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CONVERGENCE OF DIRECTED RANDOM GRAPHS TO THE POISSON-WEIGHTED INFINITE TREE

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Abstract

We consider a directed graph on the integers with a directed edge from vertex i to j present with probability n^{-1}, whenever $i < j$, independently of all other edges. Moreover, to each edge (i, j) we assign weight $n^{-1}(j - i)$. We show that the closure of vertex 0 in such a weighted random graph converges in distribution to the Poisson-weighted infinite tree as $n \to \infty$. In addition, we derive limit theorems for the length of the longest path in the subgraph of the Poisson-weighted infinite tree which has all vertices at weighted distance of at most ρ from the root.

Keywords: Directed random graph; Poisson-weighted infinite tree; rooted geometric graph

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1. Introduction and main results

In this paper we consider a directed version of the standard Erdős–Rényi random graph [7] on n vertices, defined as follows. For each pair $\{i, j\}$ of distinct positive integers, $i, j \le n$, toss a coin with probability of heads equal to p, $0 < p < 1$, independently from pair to pair; if heads shows up then introduce an edge directed from $\min(i, j)$ to $\max(i, j)$. A path in a directed graph is a sequence of increasing vertices which are successively connected and the length of a path is the number of edges in the path. The longest path in the graph is a path with maximal length. The length of the longest path in a directed random graph, sometimes also called the height of a graph, was the subject in a number of papers, see [9] and [12]–[15]. Newman [15] and Itoh and Krapivsky [14] showed that the length of the longest path starting from vertex 1 in a directed random graph on n vertices with edge probability n^{-1}, rescaled by n^{-1}, converges to the constant e. In both papers, the authors also describe the limiting object of the closure of vertex 1. This limiting object can be found in the literature under the name Poisson-weighted infinite tree.

Let $U = \{\varnothing\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k$ and, if $u, v \in U$, denote by uv the concatenated element and $\varnothing u = u \varnothing = u$. The Ulam–Harris tree is the infinite rooted tree with vertex set U, the root in the vertex with label $\varnothing$, and an edge joining vertices u and uj for any $u \in U$ and $j \in \mathbb{N}$. This, in the language of trees, means that if $u \in U$ then the children of u are $u1, u2, u3, \ldots$.

The *Poisson-weighted infinite tree* (PWIT) is the Ulam–Harris tree with weight function w on edges defined as follows. Assign to each vertex $u \in U$ an independent realization of a Poisson process with rate 1. Let the arrival times of the Poisson process assigned to vertex u be $\{\xi_j^u, j \in \mathbb{N}\}$. Then the weight function w of the edges from the vertex u is given by $w(u, uj) = \xi_j^u$, $j \in \mathbb{N}$.

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In their survey, Aldous and Steele [4] defined rooted geometric graphs and their convergence, a framework which we use here in order to prove rigorously that the limit graph of directed random graphs is the PWIT.

Let $G = (V, E)$ be a connected graph with a finite or countably infinite vertex set V, edge set E, and associated function $w: E \to (0, \infty]$ for weights of the edges of G. Moreover, for any pair of vertices u and v, the distance between u and v is defined as the infimum over all paths from u to v of the sum of the weights of the edges in the path. The graph G is called a *geometric graph* if the weight function makes G locally finite in the sense that for each vertex v and each $\rho > 0$ the number of vertices within distance ρ from v is finite. If, in addition, there is a distinguished vertex v, we say that G is a *rooted geometric graph* with root v. The set of rooted geometric graphs will be denoted by $\mathcal{G}_*$.

For any $\rho > 0$ and for any rooted geometric graph G denote by $N_\rho(G)$ the graph whose vertex set $V_\rho(G)$ is the set of vertices of G that are at a distance of at most ρ from the root and whose edge set consists of just those edges of G that have both vertices in $V_\rho(G)$. We say that ρ is a *continuity point* of G if no vertex of G is exactly at a distance ρ from the root of G.

A *graph isomorphism* between rooted geometric graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $\Phi: V \to V'$ such that $(\Phi(u), \Phi(v)) \in E'$ if and only if $(u, v) \in E$ and Φ maps the root of G to the root of G'. A *geometric isomorphism* between rooted geometric graphs G and G' is a graph isomorphism Φ between G and G' which preserves edge weights, that is, $w'(\Phi(e)) = w(e)$ for all $e \in E$ (where $\Phi(e)$ denotes $(\Phi(u), \Phi(v))$ for $e = (u, v)$).

We say that G_n *converges* (locally) to G in $\mathcal{G}_*$ if for each continuity point ρ of G there is an $n_0 = n_0(\rho, G)$ such that for all $n \geq n_0$ there exists a graph isomorphism $\Phi_{\rho,n}$ from the rooted geometric graph $N_\rho(G)$ to the rooted geometric graph $N_\rho(G_n)$ such that for each edge e of $N_\rho(G)$ the weight of $\Phi_{\rho,n}(e)$ converges to the weight of e.

In Appendix A we propose a distance function d which makes $\mathcal{G}_*$ into a complete separable metric space and for which the notion of convergence in the metric space is equivalent with the definition of convergence above. In this setting we can use the usual definition of convergence in distribution, see, e.g. [8].

