^k}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)(2^{i_1+\cdots+i_\ell+k}-1)} \\
&\quad\leq \sum_{\substack{1\leq i_1,i_2,\ldots,i_\ell \\ 1\leq k_1,k_2,\ldots,k_{\ell-1} \\ \sum_{i=1}^{\ell-1} k_i<k}} \frac{2^{2\ell}}{2^{2\ell i_1}2^{2(\ell-1)i_2}\cdots 2^{2i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\cdots 2^{2k_{\ell-1}}} \\
&\quad= \frac{2^{2\ell}}{(4^\ell-1)(4^{\ell-1}-1)^2(4^{\ell-2}-1)^2\cdots(4-1)^2}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2^{2\ell}2^{2\ell-1}}{4^{\ell^2}}\\
&= \frac{1}{2^{2\ell^2-4\ell+1}}.
\end{aligned}
\tag{4}
$$

Let

$$c_\ell = \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}} \sum_{1\le k_1,k_2,\dots,k_{\ell-1}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}.$$

Similarly as in (4) one can show that $c_\ell \le 1/(2^{2\ell^2-4\ell+2})$ and, moreover,

$$
\begin{aligned}
&|c_\ell - 2^k C(k,l)|\\
&\quad\le \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}(2^{i_1+\cdots+i_\ell+k}-1)}\\
&\qquad\times \sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1} k_i<k}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}\\
&\qquad+ \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}}\\
&\qquad\quad\times \sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1} k_i\ge k}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}\\
&\quad\le \frac{1}{2^k} \sum_{1\le i_1,i_2,\dots,i_\ell}\ \sum_{1\le k_1,k_2,\dots,k_{\ell-1}} \frac{2^{2\ell}}{2^{(2\ell+1)i_1}2^{(2\ell-1)i_2}\dots 2^{3i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\dots 2^{2k_{\ell-1}}}\\
&\qquad+ \sum_{1\le i_1,i_2,\dots,i_\ell}\ \sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1} k_i\ge k}} \frac{2^{2\ell-1}}{2^{2\ell i_1}2^{2(\ell-1)i_2}\dots 2^{2i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\dots 2^{2k_{\ell-1}}}\\
&\quad\le \frac{1}{2^k}\frac{1}{2^{2\ell^2-3\ell+1}}\\
&\qquad+ \frac{1}{2^{2k}}\frac{2^{2\ell-1}}{(4^\ell-1)(4^{\ell-1}-1)\cdots(4-1)} \sum_{1\le k_1,k_2,\dots,k_{\ell-2}} \frac{2}{2^{2(\ell-2)k_1}2^{2(\ell-3)k_2}\dots 2^{2k_{\ell-2}}}\\
&\quad\le \frac{1}{2^k}\frac{1}{2^{2\ell^2-3\ell+1}} + \frac{1}{2^{2k}}\frac{1}{2^{2\ell^2-6\ell+4}}\\
&\quad= O\left(\frac{1}{2^k}\frac{1}{2^{\ell^2}}\right).
\end{aligned}
\tag{5}
$$

Since $c_\ell \le 1/(2^{2\ell^2-4\ell+2})$ then $\sum_{\ell=k+1}^{\infty} c_\ell \le 1/(2^{2k^2-1})$. This, together with (5), gives

$$2^k\mathbb{P}(N=k) = \prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)\sum_{\ell=1}^{k} 2^k C(k,\ell) = \prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)\sum_{\ell=1}^{k} c_\ell + O\left(\frac{1}{2^k}\right) = c + O\left(\frac{1}{2^k}\right),$$

where $c = \prod_{n=1}^{\infty}(1-1/2^n)\sum_{\ell=1}^{\infty} c_\ell$. $\square$

4. Distribution of the random variable L

In this section we study the distribution of the index of the last point in the sequence $(S_n)_{n\geq 0}$, which is less than or equal to $S_0 = 0$; that is, the distribution of the random variable $L = \max\{n\colon S_n \leq 0\}$. Again, from (2), we have $\mathbb{P}(L = \infty) = \frac{1}{2}$. Just as in the previous section, we find the exact distribution of this random variable as well as an asymptotic result.

Theorem 2. *We have* $\mathbb{P}(L = 0) = \prod_{n=1}^{\infty}(1 - 1/2^n)$ *and, for* $k \geq 1$,

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}} \prod_{n=k+2}^{\infty}\left(1 - \frac{1}{2^n}\right).$$

Moreover, for $k \geq 0$,

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}} + O\left(\frac{1}{2^{2k}}\right).$$

Proof. The value of $\mathbb{P}(L = 0)$ follows directly from Theorem 1 since $\{N = 0\} = \{L = 0\}$.

If $L = k$, $k \geq 1$, for the sequence $(\pi(S_n))_{n\geq 0} \in \mathcal{S}$ it holds that $\pi(S_k) < 0$ and $\pi(S_{k+\ell}) > 0$ for all $\ell \geq 1$. Thus, for the corresponding sequence $(\delta_n)_{n\geq 1}$, which was defined in Lemma 2, we have $\delta_{k+1} = 1$ and $\delta_{k+\ell+1} = 0$ for all $\ell \geq 1$. The only constraint for the first k members in both sequences, other than the constraints given by the definition of the sequences, is that $\pi(S_k) < 0$. This has probability $\frac{1}{2}$ due to symmetry. Thus, by Lemma 2, we have

$$\begin{aligned}
\mathbb{P}(L = k) &= \sum_{\substack{(i_n)_{n\geq 0}\in\mathcal{S} \\ i_k<0,\, i_{k+\ell}>0 \text{ for } \ell\geq 1}} \mathbb{P}((\pi(S_n))_{n\geq 0} = (i_n)_{n\geq 0}) \\
&= \frac{1}{2} \sum_{\delta_1,\delta_2,\dots,\delta_k\in\{0,1\}} \prod_{n=1}^{k}\left(1 - \frac{1}{2^n}\right)^{1-\delta_n}\left(\frac{1}{2^n}\right)^{\delta_n} \frac{1}{2^{k+1}} \prod_{n=k+2}^{\infty}\left(1 - \frac{1}{2^n}\right) \\
&= \frac{1}{2^{k+2}} \prod_{n=k+2}^{\infty}\left(1 - \frac{1}{2^n}\right).
\end{aligned}$$

Furthermore, we can write

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}}\left(1 - \sum_{n=k+2}^{\infty} \frac{1}{2^n} + \cdots\right) = \frac{1}{2^{k+2}}\left(1 - \frac{1}{2^{k+1}} + \cdots\right) = \frac{1}{2^{k+2}} + O\left(\frac{1}{2^{2k}}\right). \qquad \square$$

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Paper IV

GREEDY WALKS ON TWO LINES

KATJA GABRYSCH

ABSTRACT. The greedy walk is a walk on a point process that always moves from its current position to the nearest not yet visited point. We consider here various point processes on two lines. We look first at the greedy walk on two independent one-dimensional Poisson processes placed on two intersecting lines and prove that the greedy walk almost surely does not visit all points. When a point process is defined on two parallel lines, the result depends on the definition of the process: If each line has a copy of the same realisation of a homogeneous Poisson point process, then the walk almost surely does not visit all points of the process. However, if each point of this process is removed with probability p from either of the two lines, independently of the other points, then the walk almost surely visits all points. Moreover, the greedy walk on two parallel lines, where each line has a copy of the same realisation of a homogeneous Poisson point process, but one copy is shifted by some small s, almost surely visits all points.

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1. INTRODUCTION

Consider a simple point process Π in a metric space (E, d). We think of Π as a collection of points (the support of the measure) and we use the notation $|\Pi \cap B|$ to indicate the number of points on the Borel set $B \subset E$. If $x \in E$, the notation $x \in \Pi$ is used instead of $\Pi \cap \{x\} \neq \emptyset$.

We define a greedy walk on Π as follows. The walk starts from some point $S_0 \in E$ and always moves on the points of Π by picking the point closest to its current position that has not been visited before. Thus a sequence $(S_n)_{n\geq 0}$ is defined recursively by

$$S_{n+1} = \arg\min \big\{ d(X, S_n) : X \in \Pi,\ X \notin \{S_0, S_1, \ldots, S_n\} \big\}. \tag{1}$$

The greedy walk is a model in queuing systems where the points of the process represent positions of customers and the walk represents a server moving towards customers. Applications of such a system can be found, for example, in telecommunications and computer networks or transportation. As described in [1], the model of a greedy walk on a point process can be defined in various ways and on different spaces. For example, Coffman and Gilbert [2] and Leskelä and Unger [6] study a dynamic version of the greedy walk on a circle with new customers arriving to the system according to a Poisson process.

The greedy walk defined as in (1) on a homogeneous Poisson process on $\mathbb{R}$ almost surely does not visit all points. More precisely, the expected number of times the walk jumps over 0 is 1/2 [4]. Foss *et al.* [3] and Rolla *et al.* [7] study two modifications of this model where they introduce some extra points on the line, which they call "rain" and "dust", respectively. Foss *et al.* [3] consider a space-time model, starting with a Poisson process at time 0. The positions and times of arrival of new points are given by a Poisson process on the half-plane. Moreover, the expected time that the walk spends at a point is 1. In this case the walk, almost surely, jumps over the starting point finitely many times and the position of the walk diverges logarithmically in time. Rolla *et al.* [7] assign to the points of a Poisson process one or two marks at random. The walk always moves to the point closest to the current position which still has at least one mark left and then removes exactly one mark from that point. The authors show that introducing points with two marks will force the walk to change sides infinitely many times. Thus, unlike the walk on a Poisson process with single marks, the walk here almost surely visits all points of the point process.

There is not much known about the behaviour of the greedy walk on a homogeneous Poisson process in higher dimensions. For example, it is an open problem whether the greedy walk on the points of a homogeneous Poisson process on $\mathbb{R}^2$ visits all points [7].

In this paper, we study various point processes defined on the union of two lines $E \subset \mathbb{R}^2$, where the distance function d on E is the Euclidean distance, $d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. For all point processes Π considered in this paper, every step of the greedy walk on Π is almost surely uniquely defined, that is for every $n \geq 0$, there is, almost surely, only one point for which the minimum (1) is obtained.

We study first a point process Π on two lines intersecting at $(0, 0)$ with independent homogeneous Poisson processes on each line. The greedy walk starts from $(0, 0)$. When the walk visits a point that is far away from $(0, 0)$, then the distance to $(0, 0)$ and to any point on the other line is large. Thus, the probability of changing lines or crossing $(0, 0)$ is small. In Section 2 we show by using the Borel-Cantelli lemma that almost surely the walk crosses $(0, 0)$ or changes lines only finitely many times, which implies that almost surely the walk does not visit all points of Π.

Thereafter, we look at the greedy walk on two parallel lines at a fixed distance r, $\mathbb{R} \times \{0, r\}$, with a point process on each line. The behaviour here depends on the definition of the process. The first case we study is a process Π consisting of two identical copies of a homogeneous Poisson process on $\mathbb{R}$, that are placed on the parallel lines. We show in Section 3 that the greedy walk does not visit all the points of Π, but it visits all the points on one side of the vertical line $\{0\} \times \mathbb{R}$ and just finitely many points on the other side.

In the second case, we modify the definition of the process above by deleting exactly one of the copies of each point with probability $p > 0$, independently from the other points, and the line is chosen with probability $1/2$. In particular, if $p = 1$ we have two independent Poisson processes on these lines. For any $p > 0$, the greedy walk almost surely visits all points. The reason is that the greedy walk skips some of the points when it goes away from the vertical line $\{0\} \times \mathbb{R}$ and those points will force the walk to return and cross the vertical line $\{0\} \times \mathbb{R}$ infinitely many times. We prove this in Section 4 using arguments from [7].

The greedy walk also visits all points of Π in the case when Π consists of two identical copies of a homogeneous Poisson process on $\mathbb{R}$ where one copy is shifted by $|s| < r/\sqrt{3}$. This is discussed in Section 5. Note that all results are independent of the choice of r.

For the greedy walk on a homogeneous Poisson process on $\mathbb{R}$, with single or double marks assigned to the points, Rolla *et al.* [7] show that, even though the walk visits all the points, the expected first crossing time of 0 is infinite. One can in a similar way show analogous results for the greedy walk on the processes on two parallel lines defined in Sections 4 and 5, but that is not included in this paper.

2. Two intersecting lines

Let $E = \{(x, y) \in \mathbb{R}^2 : y = m_1 x \text{ or } y = m_2 x\}$ where $m_1, m_2 \in \mathbb{R}$, $m_1 \neq m_2$. The point $(0, 0)$ divides E into four half-lines. Let d be the Euclidean distance and denote the distance of a point (x, y) from the origin by $||(x, y)|| = d((0, 0), (x, y))$.

Let Π^1, Π^2 be two independent Poisson processes on $\mathbb{R}$ with rate 1. Then, for any $a, b \in \mathbb{R}$ let

$$\left|\Pi \cap \left\{(x, m_i x) : x\sqrt{1+m_i^2} \in (a,b)\right\}\right| = \left|\Pi^i \cap (a,b)\right|,$$

so that the distances between the points of Π^1 and Π^2 are preserved in E. The greedy walk $(S_n)_{n\geq 0}$ on Π defined by (1) starts from $(0,0)$.

Theorem 1. *Almost surely, the greedy walk does not visit all points. More precisely, the greedy walk almost surely visits only finitely many points on three half-lines.*

Proof. Let $B_{(0,0)}(R)$ be a ball in $\mathbb{R}^2$ of radius R around point $(0,0)$. Then

$$|\Pi \cap B_{(0,0)}(R)| = |\Pi^1 \cap (-R,R)| + |\Pi^2 \cap (-R,R)| < \infty \qquad \text{a.s.}$$

Thus, $\lim_{n\to\infty} ||S_n|| = \infty$ almost surely. To show that the walk does not visit all points of Π, it suffices to prove that the sequence $(S_n)_{n\geq 0}$ changes half-lines only finitely many times.

We can define a subsequence of the times when the walk changes half-lines as follows:

$$\begin{aligned}
j_0 &= 0,\\
j_n &= \inf\{k > j_{n-1} : S_k \text{ and } S_{k+1} \text{ are not on the same half-line}\\
&\qquad\qquad \text{and } \ ||S_k|| > \max\{n, ||S_{j_{n-1}}||\}\},\\
k_0 &= 0,\\
k_n &= \inf\{k \leq j_n : S_k, S_{k+1}, \ldots, S_{j_n} \text{ are on the same half-line}\},
\end{aligned}$$

where $\inf\emptyset = \infty$. Let $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ be the corresponding subsequences of $(S_n)_{n\geq 0}$, that is,

$$U_n = S_{k_n} \text{ and } V_n = S_{j_n},$$

when $j_n, k_n < \infty$, and $U_n = (1,1)$, $V_n = (0,0)$ otherwise. Moreover, define the events

$$B_n = \{||U_n|| \leq ||V_n||\} \text{ and } C_n = \{||U_n|| > ||V_n||\}.$$

If the greedy walk changes half-lines infinitely often, then $j_n < \infty$ for all n and exactly one of the events B_n and C_n occurs for each n.

Assume that C_n occurs for some n such that $j_n < \infty$. Let $X = S_{k_n - 1}$ be the last visited point before U_n. Then, by the definition of the sequence $(V_n)_{n\geq 0}$, $||X|| \leq \max\{n, ||V_{n-1}||\} < ||V_n||$. From $||U_n|| > ||V_n|| > ||X||$ it follows that $d(X, V_n) < d(X, U_n)$, which contradicts the definition of the greedy walk, and therefore also the assumption that C_n occurs.

Assume now that B_n occurs for some n such that $j_n < \infty$. Denote by α the acute angle between the lines $y = m_1 x$ and $y = m_2 x$. By the definition of the sequence $(V_n)_{n\geq 0}$, the walk up to time j_n never changed lines from a point whose distance from the origin is greater than $||V_n||$. Moreover, because of the assumption $||U_n|| \leq ||V_n||$, between times k_n and j_n the walk never visited the points further away than V_n on the corresponding half-line. Thus, the points on this half-line outside $B_{(0,0)}(||V_n||)$ are not yet visited. The distance from V_n to the other line is $||V_n|| \sin\alpha$. Since the walk changes lines after visiting the point V_n, we can conclude that there are no unvisited points of Π in $B_{V_n}(||V_n|| \sin\alpha)$ at time n. Hence, for $n \geq n_0$, where n_0 is such that

$n_0 \sin\alpha > 1$, we have

$$
\begin{aligned}
B_n &\subset \bigcup_{i=1}^{2} \left\{\Pi^i \cap (||V_n||, ||V_n||(1+\sin\alpha)) = \emptyset\right\} \cup \left\{\Pi^i \cap (-||V_n||(1+\sin\alpha), -||V_n||) = \emptyset\right\} \\
&\subset \bigcup_{i=1}^{2} \bigcup_{R=n}^{\infty} \left\{\Pi^i \cap (R+1, R(1+\sin\alpha)) = \emptyset\right\} \cup \left\{\Pi^i \cap (-R(1+\sin\alpha), -(R+1)) = \emptyset\right\}.
\end{aligned}
$$

Then,

$$\mathbb{P}(B_n) \le 4 \sum_{R=n}^{\infty} e^{-R\sin\alpha+1} = \frac{4e^{-n\sin\alpha+1}}{1-e^{-\sin\alpha}}$$

and

$$\sum_{n=n_0}^{\infty} \mathbb{P}(B_n) \le \frac{4e^{-n_0\sin\alpha+1}}{(1-e^{-\sin\alpha})^2} < \infty.$$

Hence, by the Borel-Cantelli lemma, $\mathbb{P}\left(B_n \text{ for infinitely many } n \ge 1\right) = 0$.

Since the event C_n does not occur for any n such that $j_n < \infty$ and B_n occurs almost surely finitely many times, there exists, almost surely, n_0 such that $j_n = \infty$ for $n \ge n_0$. Thus, the greedy walk changes lines finitely many times. □

Remark 1. The theorem holds also when Π^1 and Π^2 are Poisson processes with different rates. Moreover, we can generalise this theorem as follows. Let E be a space of finitely many intersecting lines (every two lines are intersecting, but not necessarily all in the same point) and Π is a point process on E consisting of a homogeneous Poisson process (with possibly different rates) on every line. Then the greedy walk starting from a point in E does not visit all points of Π. Similarly as above, one can show that the walk changes lines finitely many times, and therefore, visits finitely many points on all but one line.

3. Two parallel lines with the same Poisson process

Let $E = \mathbb{R} \times \{0, r\}$ and let d be the Euclidean distance. Sometimes we refer to the lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{r\}$ as line 0 and line r, respectively.

We define a point process Π on E in the following way. Let $\widehat{\Pi}$ be a homogeneous Poisson process on $\mathbb{R}$ with rate 1 and let $|\Pi \cap (B \times \{0\})| = |\Pi \cap (B \times \{r\})| = |\widehat{\Pi} \cap B|$ for all Borel sets $B \subset \mathbb{R}$. Denote the points of the process $\widehat{\Pi}$ by

$$\ldots < X_{-2} < X_{-1} \le 0 < X_1 < X_2 < \ldots.$$

The greedy walk on Π starts from the point $S_0 = (0,0)$, which is with probability 1 not a point of Π.

