Programming Languages F28PL2, Lecture 2

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Languages and formal grammars

Recall that a language is a set of symbols/tokens and a set of (possibly empty) strings of tokens.

We will let α, β, γ range over strings.

This is a computing course, so we need to think not only about what a language is, but also about how a language may be generated.

We generate languages using formal grammar. Using a formal grammar we can:

- Verify whether a sentence is in our language.
- Synthesise legal programs.

Terminology

- Write V for the set of symbols (V for ‘vocabulary’).
- We may partition (split) the set V into two subsets: of terminal and nonterminal symbols. (Why we do this will become clear later.)
- Write V^* for the set of all strings of elements of V (including the empty string). Call this the closure of V.
- Write V^+ for the set of non-empty strings of elements of V.
- Write ε for the empty string — often written informally as ‘ ”” ’.

If $V = \{a, b, r, c, d, r\}$ (the set containing a, b, r, c, d, and r), then is $abracadabra \in V^*$?

Is ε always in V^*? How about V^+?

Example

Suppose a vocabulary $V = \{0, 1, +, -, *, (,), \langle\text{exp}\rangle\}$.

Suppose $\langle\text{exp}\rangle$ is nonterminal and all the other symbols are terminal.

Example sentences in V are (just elements of V^*):

- ε, the empty string.
- $1 + 1$.
- $(1 + 1)$ and $(1 + (1))$.
- $(((($ and $(())) * --$.
- Is $(1 + 2 + \langle\text{exp}\rangle)$ in V^*?

Terminology

Recall that α, β, γ range over strings.

A production rule is a pair $\alpha ::= \beta$.

Suppose a vocabulary $V = \{0, 1, +, -, *, (,), \langle\mathsf{exp}\rangle\}$.

Example production rules are:

$$
\begin{aligned}
\langle\mathsf{exp}\rangle &::= 0 \\
\langle\mathsf{exp}\rangle &::= 1 \\
\langle\mathsf{exp}\rangle &::= -\langle\mathsf{exp}\rangle \\
\langle\mathsf{exp}\rangle &::= (\langle\mathsf{exp}\rangle) \\
\langle\mathsf{exp}\rangle &::= \langle\mathsf{exp}\rangle + \langle\mathsf{exp}\rangle \\
\langle\mathsf{exp}\rangle &::= \langle\mathsf{exp}\rangle * \langle\mathsf{exp}\rangle
\end{aligned}
$$

Production rules

We write a sequence

$$\alpha ::= \beta_1, \ldots, \alpha ::= \beta_n \quad \text{as just}$$
$$\alpha ::= \beta_1 \mid \ldots \mid \beta_n.$$

For example:

$$\begin{array}{ll} \langle \text{exp} \rangle & ::= 0 \\ \langle \text{exp} \rangle & ::= 1 \\ \langle \text{exp} \rangle & ::= -\langle \text{exp} \rangle \\ \langle \text{exp} \rangle & ::= (\langle \text{exp} \rangle) \\ \langle \text{exp} \rangle & ::= \langle \text{exp} \rangle + \langle \text{exp} \rangle \\ \langle \text{exp} \rangle & ::= \langle \text{exp} \rangle * \langle \text{exp} \rangle \end{array} \quad \text{becomes} \quad \begin{array}{ll} \langle \text{exp} \rangle & ::= 0 \mid 1 \mid \langle \text{exp} \rangle + \langle \text{exp} \rangle \mid \\ & \qquad -\langle \text{exp} \rangle \mid \langle \text{exp} \rangle * \langle \text{exp} \rangle \mid (\langle \text{exp} \rangle) \end{array}$$