Consider now an extension of a directed random graph to the infinite, countable vertex set with labels $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and with edge probability n^{-1}. As before, any two vertices are connected with an edge with probability n^{-1}, independently of the other edges, and an edge is always directed from the vertex with the smaller label to the vertex with the larger label. We define the weight of an edge in this graph as the absolute difference between its labels rescaled by n^{-1}, that is, if $e = (i, j)$ is an edge of the graph then its weight is $w_n(e) = n^{-1}|j - i|$. The *closure of vertex* 0 in a directed random graph is its subgraph consisting of all vertices which are connected with vertex 0 through a path (including vertex 0) and all the edges between these vertices. We call vertex 0 the root of the closure of vertex 0.

Our main result is the following theorem.

Theorem 1. *Let T_n, $n \in \mathbb{N}$, be the closure of vertex* 0 *in a directed random graph on $\mathbb{N}_0$, with edge probability n^{-1} and weight function w_n. Then T_n converges in distribution to a Poisson-weighted infinite tree as $n \to \infty$.*

We show convergence in distribution of the closure of vertex 0 of the directed random graphs on an infinite vertex set to the PWIT, instead for their finite versions. A reason is that directed random graphs on n vertices with weight function w_n have all vertices at the distance at most 1 from the root and their weak limit is a subgraph of a PWIT T, $N_1(T)$. To obtain a PWIT in the limit, one could consider finite directed random graphs but then the number of vertices need to grow faster than linearly.

The PWIT appears also as a weak limit of a randomly rooted complete graph on n vertices with independent and exponentially distributed edge weights with mean n. Aldous [3] proved this limit theorem and used it in a study of a random assignment problem with n jobs, n machines, and the costs of performing a job on each machine being independent and exponentially distributed with mean n.

The PWIT was also studied in the domain of branching random walks [1]. A branching random walk is an extension of the Galton–Watson branching process obtained by associating to each individual of the process a real-valued random variable. In a tree generated by the branching process, the displacement of a particle is defined as the sum of random variables associated to the particle's ancestors, from the root to the particle itself. An object of interest in a branching random walk is the minimal displacement of particles in the nth generation, denoted by M_n.

Let L_x be the length of the longest path in the PWIT, between the root vertex and all the vertices at a distance of at most x from the root. Itoh and Krapivsky [14] gave arguments supporting the assertion that the distribution of L_x attains travelling wave behaviour for large x. It is assumed that the convergence to a travelling wave is a general phenomenon for the minimal displacement in branching random walks, but only a few examples exist where it is explicitly proven [6], [11]. A consequence of the travelling wave behaviour in the limit is tightness of the minimal displacements, that is, the tightness of the family $\{M_n - \text{median}(M_n), n \geq 1\}$, which was also separately studied in [10]. A stronger result than tightness is the existence of an exponential upper bound for the deviation probability for $M_n - \text{median}(M_n)$, which was obtained in [1], [2], and [11].

In this paper we obtain the asymptotic expression for the median of the length of the longest path L_x in the PWIT and show that the distribution of L_x has uniform exponential tails.

Theorem 2. *The median of the length of the longest path of the PWIT, for $x \geq 1$, is*

$$\text{median}(L_x) = x\mathrm{e} - \tfrac{3}{2}\log x + O(1).$$

Theorem 3. *For the length of the longest path of the PWIT, for all $x \geq 1$, $y \geq 0$, and any $c_1 < 1/\mathrm{e}$, there exists a constant C_1, dependent only on c_1, so that*

$$\mathbb{P}(L_x \leq \text{median}(L_x) - y) \leq C_1 \mathrm{e}^{-c_1 y},$$

and for any $c_2 < 1$ there exists a constant C_2, dependent only on c_2, so that

$$\mathbb{P}(L_x \geq \text{median}(L_x) + y) \leq C_2 \mathrm{e}^{-c_2 y}.$$

In particular,

$$\mathbb{E}L_x = x\mathrm{e} - \tfrac{3}{2}\log x + O(1) \quad \textit{and} \quad \text{var}(L_x) = O(1).$$

The paper is organized as follows. In Section 2 we show that the studied random graphs are almost surely rooted geometric graphs. Theorem 1 is proven in Section 3 and the proofs of Theorem 2 and Theorem 3 are given in Section 4. In Section 3 we use a definition of distance between two rooted geometric graphs, which we study in more detail in Appendix A.

2. Rooted geometric graphs

The closure of vertex 0 in a directed random graph on vertex set $\mathbb{N}_0$ with edge probability n^{-1} is a graph with the root at vertex 0 and a countably infinite vertex set. The weight function w_n makes the graph locally finite as, for any $\rho > 0$, the number of vertices within the distance ρ from a vertex v of the graph is at most $2\lfloor \rho n \rfloor + 1$.

The PWIT is also a rooted graph with a countably infinite vertex set. To show that the PWIT is almost surely a rooted geometric graph, it is left to prove that is almost surely locally finite.

We start by a connection between the PWIT and the Yule process. The *Yule process* $\{Y(t),\ t \geq 0\}$ is a branching process which starts at time 0 with one particle, $Y(0) = 1$, and each particle, independently of all others and of the past of the process, after an exponential waiting time with mean 1 splits into two particles. The number of particles present at time t is $Y(t)$.

Lemma 1. *Let $\{Y(t),\ t \geq 0\}$ be the Yule process and let T be the PWIT. Let, for $t > 0$, $|V_t(T)|$ be the number of vertices of the graph T at a distance of at most t from the root. Then*

$$\{Y(t), t \geq 0\} \stackrel{\mathrm{D}}{=} \{|V_t(T)|,\ t \geq 0\},$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution.