Theorem 2. *Almost surely, the greedy walk does not visit all points of* Π. *More precisely, the greedy walk almost surely visits all points on one side of the vertical line* $\{0\} \times \mathbb{R}$ *and finitely many points on the other side.*

Proof. We can divide the points $(X_i)_{i\in\mathbb{Z}}$ into clusters, so that successive points in the same cluster have a distance less than r and the distance between any two points in different clusters is greater than r. Let $(\tau_i)_{i\in\mathbb{Z}}$ be the indices of the closest point to 0 in each cluster. More specifically, $\tau_0 = -1$ if $|X_{-1}| < X_1$ and $\tau_0 = 1$ otherwise, $(\tau_i)_{i\ge1}$ is the unique sequence of integers such that $X_{\tau_i} - X_{\tau_i - 1} > r$ and $X_k - X_{k-1} \le r$ for $\tau_{i-1} < k < \tau_i$, and, similarly, $(\tau_{-i})_{i\ge1}$ is a sequence of integers such that $X_{\tau_{-i}+1} - X_{\tau_{-i}} > r$ and $X_{k+1} - X_k \le r$ for $\tau_{-i-1} < k < \tau_{-i}$. Moreover, we call the cluster containing the

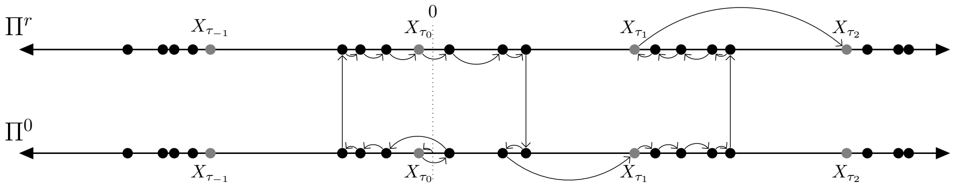

FIGURE 1. The points closest to 0 in each cluster form the sequence $(X_{\tau_i})_{i\in\mathbb{Z}}$. Once the greedy walk enters a cluster, it visits successively all the points of that cluster before it moves to the next cluster.

point X_{τ_i} cluster i. See Figure 1 for an example of clusters $-1, 0, 1$ and 2. The points $\{X_{\tau_{-1}}, X_{\tau_0}, X_{\tau_1}, X_{\tau_2}\} \times \{0, r\}$ are marked with gray colour.

The greedy walk starting from $(0,0)$ visits several points around $(0,0)$ and then moves to line r from one of the outermost points of cluster 0 on line 0. Then it visits all points of cluster 0 on line r and if there are points left it changes lines again to visit the remaining points on line 0 before moving to the next cluster. Later, when the greedy walk is in cluster i, $i \neq 0$, it always visits first a point at X_{τ_i}, that is $(X_{\tau_i}, 0)$ or (X_{τ_i}, r). Then the walk visits successively all the points of cluster i on the same line and it changes lines at the other outermost point of the cluster. Thereafter the walk visits the corresponding points on the other line in reverse order until it reaches a point at X_{τ_i}. Thus, the walk visits all points of cluster i consecutively and it ends at the starting position X_{τ_i}, but on the other line. Therefore, to know whether the walk visits all points, it is enough to know the positions of the points in cluster 0 and the position of the points at $X_{\tau_i}, i \neq 0$. Since the points of cluster 0 almost surely do not change the asymptotic behaviour of the walk, for the proof we look at the greedy walk $(\widetilde{S}_n)_{n\geq 0}$ on $(X_{\tau_i})_{i\in\mathbb{Z}}$ with 0 as starting point. Note that the distances $X_{\tau_{i+1}} - X_{\tau_i}$, $i \in \mathbb{Z} \setminus \{-1, 0\}$, are independent and identically distributed random variables with finite expectation.

To visit all points $(X_{\tau_i})_{i\in\mathbb{Z}}$, the greedy walk needs to cross over 0 infinitely many times. Let A_m be the event that the walk $(\widetilde{S}_n)_{n\geq 0}$ crosses 0 after visiting a point in the interval $(rm, r(m+1))$, that is

$$A_m = \{\exists\ n : rm \leq \widetilde{S}_n < r(m+1) \text{ and } \widetilde{S}_{n+1} < 0\}.$$

This can be written as

$$\begin{aligned} A_m &= \{\exists\ n, i : rm \leq X_{\tau_i} < r(m+1),\ \widetilde{S}_n = X_{\tau_i} \text{ and } \widetilde{S}_{n+1} < 0\} \\ &\subset \{\exists\ i : rm \leq X_{\tau_i} < r(m+1),\ X_{\tau_{i+1}} - X_{\tau_i} > rm\} \\ &= \bigcup_{i\geq 0} \{rm \leq X_{\tau_i} < r(m+1),\ X_{\tau_{i+1}} - X_{\tau_i} > rm\}. \end{aligned}$$

For $i \geq 1$, the random variable $X_{\tau_{i+1}} - X_{\tau_i}$ is independent of X_{τ_i} and it has the same distribution as $X_{\tau_2} - X_{\tau_1}$. Moreover, $X_{\tau_i} \in [rm, r(m+1))$ for at most one i. Thus, we have

$$\begin{aligned}
\mathbb{P}(A_m) &\leq \mathbb{P}(rm \leq X_{\tau_0} < r(m+1)) + \sum_{i=1}^{\infty} \mathbb{P}(rm \leq X_{\tau_i} < r(m+1),\ X_{\tau_{i+1}} - X_{\tau_i} > rm) \\
&\leq \mathbb{P}(rm \leq X_{\tau_0} < r(m+1)) + \sum_{i=1}^{\infty} \mathbb{P}(rm \leq X_{\tau_i} < r(m+1))\mathbb{P}(X_{\tau_2} - X_{\tau_1} > rm) \\
&= \mathbb{P}(rm \leq X_{\tau_0} < r(m+1)) + \mathbb{P}(X_{\tau_2} - X_{\tau_1} > rm) \sum_{i=1}^{\infty} \mathbb{E}\left(\mathbf{1}_{\{rm \leq X_{\tau_i} < r(m+1)\}}\right) \\
&\leq \frac{1}{2} e^{-2rm}(1 - e^{-2r}) + \mathbb{P}\left(X_{\tau_2} - X_{\tau_1} > rm\right).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{m=1}^{\infty} \mathbb{P}(A_m) \leq \sum_{m=1}^{\infty} \frac{1}{2} e^{-2rm}(1 - e^{-2r}) &+ \sum_{m=1}^{\infty} \mathbb{P}(X_{\tau_2} - X_{\tau_1} > rm) \\
< \frac{1}{2} e^{-2r} &+ \frac{1}{r}\mathbb{E}(X_{\tau_2} - X_{\tau_1}) < \infty.
\end{aligned}$$

Now the Borel-Cantelli lemma implies that

$$\mathbb{P}(A_m \text{ for infinitely many } m \geq 1) = 0.$$

Hence, the walk $(\widetilde{S}_n)_{n\geq 0}$ almost surely crosses 0 finitely many times. Therefore, also the walk $(S_n)_{n\geq 0}$ crosses the vertical line $\{0\} \times \mathbb{R}$ finitely many times and visits just finitely many points on one side of that line. □

4. Two parallel lines with thinned Poisson processes

Let $E = \mathbb{R} \times \{0, r\}$ and let d be the Euclidean distance. Moreover, let $0 < p \leq 1$. We define a point process Π on E as follows.

Let $\widehat{\Pi}$ be a homogeneous Poisson process with rate 1. Let Π^0 and Π^r be two (dependent, in general) thinnings of $\widehat{\Pi}$ generated as follows. For all $X \in \widehat{\Pi}$, do one of the following:

- With probability $1 - p$, duplicate the point X and make it a point of both processes Π^0 and Π^r.
- With probability p, assign X to either Π^0 or Π^r (but not both), with probability $1/2$ each.

Let now Π be the point process on E with $\Pi \cap (B \times \{0\}) = (\Pi^0 \cap B) \times \{0\}$ and $\Pi \cap (B \times \{r\}) = (\Pi^r \cap B) \times \{r\}$ for all Borel sets $B \subset \mathbb{R}$.

We study the greedy walk on Π defined by (1) that starts at $S_0 = (0, 0)$. Note that $(0, 0) \notin \Pi$ with probability 1. Sometimes in the proofs we consider the greedy walk starting from another point $x \in E$. We emphasise this by writing the superscript x in the sequence $(S_n^x)_{n\geq 0}$.

If $p = 0$, Π^0 and Π^r are identical. This was studied in Section 3 where we showed that the greedy walk does not visit all points of the process. In this section we consider $p > 0$ and we get the opposite result:

Theorem 3. *For any $p > 0$, the greedy walk visits all points of Π almost surely.*

For $p = 1$, the processes Π^0 and Π^r defined above are two independent Poisson processes with rate $1/2$. When $0 < p < 1$, the process Π has some "double" points and Π^0 and Π^r are not independent. However, for $p = 1$ and $p \in (0, 1)$ the behaviour of the walk is similar and, thus, we study these cases together.

Let us now introduce some notation and definitions that we use throughout this section. We call the projections of elements of E on their first coordinate *shadows* of the elements of E on $\mathbb{R}$ and we denote it by $\widehat{\cdot}$. For example, the shadow $\widehat{x}$ of a point $x = (x_1, x_2) \in E$ is x_1. The point process Π is defined from the process $\widehat{\Pi}$ that can be seen as a shadow of Π, that is, $x \in \widehat{\Pi}$ if and only if there exists $\iota \in \{0, r\}$ such that $(x, \iota) \in \Pi$. Also, $(\widehat{S}_n)_{n\geq 0}$ is the shadow of the greedy walk $(S_n)_{n\geq 0}$, that is, $(\widehat{S}_n)_{n\geq 0}$ contains just the information about the first coordinates of the locations of the walk.

Let $\Pi_n = \Pi \setminus \{S_0, S_1, S_2, \dots, S_{n-1}\}$ be the set of all points of Π that are not visited before time n. Let Π_n^0 and Π_n^r be the shadows of $\Pi_n \cap (\mathbb{R} \times \{0\})$ and $\Pi_n \cap (\mathbb{R} \times \{r\})$, respectively. Moreover, $\widehat{\Pi}_n$ denotes the shadow of Π_n, that is, the set of the first coordinates of the points of Π_n.

Define the *shift operator* $\theta_{(x,\iota)}$, $(x, \iota) \in E$, on Π, by $\theta_{(x,0)}\Pi = \big((\Pi^0 - x) \times \{0\}\big) \cup ((\Pi^r - x) \times \{r\})$ and $\theta_{(x,r)}\Pi = ((\Pi^r - x) \times \{0\}) \cup \big((\Pi^0 - x) \times \{r\}\big)$. Define also the *mirroring operator* σ by $\sigma\Pi = \big((-\Pi^0) \times \{0\}\big) \cup ((-\Pi^r) \times \{r\})$.

For any subset A of $\mathbb{R}$ define

$$T_A = \inf\{n \geq 1 : \widehat{S}_n \in A\}$$

to be the first time the shadow of the greedy walk enters A and write T_x for $T_{\{x\}}$. Let $T_x^R = \inf\{n \geq 0 : \widehat{\Pi}_{n+1}(x) = 0\}$, that is, T_x^R is the time when both points $(x, 0)$ and (x, r) are visited. If exactly one of $(x, 0)$ and (x, r) is in Π then $T_x^R = T_x$.

Note that for any $x > 0$ we have $T_{[x,\infty)} < \infty$ or $T_{(-\infty,0)} < \infty$ because $\widehat{\Pi}[0, x] < \infty$, almost surely, and the walk exits $[0, x] \times \{0, r\}$ in a finite time. Define now the variable D_x, $x \in \widehat{\Pi}$, $x > 0$, as follows. If $T_{(-\infty,0)} < T_{[x,\infty)}$ then $D_x = 0$. Otherwise, $T_{[x,\infty)} < T_{(-\infty,0)}$ and $0 < \widehat{S}_1, \widehat{S}_2, \dots, \widehat{S}_{T_{[x,\infty)}-1} < x$, $\widehat{S}_{T_{[x,\infty)}} \geq x$. We can label the remaining points of $\widehat{\Pi}_{T_{[x,\infty)}}$ in the interval $(0, x)$ by $z_1, z_2, \dots, z_{n-1}$ so that

$$0 = z_n < z_{n-1} < \cdots < z_1 < z_0 = x. \tag{2}$$

Let then

$$D_x = \max_{0\leq i\leq n-1} \{(z_i - z_{i+1}) - (x - z_i)\} = \max_{0\leq i\leq n-1} \{2z_i - z_{i+1} - x\}. \tag{3}$$

Since $2z_0 - z_1 - x = x - z_1 > 0$ and $2z_i - z_{i+1} - x \leq 2z_i - x \leq x$ for $0 \leq i \leq n-1$, we have $0 < D_x \leq x$. The variable D_x measures how big should be the distance between x and the closest point to x in $\widehat{\Pi} \cap (x, \infty)$, so that the walk possibly visits a point in $(-\infty, 0) \times \{0, r\}$ before visiting any point in $(x, \infty) \times \{0, r\}$.

We prove Theorem 3 in a similar way as Rolla *et al.* [7] prove that the greedy walk visits all marks attached to the points of a homogeneous Poisson process on $\mathbb{R}$, where each point has one mark with probability p or two marks with probability $1-p$. The idea of the proof is the following. We define first a subset Ξ of Π^0, which is stationary and ergodic. Then, using the definition and properties of Ξ, we are able to show that there exists $d_0 > 0$ such that, almost surely, $D_X < d_0$ for infinitely many $X \in \widehat{\Pi}$. This we use to show that events A_k, which we define later in (6), occur for infinitely many $k > 0$, almost surely. Then we show that whenever A_k occurs, the greedy walk visits $(-\infty, 0) \times \{0, r\}$ in a finite time. Therefore, we can conclude that $T_{(-\infty,0)} < \infty$, almost surely. Finally, using repeatedly the fact that $T_{(-\infty,0)} < \infty$, almost surely, we are able to show that the greedy walk on Π crosses the vertical line $\{0\} \times \mathbb{R}$ infinitely many times and, thus, almost surely visits all points of Π.

Let us first discuss general properties of the greedy walk on E that do not depend on the definition of the point process Π. If the greedy walk visits two points (x, ι) and (y, ι) on a line $\iota \in \{0, r\}$, without changing lines in between those visits, then it visits also all the points between x and y on line ι. So the walk clears some intervals on the lines, but because of changing the lines it possibly leaves some unvisited points between those intervals. The following lemma shows that if the walk omits two points on different lines between visited intervals, then the horizontal distance between those points is greater than r. Moreover, looking from a point that is to the right of those two points, the closer point is always the point that is more to the right. Thus, when the walk returns to a partly visited set, it always visits first the rightmost remaining point in this set.

Lemma 1. *Let $a = \min_{0\le i\le n} \widehat{S}_i$ and $b = \max_{0\le i\le n} \widehat{S}_i$.*

(a) *Let $a \le x < y \le b$ such that $(x,\iota_x),(y,\iota_y) \in \Pi_{n+1}$ and $\iota_x \ne \iota_y$. Then $y-x > r$.*
(b) *Let $a \le x < y \le z \le b$ such that $(x,\iota_x),(y,\iota_y) \in \Pi_{n+1}$, $(z,\iota_z) \in \Pi_n$. Then $d((y,\iota_y),(z,\iota_z)) < d((x,\iota_x),(z,\iota_z))$.*

Proof. (a) Let x, y be as in the lemma and suppose on the contrary that $y - x \le r$. Let $I_1 = ((a,x)\times\{\iota_x\})\cup((a,y)\times\{\iota_y\})$ and $I_2 = ((x,b)\times\{\iota_x\})\cup((y,b)\times\{\iota_y\})$. Since the greedy walk has visited points at a and b up to time n, but not the points (x,ι_x) and (y,ι_y), the walk before time n moved from I_1 to I_2 or in the opposite direction. Because of the assumption $y - x \le r$, a point in I_1 is closer to (x,ι_x) or (y,ι_y) than to any point in I_2, and conversely. Thus, if $y - x \le r$ it is impossible to visit a point at a and a point at b without visiting either (x,ι_x) or (y,ι_y), which contradicts the assumptions of the lemma.

(b) If $\iota_y = \iota_z$ then $d((y,\iota_y),(z,\iota_z)) = z - y < z - x \le d((x,\iota_x),(z,\iota_z))$. If $\iota_y \ne \iota_z$ and $\iota_x = \iota_y$ then $d((y,\iota_y),(z,\iota_z))^2 = (z-y)^2 + r^2 < (z-x)^2 + r^2 \le d((x,\iota_x),(z,\iota_z))^2$. If $\iota_y \ne \iota_z$ and $\iota_x \ne \iota_y$, it follows from part (a) of this lemma that $y - x > r$. This yields

$$\begin{aligned} d((y,\iota_y),(z,\iota_z))^2 &= (z-y)^2 + r^2 < (z-y+r)^2 \\ &< (z-y+y-x)^2 = d((x,\iota_x),(z,\iota_z))^2. \end{aligned}$$ □

Let $(c,\iota_c) \in \Pi$, $c > 0$, be a point far enough from $(0,0)$. If (c,ι_c) is not visited when the walk goes for the first time from $(-\infty,c)\times\{0,r\}$ to $(c,\infty)\times\{0,r\}$, then this point is visited before the walk returns to $(-\infty,c)\times\{0,r\}$. Furthermore, as we show in the following lemma, when the walk is finally at the location (c,ι_c), then there is no unvisited points left in $(c,\max_{0\le i\le T_c^R}\widehat{S}_i]\times\{0,r\}$. If (c,ι_c) is close to $(0,0)$ and $\iota_c = r$, then this is not always true because the greedy walk might jump several times over $(0,0)$ (and $(c,0)$) without visiting any point on line r.

Lemma 2. *Let $a = \min_{0\le i\le n}\widehat{S}_i$ and $b = \max_{0\le i\le n}\widehat{S}_i$. Let $a \le c \le b$, $\iota_c \in \{0,r\}$, such that $(c,\iota_c) \in \Pi_{n+1}$. If $\widehat{S}_n \ge c$ and (c,ι_c) is visited before any other point of Π_n in $(-\infty,c)\times\{0,r\}$, then all points of Π in $(c,\max_{0\le i\le T_c^R}\widehat{S}_i]\times\{0,r\}$ are visited before time T_c^R.*

Proof. Let us denote $\max_{0\le i\le T_c^R}\widehat{S}_i$ by M. It is easy to see that the lemma is true for $c = M$. Thus, assume that $M > c$ and let j be the first time for which $\widehat{S}_j = M$. Then, for $j+1 \le k \le T_c^R - 1$, S_k is in $(c,M]\times\{0,r\}$.