Production rules

We can use production rules to produce sentences. For example:

$$\begin{aligned} \langle \text{exp} \rangle &\Rightarrow -\langle \text{exp} \rangle \\ &\Rightarrow -(\langle \text{exp} \rangle) \\ &\Rightarrow -(\langle \text{exp} \rangle + \langle \text{exp} \rangle) \\ &\Rightarrow -(1 + \langle \text{exp} \rangle) \\ &\Rightarrow -(1+1) \end{aligned}$$

$$\begin{aligned} \langle \text{exp} \rangle &\Rightarrow \langle \text{exp} \rangle + \langle \text{exp} \rangle \\ &\Rightarrow 1 + \langle \text{exp} \rangle \\ &\Rightarrow 1 + \langle \text{exp} \rangle * \langle \text{exp} \rangle \\ &\Rightarrow 1 + 0 * \langle \text{exp} \rangle \\ &\Rightarrow 1 + 0 * 1 \end{aligned}$$

So, starting from the nonterminal $\langle \text{exp} \rangle$, we can generate many different sentences.

Grammars

Formally, a grammar is a 4-tuple of:

- $\mathbb{N}$ a set of nonterminal symbols.
- $\mathbb{T}$ a set of terminal symbols, disjoint from $\mathbb{N}$.
- A start symbol, in $\mathbb{N}$.
- A set of productions $\alpha ::= \beta$.

Notational conventions

Some important notational conventions which you are required to just know:

$A, B, C, S, T, \langle \text{exp} \rangle, \ldots$ range over nonterminals ($\mathbb{N}$).

$a, b, c, \ldots$ range over terminals ($\mathbb{T}$).

We call $\mathbb{N} \cup \mathbb{T}$ a vocabulary. X, Y, Z range over $\mathbb{N} \cup \mathbb{T}$.

Strings of terminals: x, y, z

Strings of terminals and/or nonterminals: $\alpha, \beta, \gamma, \ldots$

Terminology

The object-language is a language, defined as the set of strings of terminals that we can produce using the production rules, starting from the start symbol.

The meta-language is the language, defined as the set of all strings of terminals or nonterminals that we can produce using the production rules, starting from the start symbol.

The meta-language contains sentences of the object-language, but it may also contain extra sentences.

Production rules

What were the terminals and non-terminals implicit in the example production rules considered previously?

What was the start symbol?

Example grammars

$$\begin{aligned}\langle\text{exp}\rangle \; &::= 0 \mid 1 \mid \langle\text{exp}\rangle + \langle\text{exp}\rangle \mid \\ &\quad - \langle\text{exp}\rangle \mid \langle\text{exp}\rangle * \langle\text{exp}\rangle \mid (\langle\text{exp}\rangle) \qquad \text{Start symbol: } \langle\text{exp}\rangle\end{aligned}$$

$$S ::= ab \mid aSb \qquad \text{Start symbol: } S$$

$$S ::= aS \mid aT \qquad \text{Start symbol: } S$$

$$T ::= b \mid bT \qquad \text{Start symbol: } T$$

This is generative grammar. Let's generate a sentence using the second example:

$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aaSbb \\ &\Rightarrow aaabbb \end{aligned}$$

Chomsky classification of grammars

Type 0 grammars contain productions of the form

$$\alpha ::= \beta.$$

α is a non-empty string of terminal and/or nonterminal symbols.

Type 0 grammars include pretty much anything.

Type 1 or context-sensitive grammars contain productions of the form

$$\alpha A \gamma ::= \alpha \beta \gamma.$$

Here A denotes a single nonterminal and β denotes an arbitrary string of terminal and/or nonterminal symbols. You can 'expand' A — subject to it occurring in the context described by α and γ.

Production rules

Things get more restrictive:

Type 2 or context-free grammars contain productions of the form

$$A ::= \gamma.$$

A denotes a single nonterminal. BNF is a language for describing Type 2 languages.

Type 3 or regular grammars contain productions of the form

$$\begin{aligned} A &::= aB \\ A &::= b \\ A &::= \epsilon. \end{aligned}$$

See also regular expressions.