Proof. For $t = 0$, $Y(0) = 1$, $V_0(T) = \varnothing$, and $|V_0(T)| = 1$. Because of the memoryless property of the exponential distribution, at any time $t \geq 0$ the time left until the next splitting occurs in the Yule process is the minimum of $Y(t)$ exponential random variables with mean 1, that is, exponential random variable with mean $Y(t)^{-1}$. Similarly for the PWIT, at time $t \geq 0$ there are $|V_t(T)|$ vertices and each vertex is assigned a Poisson process which defines the weights of the edges. The closest vertex to the root that is outside $V_t(T)$ is determined by the position of the first point after t in these $|V_t(T)|$ Poisson processes. Thus, the first vertex with distance from the root greater than t is at the distance $t + X$ from the root, where X is exponentially distributed random variable with mean $|V_t(T)|^{-1}$. □

Proposition 1. *The PWIT is almost surely a locally finite graph.*

Proof. Let T be a PWIT and let $\{Y(t), t \geq 0\}$ be the Yule process. For the Yule process is $\mathbb{E}(Y(t)) = \mathrm{e}^t$ [5, Chapter III.4] and, thus, by Lemma 1, we have $\mathbb{E}(|V_t(T)|) = \mathrm{e}^t$. Hence, the number of vertices within the distance t from the root is, almost surely, finite.

Let v be a vertex of T and denote by r the distance between v and the root. Note that $r < \infty$. Then the set of vertices that are within distance ρ from vertex v is a subset of $V_{r+\rho}(T)$. Therefore, there exists almost surely finitely many such vertices and the PWIT is almost surely a locally finite graph. □

3. Convergence to the PWIT

In this section we prove Theorem 1. The first step is to show that for large enough n, with high probability, the directed graph on the segment $[0, \rho]$ is a tree, which suggests that the limiting object should also be a tree.

Lemma 2. *Let T_n, $n \in \mathbb{N}$, be the closure of vertex 0 in a directed random graph on $\mathbb{N}_0$, with edge probability n^{-1} and weight function w_n. For $\rho > 0$,*

$$\mathbb{P}(N_\rho(T_n) \textit{ is a tree}) \to 1 \quad \textit{as } n \to \infty.$$

Proof. Note first that for $\rho > 0$ it holds that $|V_\rho(T_n)| \le \lfloor n\rho \rfloor + 1$. For vertices i and j, $i < j$, we denote by $i \rightsquigarrow j$ the event that there exists a path from vertex i to vertex j and by $i \overset{2}{\rightsquigarrow} j$ the event that there exist two nonintersecting paths from vertex i to j, where by nonintersecting we mean that the paths do not have any common vertices except i and j. Define now for $n \in \mathbb{N}$ and $\rho > 0$ the events

$$A^{\rho,n} = \{N_\rho(T_n) \text{ is not a tree}\}$$

and for $i, j \in \mathbb{N}_0$, $i < j$,

$$A^{\rho,n}_{i,j} = \{i, j \in V_\rho(T_n),\ i \overset{2}{\rightsquigarrow} j\}.$$

Obviously, when $j = i + 1$ the events $A^{\rho,n}_{i,j}$ are empty. If the graph $N_\rho(T_n)$ is not a tree, there exist vertices $i_1, i_2, j \in V_\rho(T_n)$, $i_1, i_2 < j$, such that (i_1, j) and (i_2, j) are edges of $N_\rho(T_n)$. Then we can find a vertex $i \in V_\rho(T_n)$, $i \le \min\{i_1, i_2\}$, such that i and j are connected with two nonintersecting paths. Therefore, we can write

$$\mathbb{P}(A^{\rho,n}) \le \sum_{0 \le i < j \le \lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{i,j}) = \sum_{j=2}^{\lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{0,j}) + \sum_{i=1}^{\lfloor n\rho \rfloor - 2} \sum_{j=i+2}^{\lfloor n\rho \rfloor} \mathbb{P}(A^{\rho,n}_{i,j}). \tag{1}$$

For $i \ge 1,\ i + 2 \le j \le \lfloor n\rho \rfloor$, we have

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}_{i,j}) &= \mathbb{P}(0 \rightsquigarrow i)\mathbb{P}(i \overset{2}{\rightsquigarrow} j) \\
&\le \sum_{k=0}^{i-1} \mathbb{P}((0, v_1, v_2, \dots, v_k, i) \text{ is a path in } N_\rho(T_n)) \\
&\quad \times \sum_{\ell=1}^{j-i-1} \mathbb{P}((i, u_1, u_2, \dots, u_{\ell_1}, j) \text{ and } (i, u'_1, u'_2, \dots, u'_{\ell_2}, j) \text{ are} \\
&\qquad\qquad \text{nonintersecting paths in } N_\rho(T_n), \ell_1 + \ell_2 = \ell) \\
&= \sum_{k=0}^{i-1} \binom{i-1}{k} p^{k+1} \sum_{\ell=1}^{j-i-1} \binom{j-i-1}{\ell} 2^{\ell-1} p^{\ell+2} \\
&< \frac{p^3}{2}(1+p)^{i-1}(1+2p)^{j-i-1}.
\end{aligned}$$