Let x be the rightmost point of $\widehat{\Pi}_{j+1}$ in the interval $[c,M]$ and let ι_x be the corresponding line of that point. Note that at time $j+1$ there is only one point left at x. If $x = c$, the claim of the lemma follows directly. Otherwise, by Lemma 1 (b), the point (x,ι_x) is closer to S_j than any other point in $[c,x)\times\{0,r\}$. Thus, $S_{j+1} = (x,\iota_x)$ and there are no points left in $[x,M]\times\{0,r\}$ after time $j+1$. Repeating the same arguments we can see that the walk successively visits all the remaining points in $[c,M]\times\{0,r\}$ until it reaches c. Hence, at time T_c^R the set $(c,M]\times\{0,r\}$ is empty. □

To show that $D_X < \infty$ for infinitely many $X \in \widehat{\Pi}$, the crucial will be the subset Ξ of Π^0 which we define as follows. For $X \in \Pi^0$, let $L^0(X) = \max\{Y \in \Pi^0 : Y < X\}$. Then let

$$\begin{aligned}\Xi = \{X \in \Pi^0 :\ & S_n^{(L^0(X),0)} \in \big([L^0(X), \infty) \times \{0, r\}\big) \setminus \{(X, 0)\} \text{ for all } n \geq 1 \text{ and} \\ & S_n^{(L^0(X),r)} \in \big([L^0(X), \infty) \times \{0, r\}\big) \setminus \{(X, 0)\} \text{ for all } n \geq 1\},\end{aligned}$$

that is, Ξ contains all points $X \in \Pi^0$ such that if we start a walk from $(L^0(X), 0)$ or $(L^0(X), r)$, then the walk stays always in $[L^0(X), \infty) \times \{0, r\}$ and it never visits $(X, 0)$. We have defined Ξ in this way because then for every $X \in \Xi$, whenever the greedy walk approaches $(X, 0)$ and (X, r) from their left, the walk passes around $(X, 0)$ and it never comes back to visit $(X, 0)$. Thus, if Ξ is non-empty, there are some points of Π that are never visited.

The point process Π is generated from the homogeneous Poisson process $\widehat{\Pi}$ and thus it is stationary and ergodic. The process Ξ is defined as a function of the points in Π and therefore Ξ is a stationary and ergodic process in $\mathbb{R}$. Therefore, Ξ is almost surely the empty set or it is almost surely a non-empty set, in which case Ξ has a positive rate. We look at these two cases in the next two lemmas.

For the first lemma we need the random variable $W_{(x,\iota)}$, $x \in \mathbb{R}$, $\iota \in \{0, r\}$, which measures a sufficient horizontal distance from (x, ι) to the rightmost point in the set $(-\infty, x) \times \{0, r\}$, so that the greedy walk starting from (x, ι) never visits that set. For $x \in \mathbb{R}$ let $\Pi^x = \Pi \cap ([x, \infty) \times \{0, r\})$. Consider for the moment the greedy walk on Π^x starting from (x, ι) defined by (1) and let

$$W_{(x,\iota)} = \inf_{i \geq 0} \{\widehat{S}_i - d(S_i, S_{i+1})\}.$$

Let $L^0(x) = \max\{Y \in \Pi^0 : Y < x\}$ and $L^r(x) = \max\{Y \in \Pi^r : Y < x\}$. If for some $x \in \mathbb{R}$, $\iota \in \{0, r\}$, $|W_{(x,\iota)}| < \infty$ and $\max\{L^0(x), L^r(x)\} < W_{(x,\iota)}$, then for all $i \geq 0$

$$\begin{aligned}\min\{d(S_i, (L^0(x), 0)), d(S_i, (L^r(x), r))\} &\geq \widehat{S}_i - \max\{L^0(x), L^r(x)\} \\ &> \widehat{S}_i - W_{(x,\iota)} \geq d(S_i, S_{i+1}).\end{aligned}$$

Therefore, the walk on Π starting from (x, ι) coincides with the walk on Π^x, because for every $i \geq 0$ the point S_i is closer to S_{i+1} than to any point in $\Pi \setminus \Pi^x$. Thus, the walk does not visit $\Pi \setminus \Pi^x$. The opposite is also true, i.e. if the walk on Π starting from (x, ι) coincides with the walk on Π^x, then $|W_{(x,\iota)}| < \infty$.

Lemma 3. *If Ξ is almost surely the empty set, then $\Psi = \{(X, \iota) \in \Pi : |W_{(X,\iota)}| < \infty\}$ is also almost surely the empty set.*

Proof. For $x \in \mathbb{R}$, let $L^0(x)$ and $L^r(x)$ be as above. Since $W_{(X,\iota)}$ is identically distributed for all $(X, \iota) \in \Pi$, Ψ is stationary and ergodic. Suppose, on the contrary, that Ψ is a non-empty set. For $d > 0$ let $\Psi_d = \{(X, r) \in \Psi : |W_{(X,r)}| < d\}$ and note that $\bigcup_{d>0} \Psi_d = \Psi \cap (\mathbb{R} \times \{r\})$. Thus, there exists d large enough so that Ψ_d is a non-empty set which is stationary and ergodic. Let $\widetilde{\Psi}_d$ be the set of all $(X, r) \in \Psi_d$ such that the points $(L^0(X), 0)$, $(L^r(X), r)$ and $(L^0(L^0(X)), 0)$ satisfy the following. First, these points are in $(-\infty, X - d) \times \{0, r\}$. Second, the point $(L^r(X), r)$ is closer to (X, r) than to $(L^0(X), 0)$. Third, the greedy walks starting from $(L^0(L^0(X)), 0)$ and $(L^0(L^0(X)), r)$ visit only the points in $[L^0(L^0(X)), \infty) \times \{0, r\}$ and these walks visit $(L^r(X), r)$ before visiting $(L^0(X), 0)$. Since these three conditions have a positive probability, which is independent of $W_{(X,r)}$, $\widetilde{\Psi}_d$ is also almost surely a non-empty set. But, by definition of Ξ, for every $(X, r) \in \widetilde{\Psi}_d$, $L^0(X) \in \Xi$ and Ξ is a non-empty set, which is a contradiction. $\square$

In the following lemma we use the random variable $\widetilde{D}_X$, $X \in \Pi_0$, which can be compared to D_X defined in (3). For $X \in \Pi^0 \setminus \Xi$ set $\widetilde{D}_X = 0$. For $X \in \Xi$ denote the points of $\Xi \cap (\infty, X]$ in decreasing order

$$\cdots < \widetilde{z}_2 < \widetilde{z}_1 < \widetilde{z}_0 = X$$

and define $\widetilde{D}_X$ as

$$\widetilde{D}_X = \sup_{i \geq 0} \{2\widetilde{z}_i - \widetilde{z}_{i+1} - X\}. \tag{4}$$

For $d_1, d_2 > 0$ define

$$\Xi_{d_1,d_2} = \{X \in \Xi : \widetilde{D}_X < d_1 \text{ and there exists } Y \in \widehat{\Pi} \text{ such that } 0 < Y - X < d_2 \text{ and } \widehat{\Pi} \cap (Y, Y + r) = \emptyset\},$$

i.e. $X \in \Xi$ belongs to Ξ_{d_1,d_2} if $\widetilde{D}_X < d_1$ and at the distance less than d_2 from X there is an interval of length r where there is no points of $\widehat{\Pi}$. We consider the empty intervals of length r in Π, because whenever $\widehat{\Pi} \cap (Y, Y + r) = \emptyset$ for some $Y \in \widehat{\Pi}$, $Y > 0$, the greedy walk is forced to visit the points $(Y, 0)$ and (Y, r), before crossing the interval and visiting a point in $[Y + r, \infty) \times \{0, r\}$.

Lemma 4. *If Ξ is almost surely a non-empty set, then there exist $d_1, d_2 > 0$ such that Ξ_{d_1,d_2} is almost surely a non-empty set. Moreover, Ξ_{d_1,d_2} is a stationary and ergodic process.*

Proof. If Ξ is non-empty, the rate δ of Ξ is positive. Then for $X \in \Xi$, $\lim_{i\to\infty} \frac{X - \widetilde{z}_i}{i} = \delta^{-1}$, almost surely. Also, $\lim_{i\to\infty} \frac{\widetilde{z}_i - \widetilde{z}_{i+1}}{i} = 0$, almost surely. Hence, for all large i, $2\widetilde{z}_i - \widetilde{z}_{i+1} - X = (\widetilde{z}_i - \widetilde{z}_{i+1}) - (X - \widetilde{z}_i) < 0$ and we can conclude that $\widetilde{D}_X$ is almost surely finite. Therefore, there exists d_1 such that $\{X \in \Pi^0 : X \in \Xi, \ \widetilde{D}_X < d_1\}$ is with positive probability a non-empty set. Moreover, since $\widetilde{D}_X$ is identically distributed for all $X \in \Xi$, $\{X \in \Pi^0 : X \in \Xi, \ \widetilde{D}_X < d_1\}$ is a stationary and ergodic process with positive rate.

Almost surely, the gap between two neighbouring points of the homogeneous Poisson process $\widehat{\Pi}$ is infinitely often greater than r. Thus, also $\Pi \cap ((Y, Y + r) \times \{0, r\}) = \emptyset$ for infinitely many $Y \in \widehat{\Pi}$ and all such Y form a stationary and ergodic process. Thus, Ξ_{d_1,d_2} is also stationary and ergodic for all $d_2 > 0$. Since $\bigcup_{d_2>0} \Xi_{d_1,d_2} = \{X \in \Pi^0 : X \in \Xi, \ \widetilde{D}_X < d_1\}$ and this is not empty when d_1 is large enough, we can choose d_2 large enough so that Ξ_{d_1,d_2} is almost surely a non-empty set. □

We study the greedy walk starting from the point $(0, 0)$, which is, almost surely, not a point of Π. From now on denote the points of $\widehat{\Pi}$ by

$$\ldots < X_{-2} < X_{-1} \leq 0 < X_1 < X_2 < \ldots$$

and denote the points of Π^0 and Π^r by

$$\ldots < X^0_{-2} < X^0_{-1} \leq 0 < X^0_1 < X^0_2 < \ldots \text{and} \ldots < X^r_{-2} < X^r_{-1} \leq 0 < X^r_1 < X^r_2 < \ldots,$$

respectively.

Now we are ready to prove that $D_X < d_0$ for infinitely many $X \in \widehat{\Pi}$. We use here the definition of Ξ and divide the proof in two parts, depending weather Ξ is almost surely the empty set or a non-empty set.

Lemma 5. *There exists $d_0 < \infty$ such that, almost surely, $D_{X_k} < d_0$ and $X_{k+1} - X_k > r$ for infinitely many $k > 0$.*

Proof. Assume first that Ξ is almost surely the empty set. Then, by Lemma 3, $\Psi = \{(X,\iota) \in \Pi : |W_{(X,\iota)}| < \infty\}$ is almost surely the empty set. Moreover, $\{(X,0) \in \Pi : X \notin \Pi^r, |W_{(X,0)}| < \infty\}$ is also the empty set. The greedy walk $(S_n^{(0,0)})_{n\geq 0}$ on $\Pi \cap ([0,\infty) \times \{0,r\})$ has the same law as the greedy walk $(S_n^{(X,0)})_{n\geq 0}$ on $\Pi \cap ([X,\infty) \times \{0,r\})$ shifted for $(X,0)$, where $X \in \Pi^0, X \notin \Pi^r$. This implies that $\mathbb{P}(|W_{(0,0)}| < \infty) = 0$ and the greedy walk $(S_n^{(0,0)})_{n\geq 0}$ almost surely visits a point in $(-\infty,0) \times \{0,r\}$ in a finite time. Then it follows from the definition of D_x that $D_x = 0$ for all large enough $x > 0$. Since $X_{k+1} - X_k > r$ for infinitely many $k > 0$, the claim of the lemma holds for any $d_0 > 0$.

Assume now that Ξ is not empty. Then, by Lemma 4, we can find d_1 and d_2 large enough so that Ξ_{d_1,d_2} is a non-empty set. We first show that $D_X \leq d_1$ for infinitely many $X \in \Xi$, $X > 0$, and then we prove that $D_{X_k} < d_1 + d_2$ and $X_{k+1} - X_k > r$ for infinitely many $k > 0$.

For $X \in \Xi$, $X > 0$, denote the points of Ξ in $[0,X]$ by $\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}-1}$ so that

$$0 = \widetilde{z}_{\widetilde{n}} < \widetilde{z}_{\widetilde{n}-1} < \cdots < \widetilde{z}_1 < \widetilde{z}_0 = X$$

and define $\widetilde{D}'_X$, an analogue of D_X and a restricted version of $\widetilde{D}_X$, as

$$\widetilde{D}'_X = \max_{0\leq i\leq \widetilde{n}-1}\{2\widetilde{z}_i - \widetilde{z}_{i+1} - X\} = \max\left\{\max_{0\leq i\leq \widetilde{n}-2}\{2\widetilde{z}_i - \widetilde{z}_{i+1} - X\}, 2\widetilde{z}_{\widetilde{n}-1} - X\right\}. \tag{5}$$

Let $\xi = \min\{Y \in \Xi :\ Y > 0\}$ and note that $z_{\widetilde{n}-1} = \xi$ for every $X \in \Xi$, $X > 0$. Then for $X \in \Xi$, $X > 2\xi$ we have $2z_{\widetilde{n}-1} - X = 2\xi - X < 0$. Since $\widetilde{D}'_X \geq 2\widetilde{z}_0 - \widetilde{z}_1 - X = X - \widetilde{z}_1 > 0$, the term $2z_{\widetilde{n}-1} - X$ does not contribute to $\widetilde{D}'_X$. From the definition of Ξ_{d_1,d_2}, we have $\widetilde{D}_X < d_1$ for infinitely many $X \in \Xi$, almost surely. When $X > 2\xi$, $\widetilde{D}'_X$ is the maximum of a finite subset of the values in (4) and thus $\widetilde{D}'_X \leq \widetilde{D}_X < d_1$ for infinitely many $X \in \Xi$, $X > 2\xi$.

We prove now that $D_X \leq \widetilde{D}'_X$ in two steps. First, we show that the points used in the definition of $\widetilde{D}'_X$ are a subset of the points used in the definition of D_X. Second, we show that adding a new point to the definition (5) decreases the value of the maximum. Let $\xi \in \Xi$, $\xi > 0$. Before visiting any point in $[\xi,\infty) \times \{0,r\}$, the greedy walk starting from $(0,0)$ visits the leftmost point on one of the lines in $[L^0(\xi),\infty) \times \{0,r\}$, where $L^0(\xi) = \max\{Y \in \Pi^0 : Y < \xi\}$. That is, the greedy walk visits $(L^0(\xi),0)$ or the closest point to the right of $(L^0(\xi),r)$ on the line r (which is $(L^0(\xi),r)$ if that point exists). By the definition of Ξ, the greedy walk starting from one of these two points never visits $(\xi,0)$ and it never visits any point to the left of $\{L^0(\xi)\} \times \{0,r\}$. Therefore, once the greedy walk starting from $(0,0)$ enters $[L^0(\xi),\infty) \times \{0,r\}$, it continues on the path of one of these two walks. Thus, the greedy walk starting from $(0,0)$ does not visit $(\xi,0)$. From this we can conclude that all points in $\{\Xi \cap (0,\infty)\} \times \{0\}$ are not visited by the greedy walk and for $X \in \Xi \cap (0,\infty)$ we have $\{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\} \subset \{z_0, z_1, \ldots, z_n\}$, where $z_0, z_1, \ldots, z_n$ are as in (2).

Let $Y \in \{z_0, z_1, \ldots, z_n\} \backslash \{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\}$ and find j such that $\widetilde{z}_j < Y < \widetilde{z}_{j-1}$. Adding Y to the set $\{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\}$ in the definition (5), removes the value $2\widetilde{z}_{j-1} - \widetilde{z}_j - X$ and adds the values $2\widetilde{z}_{j-1} - Y - X$ and $2Y - \widetilde{z}_j - X$. Since, $2\widetilde{z}_{j-1} - Y - X < 2\widetilde{z}_{j-1} - \widetilde{z}_j - X$ and $2Y - \widetilde{z}_j - X < 2\widetilde{z}_{j-1} - \widetilde{z}_j - X$, the point at Y added to $\{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\}$ decreases the value of $\widetilde{D}'_X$ or leaves it unchanged. Since both D_X and $\widetilde{D}'_X$ are defined on $\{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\}$, but D_X has also points $\{z_0, z_1, \ldots, z_n\} \setminus \{\widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_{\widetilde{n}}\}$ in the definition, we can conclude that $D_X \leq \widetilde{D}'_X$ for all $X \in \Xi$, $X > 0$. Thus $D_X < d_1$ for infinitely many $X \in \Xi$, $X > 0$.

This together with Lemma 4 implies that for infinitely many $X \in \Xi \cap (0,\infty)$, $D_X < d_1$ and there exists $Y \in \widehat{\Pi}$ such that $0 < Y - X < d_2$ and $\Pi((Y, Y+r) \times \{0,r\}) = 0$. Choose one such X and let k be such that $X_k - X < d_2$ and $X_{k+1} - X_k > r$. If $T_{(-\infty,0)} < T_{[X_k,\infty)}$, then $D_{X_k} = 0 < d_1 + d_2$. Otherwise, $T_{[X_k,+\infty)} < T_{(-\infty,0)}$, and we can denote the points of $\widehat{\Pi}_{T_{[X_k,\infty)}}$ in $(0, X_k)$ by $z_1, z_2, \ldots, z_{n-1}$ so that $0 = z_n < z_{n-1} < \cdots < z_1 < z_0 = X_k$. By the definition of Ξ, the point $(X, 0)$ is never visited by the walk and thus there exists j such that $z_j = X$. Then we have

$$
\begin{aligned}
D_{X_k} &= \max_{0\le i\le n-1} \{2z_i - z_{i+1} - X_k\} \\
&\le \max\{\max_{j\le i\le n-1} \{2z_i - z_{i+1} - X\} - (X_k - X), \max_{0\le i\le j-1} \{2z_i - z_{i+1} - X_k\}\} \\
&\le \max\{D_X, X_k - X\} \le D_X + X_k - X < d_1 + d_2,
\end{aligned}
$$

where in the second inequality we use the fact that, by the definition of the points in Ξ, the walk does not visit any point in $(0, X) \times \{0, r\}$ after time $T_{[X,\infty)}$ and therefore $D_X = \max_{j\le i\le n-1} \{2z_i - z_{i+1} - X\}$.

Let now $d_0 = d_1 + d_2$. Since there are, almost surely, infinitely many $X \in \Xi$ and k such that $D_X < d_1$, $X_k - X < d_2$ and $X_{k+1} - X_k > r$, it follows that $D_{X_k} < d_0$ and $X_{k+1} - X_k > r$ for infinitely many $k > 0$, almost surely, which proves the claim of the lemma. □

Since $D_{X_k} < d_0$ for infinitely many k, almost surely, one should expect that also $X_{k+1} - X_k > d_0 > D_{X_k}$ for infinitely many k. That is exactly what we show next, but let us first state the extended Borel–Cantelli Lemma which we use in the proof.

Lemma 6 (Extended Borel–Cantelli lemma, [5, Corollary 6.20])**.** *Let $\mathcal{F}_n$, $n \ge 0$, be a filtration with $\mathcal{F}_0 = \{0, \Omega\}$ and let $A_n \in \mathcal{F}_n, n \ge 1$. Then a.s.*

$$\{A_n \text{ i.o.}\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}[A_n \mid \mathcal{F}_{n-1}] = \infty \right\}.$$

Lemma 7. *Almost surely, the events*

$$A_k = \{X_{k+1} - X_k > D_{X_k} - X_{-1} + r\} \tag{6}$$

occur for infinitely many $k > 0$.

Proof. For $k > 0$ let $j_k^0 = \max\{i : X_i^0 \le X_k\}$ and $j_k^r = \max\{i : X_i^r \le X_k\}$. Furthermore, define the σ-algebra

$$\mathcal{F}_k = \sigma((X_{-1}^0, 0), (X_0^0, 0), \ldots, (X_{j_k^0}^0, 0), (X_{-1}^r, r), (X_0^r, r), (X_1^r, r), \ldots, (X_{j_k^r}^r, r))$$

and denote by T_A^σ and D_x^σ analogues of T_A and D_x for the greedy walk on the set of points which generates $\mathcal{F}_k$. Assume $T_{[X_k,\infty)} < T_{(-\infty,0)}$. Then, the greedy walk on Π and the walk on the restricted set are the same until time $T_{[X_k,\infty)}$. If $X_{k+1} - X_k > r$ then $\widehat{S}_{T_{[X_k,\infty)}} = X_k$, $T_{X_k}^\sigma = T_{[X_k,\infty)}$ and $D_{X_k}^\sigma = D_{X_k}$.

Let $A_k^\sigma = \{X_{k+1} - X_k > D_{X_k}^\sigma - X_{-1} + r\}$ and observe that $A_k^\sigma \in \mathcal{F}_{k+1}$. For $d_0 > 0$ we have

$$
\begin{aligned}
\mathbb{P}\left(A_k^\sigma \mid \mathcal{F}_k\right) &\ge \mathbb{P}\left(D_{X_k}^\sigma < d_0, X_{k+1} - X_k > d_0 - X_{-1} + r \mid \mathcal{F}_k\right) \\
&= \mathbf{1}_{\{D_{X_k}^\sigma < d_0\}} \mathbb{P}\left(X_{k+1} - X_k > d_0 - X_{-1} + r \mid \mathcal{F}_k\right) \\
&= \mathbf{1}_{\{D_{X_k}^\sigma < d_0\}} e^{-(d_0 - X_{-1} + r)} \quad \text{a.s.}
\end{aligned}
$$

The first equality above holds because $\{D_{X_k}^\sigma < d_0\} \in \mathcal{F}_k$. The second equality follows from the facts that $X_{-1} \in \mathcal{F}_k$ and $X_{k+1} - X_k$ is exponentially distributed with mean

1 and independent of $\mathcal{F}_k$. By Lemma 5, there exists d_0 such that $D_{X_k} < d_0$ and $X_{k+1} - X_k > r$ for infinitely many k, almost surely. Since, $D^\sigma_{X_k} = D_{X_k}$ whenever $X_{k+1} - X_k > r$, also $D^\sigma_{X_k} < d_0$ for infinitely many k and, thus,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(A^\sigma_k \mid \mathcal{F}_k\right) = \infty \quad \text{a.s.}$$

It follows now from the extended Borel-Cantelli lemma (Lemma 6) that

$$\mathbb{P}(A^\sigma_k \text{ for infinitely many } k \geq 1) = 1.$$

Since $A^\sigma_k \subset \{X_{k+1} - X_k > r\}$ and $A_k = A^\sigma_k$ whenever $X_{k+1} - X_k > r$, also

$$\mathbb{P}(A_k \text{ for infinitely many } k \geq 1) = 1. \qquad \square$$

Whenever A_k occurs, as we show in the next lemma, the greedy walk is forced to visit $(-\infty, 0) \times \{0, r\}$ before visiting $[X_{k+1}, \infty) \times \{0, r\}$. Note that the arguments do not depend on the definition of the point process Π.

Lemma 8. *Almost surely, $T_{(-\infty,0)} < \infty$.*

Proof. By Lemma 7, the events $A_k = \{X_{k+1} - X_k > D_{X_k} - X_{-1} + r\}$ occur for infinitely many k, almost surely. To prove the lemma, it suffices to prove that whenever A_k occurs, then $T_{(-\infty,0)} < T_{[X_{k+1},\infty)}$. Because the walk exits $[0, X_{k+1}] \times \{0, r\}$ in, almost surely, finite time, it follows that $T_{(-\infty,0)} < \infty$, almost surely.

Assume that $T_{[X_k,\infty)} < T_{(-\infty,0)}$ and A_k occurs for some k. Then $X_{k+1} - X_k > D_{X_k} - X_{-1} + r > r$ and a point in $(0, X_k) \times \{0, r\}$ is closer to a point at X_k than to any point in $[X_{k+1}, \infty) \times \{0, r\}$. Hence, $\widehat{S}_{T_{[X_k,\infty)}} = X_k$.

Denote the remaining points of $\widehat{\Pi}_{T_{[X_k,\infty)}}$ in the interval $[0, X_k]$ as in (2). Note that at time $T_{[X_k,\infty)}$ there is exactly one unvisited point left at each position $z_1, z_2, \ldots, z_{n-1}$ and denote by $\iota_1, \iota_2, \ldots, \iota_{n-1}$ the corresponding lines of these points. If there is only one point at X_k, let ι_0 be the line of this point. If there are two points at X_k, let ι_0 be the line of the point that is not visited at time T_{X_k}. The point $S_{T_{X_k}}$ is closer to the second point at X_k, if such exists, than to any point in $(X_{k+1}, \infty) \times \{0, r\}$ because $X_{k+1} - X_k > r$ or to any of the remaining points with shadows at $z_1, z_2, \ldots, z_{n-1}$ because of Lemma 1 (b).

For $i = 0, 1, \ldots, n-2$, from the definition of D_{X_k} (3) we have $D_{X_k} \geq 2z_i - z_{i+1} - X_k$ and

$$\begin{aligned}
d((z_i, \iota_i), (z_{i+1}, \iota_{i+1}))^2 &\leq (z_i - z_{i+1})^2 + r^2 < (z_i - z_{i+1} + r)^2 \leq (D_{X_k} + X_k - z_i + r)^2 \\
&< (D_{X_k} - X_{-1} + r + X_k - z_i)^2 < (X_{k+1} - z_i)^2 \\
&\leq d((z_i, \iota_i), (X_{k+1}, \iota_i))^2.
\end{aligned}$$

Thus the point (z_i, ι_i) is closer to (z_{i+1}, ι_{i+1}) than to any point in $[X_{k+1}, \infty) \times \{0, r\}$. Moreover, by Lemma 1 (b), (z_{i+1}, ι_{i+1}) is closer to (z_i, ι_i) than any other point in $[0, z_{i+1}) \times \{0, r\}$. Thus, when the walk is at (z_i, ι_i) it visits (z_{i+1}, ι_{i+1}) next, except if $z_i < r$ and there is a point at $(-\infty, 0) \times \{0\}$ which is closer. In the latter case we have $T_{(-\infty,0)} < T_{[X_{k+1},\infty)}$.

Assume that the walk visits successively the points at $z_1, z_2, \ldots, z_{n-1}$. Hence, all points in $(z_{n-1}, X_{k+1}) \times \{0, r\}$ are visited. When the greedy walk is at (z_{n-1}, ι_{n-1}), the closest unvisited point is in $(-\infty, 0) \times \{0, r\}$, because a point with shadow X_{-1} is closer to (z_{n-1}, ι_{n-1}) than any point in $[X_{k+1}, \infty) \times \{0, r\}$. This follows from

$$
\begin{aligned}
d((z_{n-1}, \iota_{n-1}), (X_{-1}, r - \iota_{n-1}))^2 &= (z_{n-1} - X_{-1})^2 + r^2 \\
&\le (D_{X_k} + X_k - z_{n-1} - X_{-1})^2 + r^2 \\
&< (D_{X_k} + X_k - z_{n-1} - X_{-1} + r)^2 < (X_{k+1} - z_{n-1})^2 \\
&= d((z_{n-1}, \iota_{n-1}), (X_{k+1}, \iota_{n-1}))^2,
\end{aligned}
$$

where in the first inequality above we use that, by the definition of D_{X_k} (3), $D_{X_k} \ge 2z_{n-1} - X_k$. Thus, the walk visits $(-\infty, 0) \times \{0, r\}$ next. Therefore, also now $T_{(-\infty,0)} < T_{[X_{k+1},\infty)}$, which completes the proof. □

Now we are ready to prove Theorem 3 where we use Lemma 8 repeatedly to show that the greedy walk crosses the vertical line $\{0\} \times \mathbb{R}$ infinitely often. Therefore the greedy walk visits all the points of Π.

Proof of Theorem 3. From Lemma 8 we have $\mathbb{P}(T_{(-\infty,0)} < \infty) = 1$, which is equivalent to

$$
\mathbb{P}\left(T_{(-\infty,0)} < \infty \mid X_{-1}^0, X_{-1}^r, X_1^0, X_1^r\right) = 1 \quad \text{a.s.} \tag{7}
$$

Moreover, the conditional probability (7) is almost surely 1 for any absolutely continuous distribution of $(X_{-1}^0, X_{-1}^r, X_1^0, X_1^r)$ on $(-\infty, 0)^2 \times (0, \infty)^2$, which is independent of $\Pi \cap \big(((X_1^0, \infty) \times \{0\}) \cup ((X_1^r, \infty) \times \{r\})\big)$.

Let $T_0 = 0$ and let T_i, $i \ge 1$, be the first time the greedy walk visits a point in $(-\infty, \min_{0 \le n \le T_{i-1}} \widehat{S}_n) \times \{0, r\}$ for i odd and in $(\max_{0 \le n \le T_{i-1}} \widehat{S}_n, \infty) \times \{0, r\}$ for i even. That is, for i odd (even) T_i is the time when the walk visits the part of Π on the left (right) of $\{0\} \times \mathbb{R}$ which is unvisited up to time T_{i-1}. We prove first that T_i is almost surely finite for all i and, thus, the greedy walk, almost surely, crosses the vertical line $\{0\} \times \mathbb{R}$ infinitely many times. Then we show that it is not possible that a point of Π is never visited, and thus the walk almost surely visits all points of Π.

Assume that T_i is finite for some even i. Let $Y_0 = \widehat{S}_{T_i}$, $Y_1^0 = \min\{Y \in \Pi^0 : Y > Y_0\}$ and $Y_1^r = \min\{Y \in \Pi^r : Y > Y_0'\}$. Furthermore, let $Y_{-1}^0 = \max\{Y \in \Pi^0 : Y < \min_{0 \le n \le T_i} \widehat{S}_n\}$ and $Y_{-1}^r = \max\{Y \in \Pi^r : Y < \min_{0 \le n \le T_i} \widehat{S}_n\}$. By the definition of Y_{-1}^0 and Y_{-1}^r, at time T_i the set $I_1 = (-\infty, Y_{-1}^0] \times \{0\} \cup (-\infty, Y_{-1}^r] \times \{r\}$ is not yet visited. Also, by definition $Y_1^0, Y_1^r > \widehat{S}_{T_i}$ and the set $I_2 = \big([Y_1^0, \infty) \times \{0\}\big) \cup \big([Y_1^r, \infty) \times \{r\}\big)$ is never visited before time T_i. Moreover, because of the strong Markov property, the distributions of $\Pi \cap I_1$ and $\Pi \cap I_2$ are independent of the points of Π outside I_1 and I_2.

Let $\Pi' = \Pi \cap (I_1 \cup I_2)$. From (7) we have that the greedy walk on $\theta_{S_{T_i}} \Pi'$ starting from $(0, 0)$, visits the set $(-\infty, 0) \times \{0, r\}$ in a finite time, almost surely. In other words, the greedy walk on Π' starting from S_{T_i}, almost surely, visits a point in I_1 at some time $T'_{i+1} < \infty$. The greedy walk on Π_{T_i}, starting from S_{T_i}, might differ from the walk on Π' if there are some points outside I_1 and I_2 that are not visited up to time T_i. Denote the shadows of these points by $c_1, c_2, \dots, c_j$, so that $\widehat{S}_{T_i} \ge c_1 > c_2 > \cdots > c_j > \max\{Y_{-1}^0, Y_{-1}^r\}$. Because of Lemma 1 (b), a point in $(\widehat{S}_{T_i}, \infty) \times \{0, r\}$ is closer to the point at c_1, than to any of the points at $c_2, c_3, \dots, c_j$. Hence the walk visits the point at c_1 before visiting any of the points at $c_2, c_3, \dots, c_j$.

Let $T_{i+1}^1 = \min\{T_{c_1}^R, T_{\widehat{I}_1}\}$, where $\widehat{I}_1 = (-\infty, \max\{Y_{-1}^{0'}, Y_{-1}^{r'}\})$. The greedy walks on Π_{T_i} and Π' starting from S_{T_i} are the same until the time T_{i+1}^1. If $T_{i+1}^1 = T_{\widehat{I}_1}$ then $T'_{i+1} = T_{i+1}^1 = T_{i+1}$ and, thus, T_{i+1} is finite. Otherwise, the point at c_1 is visited before any point in I_1, so $T_{c_1}^R < T'_{i+1}$ and T_{c_1} is, almost surely, finite. In this case, similarly as above, let us define $Y_0^{c_1} = c_1$, $Y_1^{0,c_1} = \min\{Y \in \Pi^0 : Y > \max_{0 \le n \le T_{c_1}} \widehat{S}_n\}$

and $Y_1^{r,c_1} = \min\{Y \in \Pi^r : Y > \max_{0\leq n\leq T_{c_1}} \widehat{S}_n\}$. Moreover, let $I_1^{c_1} = I_1$ and let $I_2^{c_2} = \Big([Y_1^{0,c_1}, \infty) \times \{0\}\Big) \cup \Big([Y_1^{r,c_1}, \infty) \times \{r\}\Big)$. From Lemma 2 we can deduce that $(c_1, \max_{0\leq n\leq T^R_{c_1}} \widehat{S}_n] \times \{0, r\}$ is empty at time $T^R_{c_1}$ and, therefore, $I_2^{c_1}$ contains all points of $\Pi_{T^R_{c_1}}$ to the right of $\{(c_1, 0), (c_1, r)\}$. Now, by the same arguments as above, it follows that the walk starting from the point at c_1 visits almost surely a point in I_1 or a point at c_2 in a finite time. If the walk, starting from a point at c_k, $1 \leq k \leq j - 1$, visits a point at c_{k+1} before I_1, we repeat the same procedure. Since there are only finitely many such points, the walk in almost surely finite time eventually visits I_1 and thus $T_{i+1} < \infty$, almost surely.

When i is odd, we can look at the walk on $\sigma\Pi_{T_i}$ starting from σS_{T_i}. Then the same procedure as above yields $T_{i+1} < \infty$, almost surely. Therefore, T_i is almost surely finite for all i. Assume now that the walk does not visit all points of Π and let (x, ι_x) be a point of Π that is never visited. Then, there is i_0 even, such that $x < \widehat{S}_{T_{i_0}}$ and $x - \min_{0\leq n\leq T_{i_0}} \widehat{S}_n > r$. Then for all $n \in (T_{i_0}, T_{i_0+1} - 1)$ such that $\widehat{S}_n \geq x$, by the choice of i_0, S_n is closer to (x, ι_x) than to any point in $(-\infty, \min_{0\leq n\leq T_{i_0}} \widehat{S}_n) \times \{0, r\}$. Also, by Lemma 1 (b), S_n is closer to (x, ι_x) than to any remaining point in $[\min_{0\leq n\leq T_{i_0}} \widehat{S}_n, x)$. Hence, the greedy walk visits (x, ι_x) before time T_{i_0+1}, which is a contradiction. □

5. Two parallel lines with shifted Poisson processes

Let $E = \mathbb{R} \times \{0, r\}$ and let d be the Euclidean distance. We define a point process Π on E in the following way. Let Π^0 be a homogeneous Poisson process on $\mathbb{R}$ with rate 1 and let Π^r be a copy of Π^0 shifted by s, $0 < |s| < r/\sqrt{3}$, i.e. $\Pi^r = \{x : x - s \in \Pi^0\}$. Then, let Π be a point process on E with $\Pi \cap (B \times \{0\}) = (\Pi^0 \cap B) \times \{0\}$ and $\Pi \cap (B \times \{r\}) = (\Pi^r \cap B) \times \{r\}$ for all Borel sets $B \subset \mathbb{R}$. We consider the greedy walk on Π defined by (1) starting from $S_0 = (0, 0)$.

We call the pair of points $(Y, 0)$ and $(Y + s, r)$ *shifted copies*. Moreover, we say that a point of Π is an *indented point* if it is further away from the vertical line $\{0\} \times \mathbb{R}$ than its shifted copy. That is, for $s > 0$, the indented points are in $(-\infty, 0) \times \{0\}$ and $(0, \infty) \times \{r\}$. For $s < 0$, the indented points are in $(0, \infty) \times \{0\}$ and $(-\infty, 0) \times \{r\}$.

We can divide the points of Π into clusters in the following way. Any two successive points on line 0 are in the same cluster if their distance is less than $\sqrt{r^2 + s^2}$, otherwise they are in different clusters. Moreover, any two points on line 0 are in the same cluster only if all points between those two points belong to that cluster. The points on line r belong to the cluster of its shifted copy. Throughout this section, we will call the closest point to the vertical line $\{0\} \times \mathbb{R}$ of a cluster on each line the *leading point* of the cluster. Every cluster has one leading indented and one leading unindented point, except the cluster around $(0, 0)$ that has possibly points in both $(-\infty, 0) \times \{0, r\}$ and $(0, \infty) \times \{0, r\}$.

In Section 3 the points were divided into clusters in a similar way and we observed that the walk always visits all points of a cluster before moving to a new cluster. This is not the case here. See Figure 2 for an example where the greedy walk moves to a new cluster before visiting all points in a current cluster. The points that are not visited during the first visit of a cluster cause the walk to jump over the vertical line $\{0\} \times \mathbb{R}$ infinitely many times. Therefore, we obtain here the same result as in Section 4:

Theorem 4. *Almost surely, the greedy walk visits all points of* Π.

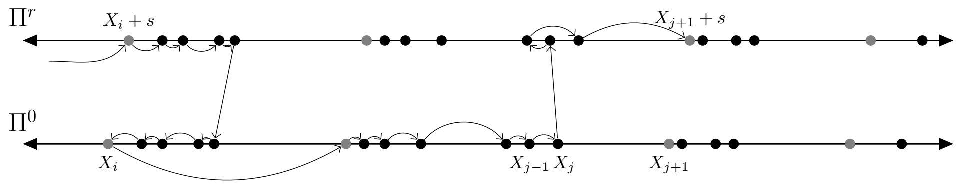

FIGURE 2. The leading points of the clusters are marked in gray. After visiting the indented leading point $(X_i + s, r)$, the walk visits successively all points of its cluster. This is not always the case when the first visited point of the cluster is the unindented leading point. In the example above, the point $(X_j, 0)$ is closer to $(X_{j-1} + s, r)$ than to $(X_j + s, r)$, so the walk moves from $(X_j, 0)$ to $(X_{j-1} + s, r)$. Later, when the walk is at $(X_j + s, r)$, the closest unvisited point is $(X_{j+1} + s, r)$, which is in a new cluster. The greedy walk moves to the next cluster before visiting all points of the current cluster.