Similarly, for $i = 0,\ 2 \le j \le \lfloor n\rho \rfloor$,

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}_{0,j}) &= \mathbb{P}(0 \overset{2}{\rightsquigarrow} j) \\
&\le \sum_{\ell=1}^{j-i-1} \mathbb{P}((0, u_1, u_2, \dots, u_{\ell_1}, j) \text{ and } (0, u'_1, u'_2, \dots, u'_{\ell_2}, j) \text{ are} \\
&\qquad\qquad \text{nonintersecting paths in } N_\rho(T_n), \ell_1 + \ell_2 = \ell) \\
&= \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} 2^{\ell-1} p^{\ell+2} \\
&< \frac{p^2}{2}(1+2p)^{j-1}.
\end{aligned}$$

Applying the above inequalities to (1) and replacing p with n^{-1}, we obtain

$$\begin{aligned}
\mathbb{P}(A^{\rho,n}) &< \sum_{j=2}^{\lfloor n\rho\rfloor} \frac{p^2}{2}(1+2p)^{j-1} + \sum_{i=1}^{\lfloor n\rho\rfloor-2}\sum_{j=i+2}^{\lfloor n\rho\rfloor} \frac{p^3}{2}(1+p)^{i-1}(1+2p)^{j-i-1} \\
&< \frac{p^2}{2}\frac{(1+2p)^{\lfloor n\rho\rfloor}-(1+2p)}{2p} + \frac{p^3}{2}\sum_{i=1}^{\lfloor n\rho\rfloor-2}(1+p)^{i-1}\frac{(1+2p)^{\lfloor n\rho\rfloor-i}-(1+2p)}{2p} \\
&< \frac{p}{4}(1+2p)^{\lfloor n\rho\rfloor} \\
&\quad + \frac{p^2}{4}(1+2p)^{\lfloor n\rho\rfloor-1}\biggl\{\biggl(1-\biggl(\frac{1+p}{1+2p}\biggr)^{\lfloor n\rho\rfloor-2}\biggr)\biggl(1-\frac{1+p}{1+2p}\biggr)^{-1}\biggr\} \\
&< \frac{p}{2}(1+2p)^{\lfloor n\rho\rfloor} \\
&< \frac{1}{2n}\mathrm{e}^{2\rho},
\end{aligned}$$

which gives

$$\mathbb{P}(N_\rho(T_n)\text{ is a tree}) = 1-\mathbb{P}(A^{\rho,n}) \ge 1-\frac{1}{2n}\mathrm{e}^{2\rho} \to 1 \quad \text{as } n\to\infty. \qquad \square$$

In the proofs of Lemma 3 and Theorem 1 we use the following definition for a distance between two geometric graphs $G_1, G_2 \in \mathcal{G}_*$:

$$d(G_1,G_2) = \int_0^\infty \frac{R(N_\rho(G_1),N_\rho(G_2))}{\mathrm{e}^\rho}\,\mathrm{d}\rho,$$

where

$$R(N_\rho(G_1),N_\rho(G_2)) = \min\biggl\{1, \min_{\substack{\{\Phi\colon \Phi\colon V_\rho(G_1)\to V_\rho(G_2)\\ \text{graph isomorphism}\}}} \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e)-w(\Phi(e))|\biggr\};$$

see Appendix A for more details.

To show convergence in distribution of rooted geometric graphs $\{G_n, n\ge 1\}$ to G, we show one of the equivalent statements of the Portmanteau theorem [8, Theorem 2.1]:

$$\lim_{n\to\infty}\mathbb{P}(G_n\in A) = \mathbb{P}(G\in A)$$

for all Borel sets A whose boundary ∂A satisfies $\mathbb{P}(G\in\partial A)=0$.

Furthermore, define the *Bernoulli-weighted infinite tree with parameter* n^{-1} (BWIT$_n$) to be the Ulam–Harris tree with weight function $\hat{w}_n$ on edges defined as follows. Let $\{X_j^u, u\in U, j\in\mathbb{N}\}$ be a sequence of independent and geometrically distributed random variables with law $\mathbb{P}(X_j^u = k) = n^{-1}(1-n^{-1})^{k-1}$, $k\in\mathbb{N}$. The weight function $\hat{w}_n$ is $\hat{w}_n(u,uj) = n^{-1}\sum_{k=1}^{j} X_k^u$.

Lemma 3. *Let $\hat{T}_n$, $n\in\mathbb{N}$, be a BWIT$_n$. Then the sequence $\hat{T}_n$ converges in distribution to a PWIT as $n\to\infty$.*

Proof. Let T and $\widetilde{T}_n$, $n\in\mathbb{N}$, be the Ulam–Harris trees with weight functions w and $\widetilde{w}_n$ on edges defined as follows. Let $\{Y_j^u, u\in U, j\in\mathbb{N}\}$ be independent and exponentially distributed random variables, $\mathbb{P}(Y_j^u\le x) = 1-\mathrm{e}^{-x}$, $x\ge 0$. For $n\in\mathbb{N}$, let $a_n = 1/\log(n/(n-1))$ and set $X^u_{n,j} = \lceil a_n Y^u_j \rceil$, $u \in U$, $j \in \mathbb{N}$. Then the random variables $\{X^u_{n,j}, u \in U, j \in \mathbb{N}\}$ have geometric distribution with parameter n^{-1}. The weight functions of the trees T and $\widetilde{T}_n$ are

$$w(u, uj) = \sum_{k=1}^{j} Y^u_k \quad \text{and} \quad \widetilde{w}_n(u, uj) = \frac{1}{n} \sum_{k=1}^{j} X^u_{n,k} \quad \text{for } u \in U,\ j \in \mathbb{N}.$$