The proof follows similarly as the proof of Theorem 3. We change here the definition of the set Ξ. In addition, the arguments in the first part of the proof of Lemma 11, where we show that if Ξ is almost surely the empty set then the walk almost surely jumps over the starting point, are different from those in Lemma 5. Furthermore, in the proof of Theorem 4 we use the fact that whenever the walk enters a cluster at its leading unindented point, then the walk always visits successively all points of the cluster. This can be explained as follows: Let (X, r) be the leading indented point and assume that (X, r) is the first point that the greedy walk visits in its cluster. (The case when the walk first visits the leading indented point on the line 0 can be handled in the same way.) Denote the closest point to (X, r) on line r be (Y, r) and let the distance between those points be $a = |X - Y|$. Then the distance between (X, r) and $(X - s, 0)$ is $\sqrt{r^2 + s^2}$ and the distance between (X, r) and $(Y - s, 0)$ is $\sqrt{r^2 + (a - s)^2}$. Because $s < \sqrt{r^2 + s^2}/2 < a - s$, whenever $a < \sqrt{r^2 + s^2}$, then (Y, r) is closer to (X, r) then these two points on line 0 and thus (Y, r) is visited next. We can argue in the same way for all the points in this cluster on line r, until the walk reaches the outermost point. The distance from the outermost point of the cluster to the closest point on line r is greater than $\sqrt{r^2 + s^2}$ and the closest unvisited point is its shifted copy. Once the walk is on line 0, it visits all remaining points of the cluster, because the distances between the successive points in the cluster are less than $\sqrt{r^2 + s^2}$, all points of the cluster on line r are already visited and the distance to any point in another cluster is greater than $\sqrt{r^2 + s^2}$.

We define now the set Ξ in a slightly different way than in Section 4. For $x \in \mathbb{R}$, let $L^0(x) = \max\{Y \in \Pi^0 : Y < x\}$. Then define

$$\Xi = s + \Big\{ X \in \Pi^0 : \; S_n^{(X,0)} \in ((X, \infty) \times \{0, r\}) \setminus \{(X + s, r)\} \text{ for all } n \geq 1 \\ \text{and } X - L^0(X) > \frac{r^2 + s^2}{2s} \Big\},$$

that is, Ξ contains all points $X + s \in \Pi^r$ such that the distance to the closest point on its right is greater than $\frac{r^2+s^2}{2s}$ and if we start a walk from $(X, 0)$, then the walk always stays in $[X, \infty) \times \{0, r\}$, but it never visits $(X + s, r)$. If $X - L^0(X) > \frac{r^2+s^2}{2s}$ and the walk approaches $(X, 0)$ and $(X + s, r)$ from their right, then the walk visits the point

$(X, 0)$ before visiting $(X + s, r)$. Hence, if $X + s \in \Xi$ and $X + s > 0$, then the point $(X + s, r)$ is never going to be visited.

Note that the set Ξ is a function of a homogeneous Poisson process and hence it is stationary and ergodic.

Let us define now the random variable W_x, $x \in \mathbb{R}$, that corresponds to the random variable $W_{x,\iota}$ from Section 4. For $x \in \mathbb{R}$, let

$$\Pi^x = \Pi \cap \left(([x, \infty) \times \{0\}) \cup ([x + s, \infty) \times \{r\})\right).$$

Consider for the moment the greedy walk on Π^x starting from $(x, 0)$ defined by (1) and let

$$W_x = \inf_{i \geq 0} \{\widehat{S}_i - d(S_i, S_{i+1})\}.$$

If $s > 0$ and if $|W_x| < \infty$ and $L^0(x) + s < W_x$, for $x \in \mathbb{R}$, then for all $i \geq 0$

$$\min\{d(S_i, (L^0(x), 0)), d(S_i, (L^0(x) + s, r))\} \geq \widehat{S}_i - L^0(x) - s > \widehat{S}_i - W_x \geq d(S_i, S_{i+1}).$$

Since the point S_i is closer to S_{i+1} than to any point in $\Pi \setminus \Pi^x$ for all $i \geq 0$, the walk on Π starting from $(x, 0)$ coincides with the walk on Π^x. Thus, the walk does not visit $\Pi \setminus \Pi^x$. The opposite is also true, i.e. if the walk on Π starting from $(x, 0)$ coincides with the walk on Π^x, then $|W_x| < \infty$. For $s < 0$ the same holds: if $|W_x| < \infty$ and $L^0(x) < W_x$, for $x \in \mathbb{R}$, then the walk on Π starting from $(x, 0)$ does not visit $\Pi \setminus \Pi^x$ and if the walk does not visit $\Pi \setminus \Pi^x$ then $|W_x| < \infty$.

Lemma 9. *If $s > 0$ and if Ξ is almost surely the empty set, then $\Psi = \{X \in \Pi^0 : |W_X| < \infty\}$ is almost surely the empty set.*

Proof. Since W_X is identically distributed for all $X \in \Pi^0$, Ψ is stationary and ergodic. Suppose, on the contrary, that Ψ is almost surely a non-empty set. For $d > 0$ let $\Psi_d = \{y \in \Pi^0 : |W_x| < d\}$ and note that $\bigcup_{d>0} \Psi_d = \Psi$. Then there exists d such that Ψ_d is almost surely a non-empty set.

Let $\widetilde{\Psi}_d$ be the set of all $X \in \Psi_d$ which satisfy the following. First, there are no points in $(X - d, X) \times \{0, r\}$. Secondly, there is $Y \in \Pi^0$, $Y < X$, such that the distance between Y and $\max\{Z \in \Pi^0 : Z < Y\}$ is greater than $\frac{r^2+s^2}{2s}$. Thirdly, the walk starting from $(Y, 0)$ stays in $(Y, \infty) \times \{0, r\}$ until it visits $(X, 0)$ and it never visits $(Y + s, r)$. Since there is a positive probability that all three conditions occur and this probability is independent of W_x, $\widetilde{\Psi}_d$ is almost surely a non-empty set. But, then by definition of Ξ, for every $X \in \widetilde{\Psi}_d$, $Y + s \in \Xi$ and Ξ is a non-empty set, which is a contradiction. $\square$

In the next two lemmas we use the random variable $\widetilde{D}_X$, $X \in \Pi^r$, which can be compared with the corresponding random variable in Section 4 defined in (4). For $X \in \Pi^r \setminus \Xi$ set $\widetilde{D}_X = 0$. For $X \in \Xi$ denote the points of $\Xi \cap (\infty, X]$ in decreasing order

$$\cdots < \widetilde{z}_2 < \widetilde{z}_1 < \widetilde{z}_0 = X$$

and define $\widetilde{D}_X$ as

$$\widetilde{D}_X = \sup_{i \geq 0} \{2\widetilde{z}_i - \widetilde{z}_{i+1} - X\}. \tag{8}$$

Also we define the set Ξ_{d_1,d_2} in the same way as in Section 4. For $d_1, d_2 > 0$ define

$$\Xi_{d_1,d_2} = \{X \in \Xi : \widetilde{D}_X < d_1 \text{ and there exists } Y \in \Pi^r \text{ such that } 0 < Y - X < d_2 \text{ and } \widehat{\Pi} \cap (Y, Y + r) = \emptyset\},$$

where $\widehat{\Pi} = \Pi^0 + \Pi^r$.

The following lemma corresponds to Lemma 4. Since the proof is very similar, it is not included here.

Lemma 10. *If $s > 0$ and if Ξ is almost surely a non-empty set, then there exists $d_1, d_2 > 0$ such that Ξ_{d_1,d_2} is almost surely a non-empty set. Moreover, Ξ_{d_1,d_2} is a stationary and ergodic process.*

We study the greedy walk starting from the point $(0,0)$, which is almost surely not a point of Π. From now on denote the points of Π^0 by

$$\ldots < X_{-2} < X_{-1} \leq 0 < X_1 < X_2 < \ldots.$$

Also, we let $\widehat{\Pi} = \Pi^0 \cup \Pi^r$ be the shadow of all the points of the process Π. As in Section 4, we denote by Π_n the set of points that are not visited until time n. Similarly, Π_n^0, Π_n^r denotes unvisited points of Π^0 and Π^r until time n, respectively, and $\widehat{\Pi}_n = \Pi_n^0 \cup \Pi_n^r$.

We define $T_A = \inf\{n \geq 0 : \widehat{S}_n \in A\}$ to be the first time the walk visits $A \times \{0, r\}$, where A is a subset of $\mathbb{R}$.

Define the variable D_x, $x \in \mathbb{R}$, $x > 0$, as follows. If $T_{(-\infty,0)} < T_{[x,\infty)}$ then $D_x = 0$. Otherwise, $0 < \widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_{T_{[x,\infty)}-1} < x$, $\widehat{S}_{T_{[x,\infty)}} \geq x$ and we label the remaining points of $\widehat{\Pi}_{T_{[x,\infty)}}$ in the interval $(0, x)$ by $z_1, z_2, \ldots, z_{n-1}$ so that

$$0 = z_n < z_{n-1} < \cdots < z_1 < z_0 = x.$$

Let then

$$D_x = \max_{0 \leq i \leq n-1} \{(z_i - z_{i+1}) - (x - z_i)\} = \max_{0 \leq i \leq n-1} \{2z_i - z_{i+1} - x\}.$$

From the definition it follows that $0 \leq D_x \leq x$. As in Section 4, this random variable measures how large at least should be the distance between $(x, 0)$ or (x, r) and the points in $(x, \infty) \times \{0, r\}$, so that the walk possibly visits a point in $(-\infty, 0) \times \{0, r\}$ before visiting any point in $(x, \infty) \times \{0, r\}$.

We prove next that $D_X < d_0$ for infinitely many $X \in \Pi^0$. The proof is divided into two parts, one discussing the case when Ξ is almost surely the empty set and another one discussing the case when Ξ is a non-empty set. The proof of the second case follows in the similar way as the second part of the proof of Lemma 5, so we are not going to write all the details here.

Lemma 11. *When $s > 0$, there exists $d_0 < \infty$ such that, almost surely, $D_{X_k+s} < d_0$ and $X_{k+1} - X_k > r + s$ for infinitely many $k > 0$.*

Proof. Assume first that Ξ is almost surely the empty set. Then, by Lemma 9, $\{X \in \Pi^0 \cap (0, \infty) : |W_X| < \infty\}$ is almost surely empty. Observe that $d((X_1, 0), (0, 0)) < d((X_1 + s, r), (0, 0))$ and if $X_{-1} + s > 0$ then $d((X_{-1}, 0), (0, 0)) < r < d((X_{-1} + s, r), (0, 0))$. Thus, if the walk starts from $(0, 0)$ and $\widehat{S}_1 > 0$, then S_1 must be $(X_1, 0)$. Assume that $\{\widehat{S}_n > 0 \text{ for all } n \geq 1\}$ occurs with positive probability. Then one of the following three events also has positive probability.

First, $\{X_{-1} + s < 0,\ \widehat{S}_n > 0 \text{ for all } n \geq 1\}$. But, this event implies that $|W_{X_1}| < \infty$ and $X_1 \in \Psi$, which has probability 0.

Secondly, $\{0 < X_{-1} + s,\ \widehat{S}_n > 0 \text{ for all } n \geq 1$ and the walk does not visit $(X_{-1} + s, r)\}$. If this event occurs then the walk does not visit any point in $(0, X_{-1} + s) \times \{r\}$, because $X_{-1} + s < r/2$ and the distance from any point in $(0, X_{-1} + s) \times \{r\}$ to $(X_{-1} + s, r)$ is smaller than to a point in $(0, \infty) \times \{0\}$. Therefore, from $S_1 = (X_1, 0)$ the walk visits just the points of Π^{X_1} and we can conclude that $|W_{X_1}| < \infty$, which is impossible.

Thirdly, $\{\widehat{S}_n > 0 \text{ for all } n \geq 1$ and the walk visits $(X_{-1} + s, r)\}$. Assume that this event occurs. There are almost surely finitely many points in $(0, X_{-1} + s] \times \{r\}$ and thus there is a time k when that interval is visited for the last time. From Lemma 2 it follows that all points in $(\widehat{S}_k, \max_{0\le i\le k}\widehat{S}_i]\times\{0,r\}$ are visited up to time k. Therefore, S_{k+1} is in $(\max_{0\le i\le k}\widehat{S}_i,\infty)\times\{0,r\}$. Since the distance from S_k to $(X_{-1},0)$ is at most $\sqrt{r^2+s^2}$, the distance from S_k to S_{k+1} is less than $\sqrt{r^2+s^2}$. Thus, we can conclude that both S_k and S_{k+1} belong to the cluster around $(X_1,0)$. Moreover, the greedy walk did not visit another cluster before time k and it moved from line 0 to line r only once.

At time k there might be some unvisited points of the cluster around $(X_1,0)$ on line r whose shifted copies are visited before time k. Those points are visited directly after S_k, because those points are closer to S_k than any unvisited point on line 0. After visiting those points, all remaining unvisited points of the cluster around $(X_1,0)$ have unvisited shifted copy. We can think about that part of the cluster as a new cluster. The walk visits the next cluster starting from the indented or the unindented leading point. If it visits first the indented point, then the walk visits consecutively all the points of that cluster and afterwards it visits the unindented leading point of the next cluster. Once the walk is at the unindented leading point $(Y,0)$, all points in $(0,Y)\times\{0,r\}$ are visited except possibly some points in $(0,X_{-1}+s]\times\{r\}$ which are never visited. Since $\widehat{S}_n>0$ for all $n\ge 1$, we can conclude that the walk after visiting $(Y,0)$ stays in Π^Y. But, then $|W_Y|<\infty$ and Ψ is not empty, which is a contradiction.

Since these three events almost surely do not occur, also $\{\widehat{S}_n>0 \text{ for all } n\ge 1\}$ does not occur. Thus $T_{(-\infty,0)}<\infty$, almost surely. This together with the definition of D_x, implies that $D_x=0$ for all large enough $x\ge 0$. Since $X_{k+1}-X_k>r+s$ for infinitely many $k>0$, the claim of the lemma holds for any $d_0>0$.

If Ξ is a non-empty set, the proof follows in the same way as the corresponding part of the proof of Lemma 5. Thus we omit the proof here and we only emphasize that points $\Xi\cap(0,\infty)$ are never visited because the condition $X-L^0(X)>\frac{r^2+s^2}{2s}$ implies that for $X\in\Xi\cap(0,\infty)$ points in $(-\infty,X-s)\times\{0,r\}$ are closer to $(X-s,0)$ than to (X,r). Thus, the point $(X-s,0)$ is visited first and then by the definition of Ξ the walk never visits (X,r). □

Using that event $D_{X_k}<d_0$ occurs for infinitely many $k>0$ and $X_{k+1}-X_k>d_0$ occurs for infinitely many $k>0$, we show in the next lemma that there are infinitely many $k>0$ such that both events occur simultaneously.

Lemma 12. *If $s>0$ then, almost surely, the events*

$$A_k=\{X_{k+1}-X_k>D_{X_k+s}-X_{-1}+r+s\}$$

occur for infinitely many $k>0$.

Proof. Let $j_s=\max\{i : X_i<-s\}$, $B_k=\{X_{j_s},X_{j_s+1},\dots,X_{-1},X_1,\dots,X_{k-1},X_k\}$ and $\mathcal{F}_k=\sigma(B_k)$. Let T^σ_A and D^σ_x be the analogues of T_A and D_x for the greedy walk on the set of points $(B_k\times\{0\})\cup((B_k+s)\times\{r\})$. When $T_{[X_k+s,\infty)}<T_{(-\infty,0)}$, the greedy walk on Π and the walk on the restricted set are the same until time $T_{[X_k+s,\infty)}$. Moreover, if $X_{k+1}-X_k>r+s$ then $\widehat{S}_{T_{[X_k+s,\infty)}}=X_k+s$, $T^\sigma_{X_k+s}=T_{[X_k+s,\infty)}$ and $D^\sigma_{X_k+s}=D_{X_k+s}$.

Let $A^\sigma_k=\{X_{k+1}-X_k>D^\sigma_{X_k+s}-X_{-1}+r+s\}$ and observe that $A^\sigma_k\in\mathcal{F}_{k+1}$. For $d_0>0$ we have

$$\begin{aligned}\mathbb{P}\left(A^\sigma_k\mid\mathcal{F}_k\right) &\ge \mathbb{P}\left(D^\sigma_{X_k+s}<d_0, X_{k+1}-X_k>d_0-X_{-1}+r+s\mid\mathcal{F}_k\right)\\ &= \mathbf{1}_{\{D^\sigma_{X_k+s}<d_0\}}\mathbb{P}\left(X_{k+1}-X_k>d_0-X_{-1}+r+s\mid\mathcal{F}_k\right)\\ &= \mathbf{1}_{\{D^\sigma_{X_k+s}<d_0\}}e^{-(d_0-X_{-1}+r+s)}\quad \text{a.s.}\end{aligned}$$

The first equality above holds because $\{D^{\sigma}_{X_k+s} < d_0\} \in \mathcal{F}_k$. The second equality follows from the facts that $X_{-1} \in \mathcal{F}_k$ and $X_{k+1} - X_k$ is exponentially distributed with mean 1 and independent of $\mathcal{F}_k$. By Lemma 11, we can choose d_0 such that $D_{X_k+s} < d_0$ and $X_{k+1} - X_k > r + s$ for infinitely many k, almost surely. Since, $D^{\sigma}_{X_k+s} = D_{X_k+s}$ whenever $X_{k+1} - X_k > r + s$, also $D^{\sigma}_{X_k+s} < d_0$ for infinitely many k and, thus,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(A^{\sigma}_k \mid \mathcal{F}_k\right) = \infty \quad \text{a.s.}$$

It follows now from Lemma 6 that

$$\mathbb{P}(A^{\sigma}_k \text{ for infinitely many } k \geq 1) = 1.$$

Since $A^{\sigma}_k \subset \{X_{k+1} - X_k > r + s\}$ and $A_k = A^{\sigma}_k$ whenever $X_{k+1} - X_k > r + s$, also

$$\mathbb{P}(A_k \text{ for infinitely many } k \geq 1) = 1. \qquad \square$$

Whenever A_k occurs, the greedy walk is forced to visit $(-\infty, 0) \times \{0, r\}$ before visiting $[X_{k+1}, \infty) \times \{0, r\}$. If $s > 0$ then from Lemma 12 it follows that $T_{(-\infty,0)} < \infty$. We prove that the same is true if $s < 0$, but in the proof we use that $T_{(-\infty,0)} < \infty$, almost surely, for $s > 0$.

Lemma 13. *Almost surely, $T_{(-\infty,0)} < \infty$.*

Proof. If $s > 0$ one can show in the same way as in Lemma 8, that if $A_k = \{X_{k+1} - X_k > D_{X_k} - X_{-1} + r + s\}$ occurs, then the walk visits $(-\infty, 0) \times \{0, r\}$ before visiting $[X_{k+1}, \infty) \times \{0, r\}$. By Lemma 12, the event A_k occurs for some k, almost surely, and hence $T_{(-\infty,0)} < \infty$, almost surely.

Furthermore,

$$\mathbb{P}(T_{(-\infty,0)} < \infty \mid X_{-1}, X_1) = 1 \quad \text{a.s.,} \tag{9}$$

for any absolutely continuous distribution of (X_{-1}, X_1) on $(-\infty, 0) \times (0, \infty)$ which is independent of $\Pi^0 \cap (X_1, \infty)$.

Assume now on the contrary that $\mathbb{P}(T_{(-\infty,0)} = \infty) > 0$ for $s < 0$. If $T_{(-\infty,0)} = \infty$, then S_1 is in $(0, \infty) \times \{0\}$ or in $(0, \infty) \times \{r\}$. If S_1 is on line 0 then S_1 is indented and the walk consecutively visits all points of its cluster. Since the walk stays in $(0, \infty) \times \{0, r\}$, the last visited point of this cluster is $(X_1 + s, r)$ and $X_1 + s > 0$. Let T be the time when the walk visits $(X_1 + s, r)$ and let $Y_0 = (X_1 + s, r)$. Moreover, let $Y_{-1} = X_{-1} + s$ and let $Y_1 = \min\{Y \in \Pi^r : Y > \max_{0 \leq n \leq T} \widehat{S}_n\}$ be the leading indented point of the next cluster on the right of Y_0. If S_1 is on line r, then the greedy walk is the same as if the walk starts from $(0, r)$. Thus let $Y_0 = (0, r)$, $T = 0$, $Y_{-1} = X_{-1} + s$ and $Y_1 = X_1 + s$.

From (9) it follows that the walk on $\theta_{Y_0}(\Pi_T)$ starting from $(0, 0)$ and given $X_{-1} = Y_{-1}$ and $X_1 = Y_1$, visits $(-\infty, 0) \times \{0, r\}$ in a finite time, almost surely. In other words, the walk on Π_T starting from Y_0 visits $(X_{-1}, 0)$ or $(X_{-1} + s, r)$ in a finite time, almost surely. This contradicts the assumption that $\mathbb{P}(T_{(-\infty,0)} = \infty) > 0$. Therefore, the claim of the lemma holds also for $s < 0$. $\square$

Proof of Theorem 4. From Lemma 13 it follows that for any $|s| < r/2$

$$\mathbb{P}(T_{(-\infty,0)} < \infty \mid X_{-1}, X_1) = 1 \quad \text{a.s.,} \tag{10}$$

for any absolutely continuous distribution of (X_{-1}, X_1) on $(-\infty, 0) \times (0, \infty)$ which is independent of $\Pi^0 \cap (X_1, \infty)$.

We prove the theorem for $s > 0$. The proof of the theorem for $s < 0$ follows in a similar way. Let us first look at the cluster around the starting point of the greedy walk $(0,0)$. If $X_1 - X_{-1} > \sqrt{r^2+s^2}$ this cluster is empty. If the cluster is not empty, it has finitely many points and the walk visits a point in another cluster in a finite time. Let T_0 be the first time the walk visits a point in another cluster ($T_0 = 1$ if the cluster around $(0,0)$ is empty).

We assume at the moment that $\widehat{S}_{T_0} > 0$. For $i \geq 1$ let T_i be the first time the greedy walk visits $(-\infty, \min_{0\leq n \leq T_{i-1}} \widehat{S}_n) \times \{0,r\}$ for i odd and $(\max_{0\leq n\leq T_{i-1}} \widehat{S}_n, \infty) \times \{0,r\}$ for i even. That is, for i odd (even) T_i is the time when the walk visits the part of Π on the left (right) of the vertical line $\{0\}\times\mathbb{R}$ which is not visited up to time T_{i-1}.

Let $Y_0 = S_{T_0-1}$ be the last visited point in the cluster around $(0,0)$ before the first visit to another cluster and let ι be the line of S_{T_0-1}. Moreover, let Y_{-1} and Y_1 be the closest not yet visited points of Π^ι to Y_0, such that their shifted copy is also not visited, that is

$$Y_{-1} = \max\{Y \in \Pi^\iota_{T_0} : Y < \min_{0\leq n<T_0} \widehat{S}_n, Y + (-1)^{\mathbf{1}_r(\iota)}\cdot s \in \Pi^{r-\iota}_{T_0}\}$$

and

$$Y_1 = \min\{Y \in \Pi^\iota_{T_0} : Y > \max_{0\leq n<T_0} \widehat{S}_n, Y + (-1)^{\mathbf{1}_r(\iota)}\cdot s \in \Pi^{r-\iota}_{T_0}\}.$$

Let

$$I_1 = \big((-\infty, Y_{-1})\times\{\iota\}\big) \cup \big((-\infty, Y_{-1} + (-1)^{\mathbf{1}_r(\iota)}\cdot s)\times\{r-\iota\}\big)$$

and

$$I_2 = \big((Y_1,\infty)\times\{\iota\}\big) \cup \big((Y_1 + (-1)^{\mathbf{1}_r(\iota)}\cdot s)\times\{r-\iota\}\big).$$

Because of the strong Markov property the distribution of Π in I_1 and I_2 is independent of the points of Π outside these sets. Now let $\Pi' = \Pi\cap(I_1\cup I_2)$. From (10), we know that the walk on $\sigma\theta_{Y_0}(\Pi')$ starting from $(0,0)$ and given $X_{-1} = Y_{-1}$ and $X_1 = Y_1$ visits $(-\infty,0)\times\{0,r\}$ in a finite time, almost surely. Hence, the walk on Π_{T_0} starting at S_{T_0} visits I_1 or a point of the cluster around $(0,0)$ in almost surely finite time. Denote that time by T_1'.

If $S_{T_1'}$ is in I_1, then $T_1 = T_1'$. Otherwise, $S_{T_1'}$ is in $((-\infty, Y_0)\times\{0,r\})\setminus I_1$ and from the definition of I_1 we can deduce that the shifted copy of the point $S_{T_1'}$ must have been visited before T_1. Set now $Y_0 = S_{T_1'}$ and redefine Y_{-1}, Y_1, I_2 and Π' with respect to the time T_1' instead of T_0.

Observe that, by Lemma 2, at time T_1' the set $(\widehat{S}_{T_1'}, \max_{0\leq n\leq T_1'} \widehat{S}_n)\times\{0,r\}$ is empty. Moreover, if there are some points in $(\max_{0\leq n\leq T_1'} \widehat{S}_n, \infty)\times\{0,r\}$ whose shifted copy is visited, then these points are on line r and belong to one cluster. The closest point with a still unvisited shifted copy is at a horizontal distance of at least $r-s$ from those points. Since the distance from $S_{T_1'}$ to $(\max_{0\leq n\leq T_1'} \widehat{S}_n, \infty)\times\{0,r\}$ is at least $\sqrt{r^2+s^2}$, the remaining points on line r which do not have a shifted copy are closer to $S_{T_1'}$ than the closest point on line 0. Thus, these points, whose shifted copies are already visited, are visited before visiting (Y_1', ι) or its shifted copy.

Now, we can conclude that the walk starting from $S_{T_1'}$ visits in almost surely finite time I_1 or a point of $\Pi\setminus(I_1\cup I_2)$, that is one of the remaining points of the cluster around $(0,0)$ or a point in $(\max_{0\leq n\leq T_1'} \widehat{S}_n, Y_1')\times\{r\}$. There are finitely many points in $\Pi\setminus(I_1\cup I_2)$ and every time the walk visits one of these points, we redefine Y_0, Y_{-1}, Y_1, I_2 and Π', and repeat the same arguments as above. Thus the walk visits I_1 in almost surely finite time.

Assume now that T_i is finite for some odd $i\geq 1$. Let $Y_0 = \widehat{S}_{T_i-1}$ and define Y_{-1}, Y_1, I_1, I_2 and Π' as before. Then points of $\Pi_{T_i}\setminus\Pi'$ are in $(Y_{-1},0)\times\{r\}$, $(0,Y_1)\times\{0\}$ or the cluster around $(0,0)$ and there are almost surely finitely many such points. By the observation above, the greedy walk visits all points in $(Y_{-1},Y_0)\times\{0\}$ before it visits I_1

and it visits all points in $(Y_0, Y_1) \times \{r\}$ before it visits a point in I_2. Denote the time when the walk visits a point of $\Pi_{T_i} \setminus \Pi'$ before visiting I_2 with T_i'. Set then $Y_0 = S_{T_i'}$ and redefine Y_{-1}', Y_1 and Π', with the respect with the time T_i'. Again, by (10), the walk on $\sigma\theta_{Y_0}\Pi'$ visits $(-\infty, 0) \times \{0, r\}$ in almost surely finite time. Thus, the walk on $\Pi_{T_i'}$ in almost surely finite time visits I_2 or another point in $\Pi_{T_i} \setminus \Pi'$. Repeating this arguments for every visited point in $\Pi_{T_i} \setminus \Pi'$, we can see that the walk eventually visits I_2 and that T_{i+1} is almost surely finite.

Similarly, one can show that if T_i, $i \geq 2$ even, is finite, then T_{i+1} is also almost surely finite. Therefore, inductively we can conclude that the walk almost surely crosses the vertical line $\{0\} \times \mathbb{R}$ infinitely many times and, thus, it eventually visits all points of Π. $\square$

Remark 2. We conjecture that Theorem 4 holds also for $|s| > r/2$. For those s the idea to cluster the points of Π does not work in the same way. For example, the greedy walk does not always visit all points of the cluster when it starts from the leading indented point of the cluster. Thus, the walk more often does not visit all points of a cluster successively and we expect that the points that are not visited during the first visit of a cluster cause the walk to return and to cross the vertical line $\{0\} \times \mathbb{R}$ infinitely often.

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Paper V

THE GREEDY WALK ON AN INHOMOGENEOUS POISSON PROCESS

KATJA GABRYSCH AND ERIK THÖRNBLAD

Abstract. The greedy walk is a deterministic walk that always moves from its current position to the nearest not yet visited point. In this paper we consider the greedy walk on an inhomogeneous Poisson point process on the real line. Our primary interest is whether the walk visits all points of the point process, and we determine sufficient and necessary conditions on the mean measure of the point process for this to happen. Moreover, we provide precise results on threshold functions for the property of visiting all points.

Keywords and phrases. greedy walk; inhomogeneous Poisson point processes; threshold
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1. Introduction and main results

Consider a simple point process Π in a metric space (E, d). We think of Π either as an integer-valued measure or as a collection of points (the support of the measure). Moreover, the process is assumed to have no accumulation points in E. We use the notation $\Pi(B)$, or, simply, ΠB, to indicate the number of points (the value of the random measure) on the Borel set $B \subset E$. If $x \in E$, we write $x \in \Pi$ instead of $\Pi\{x\} = 1$.

The greedy walk on Π is defined as follows. Let $S_0 \in E$ and $\Pi_0 = \Pi$. Define, for $n \geq 0$,

$$S_{n+1} = \arg\min\{d(S_n, X) : X \in \Pi_n\},$$
$$\Pi_{n+1} = \Pi_n \setminus \{S_{n+1}\}.$$

The set Π_n denotes the set of unvisited points of Π up until (and including) time n and we write $\Pi_\infty = \bigcap_{n=1}^\infty \Pi_n$ for the set of points that are never visited by the walk. Note that, once the underlying environment Π is fixed, the process $(S_n)_{n=0}^\infty$ is deterministic (except possibly for ties which need to be broken, but these will almost surely not occur in our setting).

The greedy walk has been studied before in the literature, with various choices of the underlying point process. When Π is a homogenous Poisson process on $\mathbb{R}$, one can show using a Borel–Cantelli–type argument that $\Pi_\infty \neq \emptyset$ with probability 1, that is, the greedy walk does not visit all points of the underlying point process. More precisely, the expected number of times the greedy walk starting from 0 changes sign is 1/2 [4]. Due to this Rolla *et al.* [7] considered a related problem, in which each point in the process can be visited either once, with probability $1 - p$, or twice, with probability p. They show that $\Pi_\infty = \emptyset$, for any $0 < p < 1$, meaning that every point is eventually visited. Another modification of the greedy walk on $\mathbb{R}$ is studied by Foss *et al.* [3]. The authors considered a dynamic version of the greedy walk, where the times and positions of new points arriving in the system are given by a Poisson process on the space-time half-plane. They show that the greedy walk still diverges to infinity in one direction and does not visit all points. In the survey paper [2], Bordenave *et al.*

state several questions about the behaviour of the greedy walk on an inhomogeneous Poisson process in $\mathbb{R}^d$. We resolve here the problem for $d = 1$.

In this paper we define Π to be an inhomogeneous Poisson process on $\mathbb{R}$ (with the Euclidean metric) with some non-atomic mean measure μ. For such a process, the number of points in disjoint measurable subsets of $\mathbb{R}$ are independent and

$$\mathbb{P}[\Pi(a,b) = k] = \frac{\mu(a,b)}{k!} e^{-\mu(a,b)}$$

for any $a < b$ and any $k \geq 0$, where, for any measurable $A \subseteq \mathbb{R}$, $\Pi(A)$ is the cardinality of the restriction of Π to the set A. This means that the number of points in any interval (a, b) is distributed like $\mathrm{Poi}(\mu(a, b))$. Sometimes, but not always, we will assume that the mean measure μ is absolutely continuous and given in terms of a measurable intensity function $\lambda : \mathbb{R} \to [0,\infty)$, so that

$$\mu(A) = \int_A \lambda(x)\,\mathrm{d}x$$

for any measurable $A \subseteq \mathbb{R}$.

Throughout we let $S_0 = 0$ (note that $0 \notin \Pi$ with probability 1), so that the walk starts in the origin. The process $(S_n)_{n=0}^{\infty}$ will be referred to as `GWIPP`. If we want to emphasise the underlying point process, the underlying mean measure, or the underlying intensity function, we write `GWIPP`(Π), `GWIPP`(μ) or `GWIPP`(λ), respectively. We say that the walk jumped over 0 if $\mathrm{sign}(S_{n+1}) \neq \mathrm{sign}(S_n)$ for some n. If the event $\{\mathrm{sign}(S_{n+1}) \neq \mathrm{sign}(S_n) \text{ i.o.}\}$ occurs, then we say that `GWIPP` is *recurrent*. Otherwise we say that `GWIPP` is *transient*. This choice of notation is explained by viewing each jump over 0 as a pseudo–visit at 0.

To avoid certain degenerate cases, we will typically impose the following two conditions on the measure μ.

(i) $\mu(-\infty, 0) = \mu(0,\infty) = \infty$.
(ii) $\mu(A) < \infty$ for all bounded measurable $A \subseteq \mathbb{R}$.

Note that the first condition is equivalent to $\Pi(-\infty, 0) = \Pi(0,\infty) = \infty$ with probability 1, and the second condition is equivalent to $\Pi(A) < \infty$ with probability 1, for any bounded measurable $A \subseteq \mathbb{R}$. The set of all measures on $\mathbb{R}$ which satisfy (i) and (ii) will be denoted $\mathcal{M}$. If $\mu \in \mathcal{M}$ and μ is given in terms of the intensity function λ, then we will abuse notation and write $\lambda \in \mathcal{M}$. In Remark 2.2 we consider situation in which condition (i) is not satisfied. Condition (ii) implies that there are no accumulation points. If a point process has some accumulation points, arg min in the definition of the greedy walk may not be well-defined for some $n \geq 1$.

If $\mu \in \mathcal{M}$, then `GWIPP`(μ) is recurrent if and only if $\Pi_\infty = \emptyset$, and `GWIPP`(μ) is transient if and only if $\Pi_\infty \neq \emptyset$. Thus the dichotomy between recurrence and transience translates to a dichotomy between "visits all points" and "does not visit all points". This stems from the fact that `GWIPP` essentially can behave in two ways. Either the points of Π are eventually dense enough that `GWIPP` eventually gets stuck on either the positive or negative half-line and go towards ∞ or $-\infty$ accordingly (i.e. transient), or the points of Π are sparse enough that there are infinitely many "sufficiently long" empty intervals on both half-lines, and that `GWIPP` switches sign infinitely many times and thus visits all points of Π (i.e. recurrent). Moreover, if `GWIPP`(μ) is transient and μ symmetric around zero, then, by symmetry, `GWIPP`(μ) goes to $+\infty$ or $-\infty$ with probability 1/2 each.

The aim of this paper is to characterise (in terms of μ or λ) when `GWIPP` is recurrent or transient. The following result does precisely this.

Theorem 1.1. *Let $\mu \in \mathcal{M}$. Then* GWIPP*(μ) is recurrent with probability 1 if*

$$\int_0^\infty \exp(-\mu(x, 2x+R))\mu(\,\mathrm{d}x) = \infty \quad \text{and} \quad \int_{-\infty}^0 \exp(-\mu(2x-R, x))\mu(\,\mathrm{d}x) = \infty,$$

for all $R \geq 0$. If either integral is finite for some $R \geq 0$, then GWIPP*(μ) is transient with probability* 1.

Let $\lambda \in \mathcal{M}$. Then GWIPP*(λ) is recurrent with probability 1 if*

$$\int_0^\infty \exp\left(-\int_x^{2x+R} \lambda(t)\,\mathrm{d}t\right)\lambda(x)\,\mathrm{d}x = \infty \quad \text{and} \quad \int_{-\infty}^0 \exp\left(-\int_{2x-R}^{x} \lambda(t)\,\mathrm{d}t\right)\lambda(x)\,\mathrm{d}x = \infty,$$

for all $R \geq 0$. If either integral is finite for some $R \geq 0$, then GWIPP*(λ) is transient with probability* 1.

The proof of this, presented in Section 2, is an application of Campbell's theorem (which allows one to determine whether random sums over Poisson processes are convergent or divergent) and the Borel–Cantelli lemmas.

We remark a few things. First, the second part of the theorem is an immediate consequence of the first. Second, this result also states that recurrence (or transience) is a zero-one event. Third, it is a straightforward consequence that taking $\lambda(t) = c$ for all $t \in \mathbb{R}$, i.e. taking Π to be a homogenous Poisson process with rate c, results in GWIPP(λ) being transient. This is well-known and another proof appears in [4].

Take now two point processes Π and Π_0, and assume GWIPP(Π) is transient. Consider $\Pi' = \Pi + \Pi_0$. Intuitively, adding more points to an already transient process only makes it "more" transient, since it will be more difficult to find long empty intervals which allow $(S_n)_{n=1}^\infty$ to change sign. Conversely, removing points from an already recurrent process makes it "more" recurrent. Since recurrence (or transience) does not depend on the point process in any finite interval around 0, as the following result shows, it suffices to look at what happens far away from the origin. (Equivalently, the convergence or divergence of the integrals in Theorem 1.1 depends only on the tail behaviour.) The proof appears in Section 2.

Lemma 1.2. *Let $\mu, \mu' \in \mathcal{M}$ and suppose there is some $K > 0$ such that $\mu'(A) \geq \mu(A)$ for all measurable $A \subseteq (-\infty, -K) \cup (K, \infty)$. If* GWIPP($\mu'$) *is recurrent with probability* 1*, then* GWIPP(μ) *is recurrent with probability* 1.

Theorem 1.1 also facilitates the identification of threshold functions for recurrence, and it transpires that the iterated logarithms are useful in this context. Before we state our next result, we need some definitions. We define the "power tower" recursively by $a \uparrow\uparrow 0 := 1$ and $a \uparrow\uparrow n := a^{a \uparrow\uparrow (n-1)}$ for any $a \in [0, \infty)$ and $n \geq 1$. Let log be the ordinary natural logarithm. We define the iterated logarithm $\log^{(n)}$, for $n \geq 1$, to be the function defined recursively by

$$\log^{(1)} t := \begin{cases} \log t & \text{if } t > 1 \\ 0 & \text{otherwise,} \end{cases}$$

and, for any $n \geq 2$,

$$\log^{(n)} t := \log^{(1)}\left(\log^{(n-1)} t\right).$$

Note that $\log^{(n)} t = 0$ for any $t \leq e \uparrow\uparrow (n-1)$.