The tree T has the law of a PWIT and $\widetilde{T}_n$ has, the same as $\hat{T}_n$, the law of a BWIT_n. Moreover, $n^{-1} X^u_{n,j} \to Y^u_j$, almost surely, as $n \to \infty$, so the weights of the edges of trees $\widetilde{T}_n$ converge, almost surely, to the weights of the corresponding edges of the tree T. Therefore, almost surely, for all continuity points ρ of T, there is $n_0 \in \mathbb{N}$, such that $N_\rho(\widetilde{T}_n)$ and $N_\rho(T)$ are graph isomorphic for all $n \geq n_0$ and $R(N_\rho(\widetilde{T}_n), N_\rho(T)) \to 0$ as $n \to \infty$. By the dominated convergence theorem, it follows that $d(\widetilde{T}_n, T) \to 0$, almost surely, as $n \to \infty$. Thus, for any Borel set A with $\mathbb{P}(T \in \partial A) = 0$, we have

$$\lim_{n\to\infty} \mathbb{P}(\hat{T}_n \in A) = \lim_{n\to\infty} \mathbb{P}(\widetilde{T}_n \in A) = \mathbb{P}(T \in A). \qquad \square$$

Proof of Theorem 1. Fix $\rho > 0$ and let $\mathcal{T}_{n,\rho}$ be the set of all rooted geometric graphs G which are trees and for which $\mathbb{P}(N_\rho(T_n) = G) > 0$. Since a vertex i of T_n is connected with any vertex in $\{i+1, i+2, \dots\}$ with probability n^{-1}, the corresponding weights of the edges from i are sums of geometrically distributed random variables with parameter n^{-1} rescaled by n^{-1}. Thus, for $G \in \mathcal{T}_{n,\rho}$ is $\mathbb{P}(N_\rho(T_n) = G) = \mathbb{P}(N_\rho(\hat{T}_n) = G)$, where $\hat{T}_n$ is a BWIT_n. Moreover, whenever the graphs $N_\rho(T_n)$ and $N_\rho(\hat{T}_n)$ are geometric isomorphic, the distance between T_n and $\hat{T}_n$ is

$$d(T_n, \hat{T}_n) = \int_\rho^\infty \frac{R(N_r(T_n), N_r(\hat{T}_n))}{\mathrm{e}^r}\, \mathrm{d}r \leq \int_\rho^\infty \frac{1}{\mathrm{e}^r}\, \mathrm{d}r = \frac{1}{\mathrm{e}^\rho}. \tag{2}$$

Let $A \subset \mathcal{G}_*$ be a Borel set with $\mathbb{P}(T \in \partial A) = 0$. Furthermore, for $\varepsilon > 0$, define $A_{+\varepsilon} = \{G \in \mathcal{G}_* \colon \text{there is } G' \in A \text{ such that } d(G, G') < \varepsilon\}$ and $A_{-\varepsilon} = (A^c_{+\varepsilon})^c$. Using these definitions and the upper bound (2), we can write

$$\mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}, N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) \leq \mathbb{P}(T_n \in A, N_\rho(T_n) \in \mathcal{T}_{n,\rho}) \leq \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}). \tag{3}$$

Using

$$\mathbb{P}(T_n \in A) = \mathbb{P}(T_n \in A, N_\rho(T_n) \in \mathcal{T}_{n,\rho}) + \mathbb{P}(T_n \in A, N_\rho(T_n) \notin \mathcal{T}_{n,\rho})$$

and (3), we obtain

$$\begin{aligned}\mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}, N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) &\leq \mathbb{P}(T_n \in A) \\ &\leq \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}) + \mathbb{P}(N_\rho(T_n) \notin \mathcal{T}_{n,\rho}).\end{aligned}$$

Letting $n \to \infty$ and using the result of Lemma 2 that $\mathbb{P}(N_\rho(\hat{T}_n) \in \mathcal{T}_{n,\rho}) = \mathbb{P}(N_\rho(T_n) \in \mathcal{T}_{n,\rho}) \to 1$ as $n \to \infty$ yields

$$\liminf_{n\to\infty} \mathbb{P}(\hat{T}_n \in A_{-\mathrm{e}^{-\rho}}) \leq \liminf_{n\to\infty} \mathbb{P}(T_n \in A) \leq \limsup_{n\to\infty} \mathbb{P}(T_n \in A) \leq \limsup_{n\to\infty} \mathbb{P}(\hat{T}_n \in A_{+\mathrm{e}^{-\rho}}).$$

Finally, letting $\rho \to \infty$ in the inequalities above and Lemma 3, we obtain

$$\lim_{n\to\infty} \mathbb{P}(T_n \in A) = \lim_{n\to\infty} \mathbb{P}(\hat{T}_n \in A) = \mathbb{P}(T \in A),$$

where T is a PWIT. $\square$

4. The length of the longest path in the PWIT

Denote by $S(v)$ the distance of a vertex v from the root of the PWIT. For $n \in \mathbb{N}$, define the minimal displacement over all individuals in the nth generation to be $M_n = \inf_{v \in \mathbb{N}^n} S(v)$ and let $\widetilde{M}_n = \operatorname{median}(M_n) = \sup\{x : \mathbb{P}(M_n < x) < \frac{1}{2}\}$. The minimal displacement of the nth generation of a PWIT was studied by Addario-Berry and Ford [1], who obtained the following results.