Proposition 1.3. *Let*

$$\lambda(t) := \frac{1}{|t| \log 2} \sum_{i=2}^{n} a_i \log^{(i)} |t|.$$

where $n \in \{2, 3, 4, \dots\}$ and $a_i \geq 0$ for all $2 \leq i \leq n$. Then GWIPP*(λ) is transient with probability* 1 *if*

- $a_2 > 1$, *or*
- $a_2 = 1, a_3 > 2$, *or*
- $a_2 = 1, a_3 = 2$, *and there exists some $m \geq 4$ such that $a_4 = 1, a_5 = 1, \dots, a_m = 1$ and $a_{m+1} > 1$.*

Otherwise, GWIPP*(λ) is recurrent with probability* 1.

One could also ask to what extent Proposition 1.3 extends to the infinite case. The statement about transience remains true, but the final statement about recurrence does not, unless one introduces some (rather mild) restrictions on the growth rate of the sequence $(a_n)_{n=2}^{\infty}$. Consider

$$\lambda(t) = \frac{1}{|t| \log 2} \sum_{i=2}^{\infty} a_i \log^{(i)} |t|. \tag{1}$$

If $a_2 = 0$ and $a_n = (e \uparrow\uparrow n)$ for $n \geq 3$, then GWIPP(λ) is transient with probability 1, even though $a_2 = 0$. However, if $a_2 = 0$ and $a_n = (2 \uparrow\uparrow n)$ for $n \geq 3$, then GWIPP(λ) is recurrent with probability 1.

The next result describes the threshold between transience and recurrence in even greater detail. In particular, it shows that taking $a_3 = 2$ and $a_2 = 1 = a_4 = a_5 = \dots$ in (1) means that GWIPP(λ) is recurrent with probability 1.

Proposition 1.4. *Let $a_3 = 2$ and $a_2 = 1 = a_4 = a_5 = \dots$ and let $(b_n)_{n=1}^{\infty}$ be non–decreasing sequence satisfying $b_n = O(e \uparrow\uparrow (n-2))$, $g : (0, \infty) \to [1, \infty)$ be a non–decreasing slowly varying function satisfying $g(e \uparrow\uparrow n) = b_n$, and let*

$$\lambda(t) := \frac{1}{|t| \log 2} \left(\sum_{i=2}^{\infty} a_i \log^{(i)} |t| + \log^{(1)} g(|t|) \right).$$

If $\sum_{n=2}^{\infty} 1/b_n = \infty$, then GWIPP*($\lambda$) is recurrent with probability* 1. *If $\sum_{n=2}^{\infty} 1/b_n < \infty$, then* GWIPP*($\lambda$) is transient with probability* 1.

The following result provides a useful tool for investigating the behaviour of a given intensity function. The idea behind the proof is essentially to find a suitable intensity function for comparison, and apply Lemma 1.2 and Proposition 1.3.

Proposition 1.5. *Let $\lambda \in \mathcal{M}$. Let $a_3 = 2$ and $a_2 = 1 = a_4 = a_5 = \dots$. If there exists some $n \geq 2$ such that*

$$\varliminf_{t \to \infty} \frac{t\lambda(t) \log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} > 1 \quad \text{or} \quad \varliminf_{t \to -\infty} \frac{|t|\lambda(t) \log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} > 1$$

then GWIPP(λ) *is transient with probability* 1. *If there exists some $n \geq 2$ such that*

$$\varlimsup_{t \to \infty} \frac{t\lambda(t) \log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} < 1 \quad \text{and} \quad \varlimsup_{t \to -\infty} \frac{|t|\lambda(t) \log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} < 1$$

then GWIPP(λ) *is recurrent with probability* 1.

Proposition 1.5 does not answer what happens if, say,

$$\lim_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} = 1 \quad \text{and} \quad \lim_{t\to-\infty} \frac{|t|\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} |t|}{a_n \log^{(n)} |t|} = 1$$

for all $n \geq 2$. As seen in Proposition 1.4, both recurrence and transience are possible in this case.

The remainder of this paper is outlined as follow. In Section 2 we prove mainly general results, including Theorem 1.1 and Lemma 1.2. In Section 3 we concentrate on more concrete threshold results, i.e. Propositions 1.3–1.5 along with related results.

2. Proof of Theorem 1.1 and related results

Throughout we will write $\Pi = \{X_i : i \in \mathbb{Z} \setminus \{0\}\}$, assuming as we may that

$$\cdots < X_{-2} < X_{-1} < 0 < X_1 < X_2 < \cdots .$$

For $k > 0$, let

$$A_k^R = \{\Pi(X_k, 2X_k + R) = 0\} = \{d(X_k, -R) < d(X_k, X_{k+1})\}$$

and

$$B_k^R = \{\Pi(2X_{-k} - R, X_{-k}) = 0\} = \{d(X_{-k}, R) < d(X_{-k}, X_{-k-1})\}.$$

The following lemma describes the connections between these events and recurrence of `GWIPP`.

Lemma 2.1. `GWIPP` *is recurrent if and only if* $\{A_k^R$ *i.o.*$\}$ *and* $\{B_k^R$ *i.o.*$\}$ *occur for all* $R \geq 0$.

Proof. For one direction, one can argue as follows. Suppose `GWIPP` is transient and $\{A_k^R$ i.o.$\}$ and $\{B_k^R$ i.o.$\}$ occur for all R. Without loss of generality, we may assume $S_n \to \infty$ as $n \to \infty$. Then there exists J such that $S_{n+1} > S_n$ for all $n > J$. Let $Y = \max\{X \in \Pi : X < \min_{0\leq k\leq J} S_k\}$ be the rightmost point never visited (such a point exists because of the assumptions of transience and $S_n \to \infty$ as $n \to \infty$) and choose R such that $R > |Y|$. Note that Y is the closest unvisited point to the left of S_n, when $n \geq J$. Since, by assumption, A_k^R occurs for some $k > J$, we have $d(X_k, Y) < d(X_k, -R) < d(X_k, X_{k+1})$. Moreover, there exists n such that $S_n = X_k$ and by the definition of the greedy walk $S_{n+1} = Y$, which is a contradiction.

For the other direction, we assume that `GWIPP` is recurrent, but that A_k^R (the argument being identical for B_k^R) occurs at most finitely many times for some $R \geq 0$. Let K be the last index for which the event A_k^R occurred. Since `GWIPP` visits all points of Π, there are indices J_1 and J_2 such that $S_{J_1} < -R$ and $S_{J_2} = X_K$. Whenever $j > \max\{J_1, J_2\}$ and $S_j > 0$, then $\Pi(S_j, 2S_j + R) > 0$. This implies that the closest unvisited point of S_j is to the right of S_j, because the closest point on the left of S_j is in $(-\infty, S_{J_1}) \subset (-\infty, -R)$, that is, at the distance greater than $S_j + R$ from S_j. Thus, after visiting $S_j > 0$ such that $j > \max\{J_1, J_2\}$, the walk stays always on the right of S_j, which contradicts the assumption that `GWIPP` is recurrent. □

This characterisation suggests that the Borel–Cantelli lemmas will be useful. In particular, we use the extended Borel–Cantelli Lemma.

Lemma 2.2 (Extended Borel–Cantelli lemma, [5, Corollary 6.20]). *Let* $\mathcal{F}_n$, $n \geq 0$, *be a filtration with* $\mathcal{F}_0 = \{0, \Omega\}$ *and let* $A_n \in \mathcal{F}_n$, $n \geq 1$. *Then a.s.*

$$\{A_n \text{ i.o.}\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}[A_n \mid \mathcal{F}_{n-1}] = \infty \right\}.$$

The convergence or divergence of the associated random series will be determined using Campbell's theorem, which provides a zero-one law when f is a non-negative measurable function.

Theorem 2.3 (Campbell's theorem, [6, Section 3.2])**.** *Let Π be a Poisson process on S with mean measure μ and let $f : S \to [0, \infty]$ be a measurable function. Then the sum*

$$\sum_{X \in \Pi} f(X)$$

is convergent with probability 1 if and only if

$$\int_S \min\{f(x), 1\}\mu(\,\mathrm{d}x) < \infty.$$

Moreover, the sum diverges with probability 1 if and only if the integral diverges.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.1 it follows that the sufficient and necessary conditions implying that the GWIPP is recurrent with probability 1, are the same as those implying that $\{A_k^R \text{ i.o.}\}$ and $\{B_k^R \text{ i.o.}\}$ occur with probability 1.

Let $\mathcal{F}_k = \sigma(X_1, X_2, \ldots, X_k)$. Then $A_k^R \in \mathcal{F}_{k+1}$ for any $R \geq 0$, and

$$\mathbb{P}[A_k^R \mid \mathcal{F}_k] = \mathbb{P}[\Pi(X_k, 2X_k + R) = 0 \mid \mathcal{F}_k] = \exp(-\mu(X_k, 2X_k + R)),$$

where the final equality holds since $X_k \in \mathcal{F}_k$, $\Pi \cap (X_k, \infty)$ is independent of $\mathcal{F}_k$ and the number of points in a measurable set $A \subseteq \mathbb{R}$ is distributed like $\mathrm{Poi}(\mu(A))$. Applying Theorem 2.3 with $f(x) = \exp(-\mu(x, 2x + R))$, we obtain

$$\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k] = \sum_{k=1}^{\infty} \exp(-\mu(X_k, 2X_k + R)) = \infty$$

with probability 1 if and only if

$$\int_0^{\infty} \exp(-\mu(x, 2x + R))\mu(\,\mathrm{d}x) = \infty.$$

Moreover, Lemma 2.2 implies that $\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k] = \infty$ a.s. if and only if $\mathbb{P}[A_k^R \text{ i.o.}] = 1$. Thus, the integral above diverges if and only if $\mathbb{P}[A_k^R \text{ i.o.}] = 1$.

Similarly, if the integral above converges, so does the sum

$$\sum_{k=1}^{\infty} \mathbb{P}[A_k^R \mid \mathcal{F}_k]$$

with probability 1, and then by Lemma 2.2, the event $\{A_k^R \text{ i.o.}\}$ does not occur with probability 1.

In the same way one can show that $\int_{-\infty}^{0} \exp(-\mu(2x - R, x))\mu(\,\mathrm{d}x) = \infty$ if and only if $\mathbb{P}[B_k^R \text{ i.o.}] = 1$, and conversely, if the integral converge then $\mathbb{P}[B_k^R \text{ i.o.}] = 0$. $\square$

Remark 2.1. We lose no generality by assuming that the greedy walk on Π starts from the origin, since recurrence/transience does not depend on the starting point. One explanation is that the distribution of the points in any finite interval around the origin does not influence the behaviour of the greedy walk far away from the origin. More precisely, if the walk that starts from $a \in \mathbb{R}$, $a > 0$ (one can argue similarly for $a < 0$), the events $\{\Pi(X_k, 2X_k - a + R) = 0\}$ and $\{\Pi(2X_{-k} - a - R, X_{-k}) = 0\}$ occur, for any $R \geq 0$, for infinitely many k if and only if $\{A_k^R \text{ i.o.}\}$ and $\{B_k^R \text{ i.o.}\}$ occur for all $R \geq 0$. As we have seen in Lemma 2.1, GWIPP(Π) is recurrent if these events occur.

In the following remark we explore what happens if μ does not satisfy condition (i).

Remark 2.2. If $\mu(-\infty,0)<\infty$ and $\mu(0,\infty)=\infty$, then it is not true that $\mathtt{GWIPP}(\mu)$ is transient if and only if $\Pi_\infty\neq\emptyset$. (This should be contrasted with the situation when $\mu\in\mathcal{M}$.) To see this, consider the intensity function $\lambda(t)=\mathbb{1}_{(-M,\infty)}(t)$, where $M>0$ will be chosen later. It is clear that $\mathtt{GWIPP}(\lambda)$ is transient with probability 1, for any M, but we will show that M can be chosen so that $0<\mathbb{P}[\Pi_\infty\neq\emptyset]<1$. Note that

$$\mathbb{P}[\Pi_\infty=\Pi_\infty\cap(-M,0)]=1.$$

so we may consider $\Pi_\infty\cap(-M,0)$ instead of Π_∞.

Let $\lambda'(t)=1$ for all $-\infty<t<\infty$ and denote by Π' the associated Poisson process and by $(S_n')_{n=1}^\infty$ the associated greedy walk. We can couple Π and Π' together so that $\Pi=\Pi'\cap(-M,\infty)$. By symmetry, we have

$$\mathbb{P}[S_n'\to\infty]=\frac{1}{2}=\mathbb{P}[S_n'\to-\infty].$$

It holds that

$$\mathbb{P}[\Pi_\infty'\cap(-M,0)=\emptyset\mid S_n'\to-\infty]=1,$$

and

$$\mathbb{P}[\Pi_\infty'\cap(-M,0)=\emptyset\mid S_n'\to\infty]>0.$$

By the coupling of Π and Π', it holds that $\Pi_\infty'\cap(-M,0)=\Pi_\infty\cap(-M,0)$ almost surely, whence

$$\begin{aligned}\mathbb{P}[\Pi_\infty\cap(-M,0)=\emptyset]&=\mathbb{P}[\Pi_\infty'\cap(-M,0)=\emptyset]\\&=\mathbb{P}[\Pi_\infty'\cap(-M,0)=\emptyset\mid S_n'\to-\infty]\mathbb{P}[S_n'\to-\infty]\\&\quad+\mathbb{P}[\Pi_\infty'\cap(-M,0)=\emptyset\mid S_n'\to\infty]\mathbb{P}[S_n'\to\infty]\\&>\frac{1}{2}.\end{aligned}$$

Therefore $\mathbb{P}[\Pi_\infty\cap(-M,0)\neq\emptyset]<\frac{1}{2}$.

For the other inequality, we first have

$$\begin{aligned}\mathbb{P}[\Pi_\infty\cap(-M,0)=\Pi\cap(-M,0)]&=\mathbb{P}[\Pi_\infty'\cap(-M,0)=\Pi'\cap(-M,0)]\\&\geq\mathbb{P}[\Pi_\infty'\cap(-\infty,0)=\Pi'\cap(-\infty,0)]\\&=\prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)\approx0.288\ldots,\end{aligned}$$

where the final equality follows from [4, Theorem 1]. Now pick M large enough that

$$\mathbb{P}[\Pi\cap(-M,0)\neq\emptyset]>1-\prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right).$$

Then

$$\begin{aligned}\mathbb{P}[\Pi_\infty\cap(-M,0)\neq\emptyset]&\geq\mathbb{P}[\Pi_\infty\cap(-M,0)=\Pi\cap(-M,0),\ \Pi_\infty'\cap(-M,0)\neq\emptyset]\\&\geq\mathbb{P}[\Pi_\infty\cap(-M,0)=\Pi\cap(-M,0)]+\mathbb{P}[\Pi_\infty'\cap(-M,0)\neq\emptyset]-1\\&>0.\end{aligned}$$

This implies that $0<\mathbb{P}[\Pi_\infty=\emptyset]<\frac{1}{2}$, even though $\mathtt{GWIPP}(\lambda)$ is transient with probability 1.

A natural question is which conditions one needs to place on μ (or λ) so that $\{A_k^R \text{ i.o.}\}$ for all $R > 0$ if and only if $\{A_k^0 \text{ i.o.}\}$. The reason why this is not an unreasonable demand is that the events $\Pi(X_k, 2X_k + R) = 0$ and $\Pi(X_k, 2X_k) = 0$ should not be too different for large X_k, since the length of the interval $(2X_k, 2X_k + R)$ becomes negligible compared to the length of $(X_k, 2X_k)$ in the limit. However, the following example shows that some extra conditions need to be placed, and that, in general, $\{A_k^0 \text{ i.o.}\}$ does not imply that $\{A_k^R \text{ i.o.}\}$ for all $R > 0$.

Remark 2.3. Let

$$\lambda(t) = \sum_{n=1}^{\infty} a_n \mathbb{1}(2^n - 2 < t < 2^n - 1)$$

for some increasing sequence $(a_n)_{n=1}^{\infty}$. Denote by C_n the event that $\Pi(2^n - 2, 2^n - 1) = 0$, for $n = 1, 2, \ldots$. Then $\mathbb{P}[C_n] = e^{-a_n}$.

If X equals the rightmost point in the interval $(2^n - 2, 2^n - 1)$, then $\Pi(X, 2X) = 0$ almost surely. This implies that X is always closer to 0 than to the leftmost point in $(2^{n+1} - 2, 2^{n+1} - 1)$. Hence, $\{A_n^0 \text{ i.o.}\}$ occurs with probability 1.

However, for $R = 3$ we have $\{A_n^3 \text{ i.o.}\} \subseteq \{C_n \text{ i.o.}\}$. Choose now the sequence $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}[C_n] = \sum_{n=1}^{\infty} e^{-a_n} < \infty.$$

By the Borel–Cantelli lemma, the probability of $\{C_n \text{ i.o}\}$ is 0, which implies that also $\mathbb{P}(A_n^3 \text{ i.o.}) = 0$. Therefore $\texttt{GWIPP}(\lambda)$ is transient even though A_n^0 occurs infinitely often with probability 1.

Denote by $\mathcal{M}_b \subseteq \mathcal{M}$ those measures $\mu \in \mathcal{M}$ with the property that for any $R \geq 0$, there exists some constant $C = C(R) > 0$, such that $\mu(x, x + R) < C$ and $\mu(-x - R, -x) < C$ for all $x \geq 0$. As the following lemma shows, this boundedness assumption disallows any examples of the type in Remark 2.3.

Lemma 2.4. *Let $\mu \in \mathcal{M}_b$. Then $\texttt{GWIPP}(\mu)$ is recurrent with probability 1 if*

$$\int_0^{\infty} \exp(-\mu(x, 2x))\mu(\,\mathrm{d}x) = \infty \quad \textit{and} \quad \int_{-\infty}^{0} \exp(-\mu(2x, x))\mu(\,\mathrm{d}x) = \infty.$$

If either integral is finite, then $\texttt{GWIPP}(\mu)$ is transient with probability 1.

Proof. Fix $R > 0$. We have

$$\begin{aligned}
\exp(-C) \int_0^{\infty} \exp(-\mu(x, 2x))\mu(\,\mathrm{d}x) &\leq \int_0^{\infty} \exp(-\mu(x, 2x) - \mu(2x, 2x + R))\mu(\,\mathrm{d}x) \\
&= \int_0^{\infty} \exp(-\mu(x, 2x + R))\mu(\,\mathrm{d}x) \\
&\leq \int_0^{\infty} \exp(-\mu(x, 2x))\mu(\,\mathrm{d}x).
\end{aligned}$$

The integral on the negative half-line can be similarly bounded. Therefore the integrals in the statement of the lemma both diverge if and only if all integrals in Theorem 1.1 diverge. This proves the claim. □

For instance, if $\mu \in \mathcal{M}$ and the maps $x \mapsto \mu(0, x)$ and $x \mapsto \mu(-x, 0)$ from $[0, \infty)$ to $[0, \infty)$ are Lipschitz, then $\mu \in \mathcal{M}_b$. Also, $\overline{\lim}_{t \to \pm\infty} \lambda(t) < \infty$ implies that $\lambda \in \mathcal{M}_b$, which gives the following corollary.

Corollary 2.5. *Suppose $\lambda \in \mathcal{M}$ and $\overline{\lim}_{t\to\pm\infty} \lambda(t) < \infty$. Then* GWIPP$(\lambda)$ *is recurrent with probability* 1 *if*

$$\int_0^\infty \exp\left(-\int_x^{2x} \lambda(t)\,\mathrm{d}t\right)\lambda(x)\,\mathrm{d}x = \infty \quad \textit{and} \quad \int_{-\infty}^0 \exp\left(-\int_{2x}^{x} \lambda(t)\,\mathrm{d}t\right)\lambda(x)\,\mathrm{d}x = \infty.$$

If either integral is finite, then GWIPP(λ) *is transient with probability* 1.

Next we prove Lemma 1.2.

Proof of Lemma 1.2. Denote by Π the point process with mean measure μ and let $(S_n)_{n=0}^\infty$ be GWIPP(Π). Similarly, denote by Π' the point process with mean measure μ' and let $(S'_n)_{n=0}^\infty$ be GWIPP(Π'). Denote the points of Π and Π' by

$$\cdots < X_{-2} < X_{-1} < 0 < X_1 < X_2 < \cdots \text{ and } \cdots < X'_{-2} < X'_{-1} < 0 < X'_1 < X'_2 < \cdots$$

respectively. Since $\mu'(A) \geq \mu(A)$ for all measurable $A \subset (-\infty,-K)\cup(K,\infty)$, we can couple Π and Π' together so that $x \in ((-\infty,-K)\cup(K,\infty))\cap\Pi$ implies that $x \in \Pi'$.

Assume, for contradiction, that GWIPP(Π') is recurrent and GWIPP(Π) is transient. Without loss of generality, we may assume that $S_n \to \infty$ as $n \to \infty$. Then there is some $M_0 \geq 1$ such that $S_{k+1} > S_k$ for all $k > M_0$, i.e. $(S_n)_{n=1}^\infty$ moves only to the right after time M_0. Assume moreover that M_0 is large enough that $S_{M_0} > K$, so that we are on the region where Π and Π' are coupled.

For the remainder of the proof, see Figure 1 for an illustration. Let $Y = \max\{X \in \Pi : X < \min_{0\leq k\leq M_0} S_k\}$, that is, let Y be the rightmost point of Π that is never visited. Note that Y is well-defined because of the transience of GWIPP(Π) and the assumption $S_n \to \infty$ as $n \to \infty$.

Since GWIPP(Π') is recurrent, $(S'_n)_{n=1}^\infty$ visits all points of $(K,\infty)\cap\Pi \subseteq (K,\infty)\cap\Pi'$ and jumps over 0 infinitely often. Thus we can find $J \geq 1$ such that $S'_J > S_{M_0}$ and $S'_{J+1} < Y$. Let $S'_J = X'_k$ and let ℓ be such that $X_\ell \leq X'_k < X_{\ell+1}$. Moreover, let $M > M_0$ be such that $S_M = X_\ell$ and $S_{M+1} = X_{\ell+1}$.

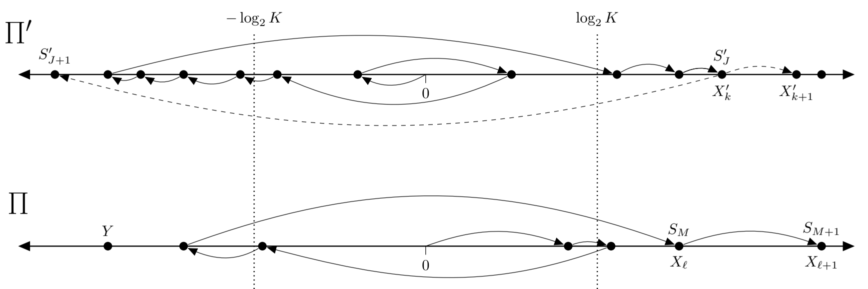

FIGURE 1. An illustration of the proof of Lemma 1.2. Note that both the positive and negative axis have been rescaled logarithmically. The proof shows that $S'_{J+1} = X'_{k+1}$ is forced, which contradicts the choice of J (which implies that $S'_{J+1} < 0$).

The coupling between Π and Π' on (K,∞) implies that $X_\ell \leq S'_J < X'_{k+1} \leq X_{\ell+1}$. Therefore $d(S'_J, X'_{k+1}) \leq d(S_M, S_{M+1}) < d(S_M, Y) \leq d(S'_J, S'_{J+1})$, which contradicts the choice of J, that is $S_{J+1} < Y$. Thus, if GWIPP(Π') is recurrent, so is GWIPP(Π). $\square$

A question that complements Lemma 1.2 is the following. Suppose $\texttt{GWIPP}(\lambda')$ is recurrent and let $\lambda = \lambda' + \lambda_0$ for some intensity function λ_0. Which conditions on λ_0 should one place to ensure that $\texttt{GWIPP}(\lambda)$ is also recurrent? That is, how many points, and where, can we add to a recurrent process without making it transient? The following lemma yields a partial answer to this.

Lemma 2.6. *Suppose* $\texttt{GWIPP}(\lambda)$ *is recurrent with probability* 1*. If, for all* $R \geq 0$*,*

$$\overline{\lim_{x\to\infty}} \int_x^{2x+R} \max\big(0, \lambda'(t) - \lambda(t)\big)\, \mathrm{d}t < \infty,$$

and

$$\overline{\lim_{x\to-\infty}} \int_{2x-R}^{x} \max\big(0, \lambda'(t) - \lambda(t)\big)\, \mathrm{d}t < \infty,$$

then $\texttt{GWIPP}(\lambda')$ *is recurrent with probability* 1*.*

Proof. For all $t \in \mathbb{R}$, let $\lambda''(t) = \max(\lambda(t), \lambda'(t))$. It suffices to show that $\texttt{GWIPP}(\lambda'')$ is recurrent, since then, by Lemma 1.2, $\texttt{GWIPP}(\lambda')$ also is recurrent. The first condition implies that for any $R \geq 0$ there exists some $C > 0$ such that

$$\int_x^{2x+R} (\lambda''(t) - \lambda(t))\, \mathrm{d}t < C$$

for all all large enough $x > 0$. Therefore

$$\int_x^{2x+R} \lambda''(t)\, \mathrm{d}t \leq \int_x^{2x+R} \lambda(t)\, \mathrm{d}t + \int_x^{2x+R} (\lambda''(t) - \lambda(t))\, \mathrm{d}t < \int_x^{2x+R} \lambda(t)\, \mathrm{d}t + C$$

for large enough x. For M large enough,

$$\begin{aligned}
\int_M^{\infty} \exp\left(-\int_x^{2x+R} \lambda''(t)\, \mathrm{d}t\right) \lambda''(x)\, \mathrm{d}x &\geq \int_M^{\infty} \exp\left(-\int_x^{2x+R} \lambda''(t)\, \mathrm{d}t\right) \lambda(x)\, \mathrm{d}x \\
&\geq \int_M^{\infty} \exp\left(-\int_x^{2x+R} \lambda(t)\, \mathrm{d}t - C\right) \lambda(x)\, \mathrm{d}x \\
&= \exp(-C) \int_M^{\infty} \exp\left(-\int_x^{2x+R} \lambda(t)\, \mathrm{d}t\right) \lambda(x)\, \mathrm{d}x \\
&= \infty,
\end{aligned}$$

where the final equality follow from the fact that $\texttt{GWIPP}(\lambda)$ is recurrent and Theorem 1.1. The integral on the negative half-line can be handled similarly. Therefore, by Theorem 1.1, $\texttt{GWIPP}(\lambda'')$ is recurrent. □

3. Treshold results

In this section we study the threshold between transience and recurrence, proving Propositions 1.3–1.5 and related results. We focus on symmetric intensity functions of the form $\lambda(t) = |t|^{-1} \log f(|t|)$, where $f : (0, \infty) \to [1, \infty)$ is a measurable function satisfying the following assumptions.

(i) The function $\log f$ is slowly varying, i.e.

$$\lim_{t\to\infty} \frac{\log f(at)}{\log f(t)} = 1$$

for any $a \geq 0$,

(ii) It holds that
$$\int_0^\infty \frac{\log f(x)}{x}\,\mathrm{d}x = \infty.$$
(iii) For any bounded measurable $A \subseteq (0,\infty)$,
$$\int_A \frac{\log f(x)}{x}\,\mathrm{d}x < \infty.$$

We denote the set of such intensity functions by $\mathcal{M}_s$. The final two assumptions imply that $\mathcal{M}_s \subseteq \mathcal{M}$, so we may apply the results developed in Section 2. Note that the intensity function $\lambda(t) = |t|^{-1}\log f(|t|)$ is symmetric about 0, so it suffices to only look at the positive half-line. However, the results in this section can be easily adapted to the case when f is not assumed to be symmetric.

We use the following standard notation. If $f, g : \mathbb{R} \to \mathbb{R}$ are two functions and there exists $C > 0$ such that $|f(x)| \le C|g(x)|$ for all large enough x, then we write $f(x) = O(g(x))$. If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, then we write $f(x) = \Theta(g(x))$.

Lemma 3.1 ([1, Theorem 1.2.1])**.** *If $A \subseteq (0,\infty)$ is a compact set and $f : (0,\infty) \to [1,\infty)$ is slowly varying, then $f(ax)/f(x) \to 1$ as $x \to \infty$, uniformly for all $a \in A$.*

Lemma 3.2. *Suppose $\lambda \in \mathcal{M}_s$. Then* `GWIPP`*($\lambda$) is recurrent with probability* 1 *if*
$$\int_0^\infty \frac{\log f(x)}{x f(x)}\,\mathrm{d}x = \infty$$
If the integral is finite, then `GWIPP`*(λ) is transient with probability* 1*.*

Proof. The set $[1,2]$ is compact and $\log f$ is slowly varying. It follows by Lemma 3.1, that
$$\int_x^{2x} \frac{\log f(t)}{t\log 2}\,\mathrm{d}t = \log f(x) + O(1).$$
Hence
$$\int_0^\infty \exp\left(-\int_x^{2x}\lambda(t)\,\mathrm{d}t\right)\lambda(x)\,\mathrm{d}x = \int_0^\infty \frac{\Theta(1)}{f(x)}\frac{\log f(x)}{x\log 2}\,\mathrm{d}x = \Theta(1)\int_0^\infty \frac{\log f(x)}{x f(x)}\,\mathrm{d}x.$$
Since $\overline{\lim}_{t\to\infty}\lambda(t) = 0$, the claim follows from Corollary 2.5. ☐

A generalisation of slow variation is that of regular variation. A function $f : (0,\infty) \to [1,\infty)$ is said to be regularly varying with index $\beta \ge 0$ if, for any $a > 0$, it holds that
$$\lim_{t\to\infty}\frac{f(at)}{f(t)} \to a^\beta.$$
A slowly varying function is then regularly varying with index $\beta = 0$. If f is regularly varying with index $\beta > 0$, then $\log f$ is slowly varying, see e.g. [1, Thm. 1.5.7]). This is true also if f is slowly varying, under the assumption that $f(t) \to \infty$ as $t \to \infty$.

The next corollary states that, if f is regularly varying with positive index, then we obtain a transitive processes with probability 1.

Corollary 3.3. *Let $f : (0,\infty) \to [1,\infty)$ be a regularly varying function with index $\beta > 0$ and let $\lambda(t) = (|t|\log 2)^{-1}\log f(|t|)$. Then* `GWIPP`*($\lambda$) is transient with probability* 1*.*

Proof. Note first that $\lambda \in \mathcal{M}_s$. Moreover, there is a slowly varying function $\ell(x)$ such that $f(x) = x^\beta \ell(x)$ (see, e.g. [1, Thm. 1.4.1]). Then for $x > 0$,

$$\frac{\log f(x)}{x f(x)} = \frac{\log f(x)}{x^{1+\beta}\ell(x)} = \frac{L(x)}{x^{1+\beta}}, \tag{2}$$

where $L(x) = \frac{\log f(x)}{\ell(x)}$ is a slowly varying function (see, e.g. [1, Thm. 1.3.6]). The function on the right hand side of (2) is integrable on $(0, \infty)$ whenever $\beta > 0$, and by Lemma 3.2, $\texttt{GWIPP}(\lambda)$ is transient. □

The intensity functions in Proposition 1.3 satisfy $\lambda \in \mathcal{M}_s$, showing that threshold behaviour occurs inside the class $\mathcal{M}_s$.

Proof of Proposition 1.3. Let

$$f(t) = \prod_{i=2}^{n} (\log^{(i-1)} t)^{a_i},$$

so that $\lambda(t) = \frac{\log f(t)}{t \log 2}$. Note that $\lambda \in \mathcal{M}_s$. Assume first that $a_2 > 0$. Then

$$\int \frac{\log f(x)}{x f(x)}\,\mathrm{d}x = \int \frac{\sum_{i=2}^{n} a_i \log^{(i)} x}{x \prod_{i=2}^{n} (\log^{(i-1)} x)^{a_i}}\,\mathrm{d}x = \Theta\left(\int \frac{\log^{(2)} x}{x \prod_{i=2}^{n} (\log^{(i-1)} x)^{a_i}}\,\mathrm{d}x \right),$$

where the final integral is the leading order term of the sum. The final integral is convergent precisely when one of the conditions in the statement is satisfied. (This is seen by repeatedly using the change of variables $x \mapsto e^x$.) By Lemma 3.2, the statement follows. If $a_2 = 0$, then consider instead $\lambda'(t) := \lambda(t) + \frac{1}{2}\frac{\log^{(2)} |t|}{|t| \log 2}$ and use the above along with Lemma 1.2 to conclude that $\texttt{GWIPP}(\lambda)$ is recurrent in this case. This completes the proof. □

Proof of Proposition 1.5. Suppose first that

$$\varliminf_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} > 1$$

for some $n \geq 2$. (Let n be minimal with this property.) Let

$$a := \frac{1}{2}\left(1 + \varliminf_{t\to\infty} \frac{t\lambda(t)\log 2 - \sum_{i=2}^{n-1} a_i \log^{(i)} t}{a_n \log^{(n)} t} \right) > 1$$

and define

$$\lambda'(t) := \begin{cases} \lambda(t), & t \leq 0 \\ \frac{1}{|t|\log 2}\left(\sum_{i=2}^{n-1} a_i \log^{(i)} |t| + a a_n \log^{(n)} |t| \right), & t > 0. \end{cases}$$

Suppose, for contradiction, that $\texttt{GWIPP}(\lambda)$ is recurrent. Since $\lambda(t) \geq \lambda'(t)$ for all t large enough, Lemma 1.2 implies that $\texttt{GWIPP}(\lambda')$ is recurrent. However, Proposition 1.3 implies that $\texttt{GWIPP}(\lambda')$ is transient. (In Proposition 1.3 we assumed that the intensity function be symmetric, but this does not change the evaluation of the integral on the positive half-axis.) This is a contradiction, so $\texttt{GWIPP}(\lambda)$ must be transient.

Now suppose the second condition holds for some $n \geq 2$. If

$$\lambda'(t) := \frac{1}{|t|\log 2}\left(\sum_{i=2}^{n} a_i \log^{(i)} |t| \right),$$

then $\lambda(t) < \lambda'(t)$ for all sufficiently large t. By Proposition 1.3, $\texttt{GWIPP}(\lambda')$ is recurrent, and Lemma 1.2 implies that $\texttt{GWIPP}(\lambda)$ is recurrent. □

Proof of Proposition 1.4. For $t > 0$ we have $\lambda(t) = \frac{\log f(t)}{t \log 2}$ with

$$\log f(t) = \sum_{i=2}^{\infty} a_i \log^{(i)}(t) + \log g(t).$$

Because of our definition of the iterated logarithm, this implies that

$$f(t) = \prod_{i=1}^{\infty} \left(\max(1, (\log^{(i)} t)^{a_{i+1}}) \right) g(t).$$

Since $b_{n-1} \le g(x)$ for any $e \uparrow\uparrow (n-1) \le x \le e \uparrow\uparrow n$, we obtain

$$\begin{aligned}
\int_0^{\infty} \frac{\log f(x)}{x f(x)} \,\mathrm{d}x &= \sum_{n=2}^{\infty} \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{\sum_{i=2}^{n} a_i \log^{(i)} x + \log g(x)}{x \prod_{i=1}^{n-1} (\log^{(i)} x)^{a_{i+1}} g(x)} \,\mathrm{d}x \\
&= \Theta(1) \sum_{n=2}^{\infty} \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{\log^{(2)} x}{x \prod_{i=1}^{n-1} (\log^{(i)} x)^{a_{i+1}} g(x)} \,\mathrm{d}x \\
&\le \Theta(1) \sum_{n=2}^{\infty} \int_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \frac{1}{x \prod_{i=1}^{n-1} (\log^{(i)} x) b_{n-1}} \,\mathrm{d}x \\
&= \Theta(1) \sum_{n=2}^{\infty} \frac{1}{b_{n-1}} \left[\log^{(n)} x \right]_{e\uparrow\uparrow(n-1)}^{e\uparrow\uparrow n} \\
&= \Theta(1) \sum_{n=2}^{\infty} \frac{1}{b_{n-1}}.
\end{aligned}$$

Using instead the bound $b_n \ge g(x)$ for any $e \uparrow\uparrow (n-1) \le x \le e \uparrow\uparrow n$, we arrive at

$$\sum_{n=2}^{\infty} \frac{1}{b_n} \le \int_0^{\infty} \frac{\log f(x)}{x f(x)} \,\mathrm{d}x \le \Theta(1) \sum_{n=2}^{\infty} \frac{1}{b_{n-1}}.$$

Applying Lemma 3.2 completes the proof. □

Remark 3.1. Proposition 1.5 does not answer what happens in, for example, the regime

$$\underline{\lim}_{t\to\infty} \frac{t\lambda(t) \log 2}{\log \log t} < 1 < \overline{\lim}_{t\to\infty} \frac{t\lambda(t) \log 2}{\log \log t}. \tag{3}$$

The intensity functions λ_1 and λ_2, to be defined next, both satisfy (3), but $\texttt{GWIPP}(\lambda_1)$ is recurrent while $\texttt{GWIPP}(\lambda_2)$ is transient. For $t > 0$ let $\lambda_1(t) = \sum_{n=1}^{\infty} \mathbb{1}(2^{2n} < t < 2^{2n+1})$ and $\lambda_2(t) = \sum_{n=1}^{\infty} n \mathbb{1}(2^n - 2 < t < 2^n - 1)$ and let λ_1 and λ_2 be symmetric around 0. We have

$$0 = \underline{\lim}_{t\to\infty} \frac{t\lambda_1(t)}{\log \log t} < \frac{1}{\log 2} < \overline{\lim}_{t\to\infty} \frac{t\lambda_1(t)}{\log \log t} = \infty$$

and

$$0 = \underline{\lim}_{t\to\infty} \frac{t\lambda_2(t)}{\log \log t} < \frac{1}{\log 2} < \overline{\lim}_{t\to\infty} \frac{t\lambda_2(t)}{\log \log t} = \infty.$$

It is easy to check that

$$\int_0^{\infty} \exp\left(-\int_x^{2x} \lambda_1(x) \right) \lambda_1(x) \,\mathrm{d}x = \infty,$$

which together with the fact that λ_1 is bounded and Corollary 2.5, yields that $\texttt{GWIPP}(\lambda_1)$ is recurrent. From Remark 2.3 we know that $\texttt{GWIPP}(\lambda_2)$ is transient.

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