Theorem 4. *The median of the minimal displacement of the nth generation of the PWIT is*

$$\widetilde{M}_n = \frac{n}{\mathrm{e}} + \frac{3}{2\mathrm{e}} \log n + O(1).$$

Theorem 5. *For the minimal displacement M_n of the PWIT, for all $n \geq 1$, $x \geq 0$, and any $c_1 < \mathrm{e}$, there exists a constant C_1, dependent only on c_1, such that*

$$\mathbb{P}(M_n \leq \widetilde{M}_n - x) \leq C_1 \mathrm{e}^{-c_1 x},$$

and for any $c_2 < 1$, there exists a constant C_2, dependent only on c_2, such that

$$\mathbb{P}(M_n \geq \widetilde{M}_n + x) \leq C_2 \mathrm{e}^{-c_2 x}.$$

The length of a path is the number of edges in the path. We denote by L_x the length of the longest path in the PWIT between all the paths of a tree starting from the root and ending in a vertex at a distance of at most x from the root. Let $\widetilde{L}_x$ be a median of L_x and let $\hat{L}_x = \max\{n : \mathbb{P}(L_x \geq n) \geq \frac{1}{2}\}$. Note that $|\hat{L}_x - \widetilde{L}_x| \leq 1$, so the asymptotic result for $\hat{L}_x$ holds for the median as well. We will show that Theorem 2 and Theorem 3 follow from Theorem 4 and Theorem 5, respectively, and from the observation: $L_x \geq n$ if and only if $M_n \leq x$.

Proof of Theorem 2. Combining the definition of $\hat{L}_x$ with the above observation, we obtain

$$\hat{L}_x = \max\bigl\{n : \mathbb{P}(L_x \geq n) \geq \tfrac{1}{2}\bigr\} = \max\bigl\{n : \mathbb{P}(M_n \leq x) \geq \tfrac{1}{2}\bigr\}.$$

The maximum in the last term above will be achieved for n such that $\widetilde{M}_n \leq x < \widetilde{M}_{n+1}$. Using Theorem 4 we can assert that the n that satisfies these inequalities is $n = x\mathrm{e} - \frac{3}{2} \log x + O(1)$, which is then the asymptotic value of $\hat{L}_x$ and $\widetilde{L}_x$. □

Proof of Theorem 3. Using the second part of Theorem 5 and definitions of $\widetilde{M}_n$ and $\widetilde{L}_x$ from Theorem 4 and Theorem 2, respectively, for any $x \geq 1$, $0 \leq y \leq \widetilde{L}_x$, and $c_2 < 1$, we have

$$\begin{aligned}
\mathbb{P}(L_x \leq \widetilde{L}_x - y) &\leq \mathbb{P}(M_{\lfloor \widetilde{L}_x - y \rfloor + 1} > x) \\
&\leq C_2 \exp\{-c_2(x - \widetilde{M}_{\lfloor \widetilde{L}_x - y \rfloor + 1})\} \\
&\leq C_2 \exp\biggl\{-c_2\biggl(x - \frac{\lfloor \widetilde{L}_x - y \rfloor + 1}{\mathrm{e}} - \frac{3}{2\mathrm{e}} \log(\lfloor \widetilde{L}_x - y \rfloor + 1) + O(1)\biggr)\biggr\} \\
&\leq C_2 \exp\biggl\{-c_2 \frac{y}{\mathrm{e}} + c_2 \frac{3}{2\mathrm{e}} \log \frac{x\mathrm{e} - (3/2) \log x}{x} + O(1)\biggr\} \\
&\leq C_1' \mathrm{e}^{-c_1' y},
\end{aligned}$$

where $c_1' = c_2/\mathrm{e}$. For $y \geq \widetilde{L}_x$ the same holds because $\mathbb{P}(L_x \leq \widetilde{L}_x - y) = 0$.

The second statement of the theorem can be obtained in an analogous way.

Applying Theorem 3 to $\mathbb{E}L_x = \sum_{n=1}^{\infty}\mathbb{P}(L_x \geq n)$, we obtain $\mathbb{E}L_x = \widetilde{L}_x + O(1)$ and then from Theorem 2, we have $\mathbb{E}L_x = x\mathrm{e} - 3\log x/2 + O(1)$. Moreover, $\widetilde{L}_x$ can be replaced by $\mathbb{E}L_x$ in the first part of Theorem 3 and this can be used to calculate $\operatorname{var}(L_x) = O(1)$. □

Appendix A. A metric on $\mathcal{G}_*$

On the set of simple rooted graphs (rooted geometric graphs with all edges of weight 1), a distance between two graphs G_1 and G_2 is sometimes defined as

$$d(G_1, G_2) = \frac{1}{2^{R(G_1,G_2)}},$$

where $R(G_1, G_2) = \inf\{\rho \in \mathbb{N}_0 \colon N_\rho(G_1) \text{ and } N_\rho(G_2) \text{ are not graph isomorphic}\}$.

We expand this formula to the definition of the distance for rooted geometric graphs in the following way. Let $G_1, G_2 \in \mathcal{G}_*$, then

$$d(G_1, G_2) = \int_0^\infty \frac{R(N_\rho(G_1), N_\rho(G_2))}{\mathrm{e}^\rho}\,\mathrm{d}\rho,$$

where

$$R(N_\rho(G_1), N_\rho(G_2)) = \min\Bigg\{1, \min_{\substack{\{\Phi\colon \Phi\colon V_\rho(G_1)\to V_\rho(G_2)\\ \text{graph isomorphism}\}}} \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e) - w(\Phi(e))|\Bigg\}.$$

Note that $R(N_\rho(G_1), N_\rho(G_2)) = 1$ if there is no graph isomorphisms from $N_\rho(G_1)$ to $N_\rho(G_2)$. Whenever there exist an isomorphism then the set of edges will be nonempty with an exception if the graphs contain just the root vertex. In the latter case, we define $R(N_\rho(G_1), N_\rho(G_2)) = 0$.

Proposition 2. *The function d is a metric on $\mathcal{G}_*$ and $(\mathcal{G}_*, d)$ is a complete separable metric space.*

Proof. The function d is symmetric and nonnegative, with $d(G_1, G_2) = 0$ if and only if the graphs G_1 and G_2 are graph isomorphic and all the corresponding edges have the same weights, i.e. if there exists a geometric isomorphism from G_1 to G_2.

Now fix $\rho > 0$ and let $G_1, G_2, G_3 \in \mathcal{G}_*$. We want to show that the triangle inequality is satisfied for the function R, which by integrating over ρ immediately gives the triangle inequality for the function d. If the graphs $N_\rho(G_1), N_\rho(G_2), N_\rho(G_3)$ are not graph isomorphic the triangle inequality holds. Otherwise, let $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ be the graph isomorphisms from $N_\rho(G_1)$ to $N_\rho(G_2)$ and from $N_\rho(G_2)$ to $N_\rho(G_3)$, respectively, for which the functions $R(N_\rho(G_1), N_\rho(G_2))$ and $R(N_\rho(G_2), N_\rho(G_3))$ attain the minima. Then, by the triangle inequality, for any edge e of $N_\rho(G_1)$, we have

$$|w(e) - w(\Phi_{\rho,1}(e))| + |w(\Phi_{\rho,1}(e)) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))| \geq |w(e) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))|.$$

Taking a maximum over edges of $N_\rho(G_1)$ yields

$$\begin{aligned} &R(N_\rho(G_1), N_\rho(G_2)) + R(N_\rho(G_2), N_\rho(G_3)) \\ &\qquad \geq \min\Big\{1, \max_{\{e\colon e \text{ edge of } N_\rho(G_1)\}} |w(e) - w(\Phi_{\rho,2}\circ\Phi_{\rho,1}(e))|\Big\} \\ &\qquad \geq R(N_\rho(G_1), N_\rho(G_3)), \end{aligned}$$

which is precisely the triangle inequality for function R.

We now proceed to show that this metric space is complete. Let $\{G_n,\ n \geq 1\}$ be a Cauchy sequence of graphs in $\mathcal{G}_*$, that is, a sequence for which for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for all $m, n \geq n_0$ the distance $d(G_n, G_m) < \varepsilon$. We want to show that there exists a graph $G \in \mathcal{G}_*$ for which $d(G_n, G) \to 0$ when $n \to \infty$.

We start by choosing a subsequence $\{G_{n_i},\ i \geq 1\}$ of the Cauchy sequence of graphs such that $d(G_{n_i}, G_{n_{i+1}}) < 2^{-i}$. Then, we have

$$1 > \sum_{i=1}^{\infty} d(G_{n_i}, G_{n_{i+1}}) = \int_0^{\infty} \sum_{i=1}^{\infty} \frac{R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}}))}{\mathrm{e}^{\rho}}\, \mathrm{d}\rho$$

and, thus, $\sum_{i=1}^{\infty} R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}})) < \infty$ for almost all ρ. For all such ρ and for all ε, $0 < \varepsilon < 1$, there exists an $i_0 = i_0(\rho, \varepsilon)$ such that $R(N_\rho(G_{n_i}), N_\rho(G_{n_{i+1}})) < \varepsilon$ for all $i \geq i_0$. Therefore, for $i \geq i_0$ the pairs of graphs $N_\rho(G_{n_i})$ and $N_\rho(G_{n_{i+1}})$ are graph isomorphic and, hence, all the graphs $\{N_\rho(G_{n_i}), i \geq i_0\}$ are graph isomorphic. Denote now by $\Phi_{\rho,i}$ a graph isomorphism from $N_\rho(G_{n_i})$ to $N_\rho(G_{n_{i+1}})$ for which the maximum of the edge weight differences is minimal. Furthermore, define for $i < j$ graph isomorphism $\Phi_{\rho,(i,j)} = \Phi_{\rho,j-1} \circ \Phi_{\rho,j-2} \cdots \circ \Phi_{\rho,i}$ and let $\Phi_{\rho,(i,i)}$ be the identity graph isomorphism. Then, for all $\varepsilon' > 0$, there exists a $j_0 = j_0(\rho, \varepsilon')$, $j_0 \geq i_0$, such that for $j > i \geq j_0$ and for any edge e of the graph $N_\rho(G_{n_i})$,

$$\begin{aligned}
|w(e) - w(\Phi_{\rho,(i,j)}(e))| &\leq \sum_{k=i}^{j-1} |w(\Phi_{\rho,(i,k)}(e) - w(\Phi_{\rho,(i,k+1)}(e))| \\
&\leq \sum_{k=i}^{j-1} R(N_\rho(G_{n_k}), N_\rho(G_{n_{k+1}})) \\
&\leq \sum_{k=i}^{\infty} R(N_\rho(G_{n_k}), N_\rho(G_{n_{k+1}})) \\
&< \varepsilon'.
\end{aligned}$$

Thus, for each edge e of the graph $N_\rho(G_{n_{i_0}})$ the sequence $\{w(\Phi_{\rho,(i_0,i)}(e)),\ i \geq i_0\}$ is a Cauchy sequence and, hence, it converges.

Construct now the limit graph G in radius ρ in the following way. Set $N_\rho(G)$ to be graph isomorphic to $N_\rho(G_{n_{i_0}})$ and let Φ_ρ be the graph isomorphism from $N_\rho(G)$ to $N_\rho(G_{n_{i_0}})$. Furthermore, set the weight of every edge e of $N_\rho(G)$ to be $w(e) = \lim_{i\to\infty} w(\Phi_{\rho,(i_0,i)} \circ \Phi_\rho(e))$. In the same way, we can construct the limit graph G in any arbitrary large radius. For each continuity point ρ of the limit graph G, the function $R(N_\rho(G_{n_i}), N_\rho(G)) \to 0$ as $i \to \infty$ and then by the dominated convergence theorem $d(G_{n_i}, G) \to 0$ as $i \to \infty$.

Moreover, we need to show that $d(G_n, G) \to 0$. For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for $m, n \geq n_0$ $d(G_n, G_m) \leq \varepsilon/2$ and there exists $i_0 = i_0(\varepsilon)$ such that for $i \geq i_0$ is $n_i \geq n_0$ and $d(G_{n_i}, G) \leq \varepsilon/2$. Then, by the triangle inequality, we have

$$d(G_n, G) \leq d(G_n, G_{n_i}) + d(G_{n_i}, G) \leq \varepsilon$$

and, thus, $d(G_n, G) \to 0$ as $n \to \infty$.

Separability follows from the facts that the geometric graphs are locally finite and rational numbers are a countable dense subset of the real line. Hence, an example of a countable dense subset of the set of the rooted geometric graphs is the set of all rooted geometric graphs with finite number of vertices and rational edge weights. □

Proposition 3. *Rooted geometric graphs G_n, $n \in \mathbb{N}$, converge to rooted geometric graph G if and only if $d(G_n, G) \to 0$ as $n \to \infty$.*

Proof. If rooted geometric graphs G_n converge to G, then $R(N_\rho(G_n), N_\rho(G)) \to 0$ as $n \to \infty$ for all continuity points ρ. Moreover, the function $R(N_\rho(G_n), N_\rho(G))e^{-\rho}$ is bounded by the integrable function $e^{-\rho}$. Since there are countably many discontinuity points, by the dominated convergence theorem,

$$d(G_n, G) = \int_0^\infty \frac{R(N_\rho(G_n), N_\rho(G))}{e^\rho}\, d\rho \to 0 \quad \text{when } n \to \infty,$$

which proves one direction of the proposition.

Assume now that $d(G_n, G) \to 0$ as $n \to \infty$. Let $\rho > 0$ be a continuity point of graph G and let x be the last discontinuity point of G before ρ and y be the first discontinuity point after ρ. Set $\delta_1 = \rho - x$ and $\delta_2 = y - \rho$. For any ε, $0 < \varepsilon < \min\{\delta_1 e^{-\rho}, \delta_2 e^{-y}\}$, there exists $n_0 = n_0(\varepsilon)$ such that $d(G_n, G) < \varepsilon$ for all $n \geq n_0$. Assume now that $N_\rho(G_n)$ and $N_\rho(G)$ are not graph isomorphic for some $n \geq n_0$. Then, we have two cases, either the graphs are not graph isomorphic for all the points of the interval $[x, \rho]$ or there is a discontinuity point x' of G_n in the interval $[x, \rho]$, such that the graphs are graph isomorphic in the interval $[x, x')$, but not graph isomorphic in the interval $[x', \rho]$. In the latter case, the graphs are not graph isomorphic for all the points in the interval $[\rho, y)$. In these two cases, we have

$$d(G_n, G) \geq \int_x^\rho \frac{1}{e^r}\, dr > \frac{\delta_1}{e^\rho} \quad \text{or} \quad d(G_n, G) \geq \int_\rho^y \frac{1}{e^r}\, dr > \frac{\delta_2}{e^y},$$

which contradicts $d(G_n, G) < \varepsilon$. Thus, the graphs $N_\rho(G_n)$ and $N_\rho(G)$ are graph isomorphic for all $n \geq n_0$. Assume now that the weights of the edges do not converge to the weights of the edges of G, i.e. that there exist an $\varepsilon' > 0$ such that for all $n_1 \geq n_0$, there is an $n \geq n_1$ for which

$$\min_{\{\Phi\colon \Phi\colon V_\rho(G) \to V_\rho(G_n) \text{ graph isomorphism}\}} \max_{\{e\colon e \text{ edge of } N_\rho(G)\}} |w(e) - w(\Phi(e))| > \varepsilon'.$$

But then the function $R(N_r(G_n), N_r(G))$ is greater than ε' for r between x and ρ and, thus, similarly as above $d(G_n, G) > \varepsilon' \delta_1 e^{-\rho}$ which is impossible. Hence, for all continuity points ρ, the graphs are graph isomorphic for large enough n and the weights of the edges converge to the weights of the corresponding edges of G, which is precisely the definition of convergence of rooted geometric graphs. □

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