Production rules

Two notions of type 3 grammar:

Left-regular	Right-regular
$A ::= Ba$	$A ::= aB$
$A ::= b$	$A ::= b$
$A ::= \epsilon$	$A ::= \epsilon.$

A right-regular grammar has the nonterminal (if any) to the right of the terminal. ‘Regular grammar’ or ‘type 3 grammar’ will mean right-regular grammar unless stated or implied otherwise.

Intuitively, a right-regular grammar is one that (reading left-to-right) produces any terminals it is going to produce first, then calls itself recursively.

Production rules

Type 3 grammars are good for identifying lexical units such as words; for instance “alphanumeric strings” or “numbers, possibly with underscores”. They are machines for extruding tokens.

Type 2 grammars are good for languages like “the language of arithmetic” or “Mary loves John”. They are machines for parsing grammatical sentences.

Most of the computer languages you know are determined by type 2 grammars (if $\langle$bool$\rangle$ then $\langle$exp$\rangle$ else $\langle$exp$\rangle$); the keywords of those languages are determined by type 3 grammars (if, then, and else).

Derivations

A little notation is useful:

$\alpha \Rightarrow \beta$ means 'β derived from α by some production'.

$\alpha \overset{p}{\Rightarrow} \beta$ means 'β derived from α by production p'.

$\alpha \overset{*}{\Rightarrow} \beta$ means 'β derived from α by zero or more productions'.

$\alpha \overset{+}{\Rightarrow} \beta$ means 'β derived from α by one or more productions'.

A type 2 (context-free) language

The language is

$$\mathbb{L} = \{a^n b^n \mid n \geq 1\}.$$

A grammar for it is

$$S ::= ab \mid aSb,$$

the start symbol is S.

Let's derive a sentence:

$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aaSbb \\ &\Rightarrow aaabbb \end{aligned}$$

Note: supports balanced bracketing!

A type 3 (regular) grammar

The language is

$$\mathbb{L} = \{a^p b^q \mid p{\geq}1, q{\geq}1\}.$$

A grammar for it is

$$S ::= aS \mid aT \qquad T ::= b \mid bT.$$

The start symbol is S.

Let's derive a sentence:

$$S \Rightarrow aS \Rightarrow aaT \Rightarrow aabT \Rightarrow aabbT \Rightarrow aabbb.$$

Note: does not support balanced bracketing.

Production rules

Suppose we want to know whether a sentence α in language $\mathbb{L}$?

One algorithm to decide this is to try to generate it by applying all possible production rules in all possible orders.

For example is $-(\mathsf{id} + \mathsf{id})$ in the language determined by this grammar:

$$\begin{aligned} \langle \mathsf{exp} \rangle \quad ::= \quad & \langle \mathsf{exp} \rangle + \langle \mathsf{exp} \rangle \mid \langle \mathsf{exp} \rangle * \langle \mathsf{exp} \rangle \mid \\ & (\langle \mathsf{exp} \rangle) \mid -\langle \mathsf{exp} \rangle \mid \mathsf{id} \end{aligned}$$

Production rules

Yes:

$$\begin{aligned}
\langle \text{exp} \rangle &\Rightarrow -\langle \text{exp} \rangle \\
&\Rightarrow -(\langle \text{exp} \rangle) \\
&\Rightarrow -(\langle \text{exp} \rangle + \langle \text{exp} \rangle) \\
&\Rightarrow -(\text{id} + \langle \text{exp} \rangle) \\
&\Rightarrow -(\text{id} + \text{id})
\end{aligned}$$

This is immensely inefficient! I am only claiming that this algorithm works in principle.

More on efficiency later.

More terminology you need to know

Phrase: a string derived from a nonterminal other than the start symbol.

Sentential form: a string derived from the start symbol.

Sentence: a sentential form without nonterminals.

How do we apply productions to form phrases, sentential forms, or sentences?

Leftmost derivation: a derivation where always the leftmost nonterminal is replaced. Gives rise to leftmost sentential form.

Rightmost derivation: a derivation where always the rightmost nonterminal is replaced. Gives rise to rightmost sentential form.

Leftmost derivation of $-(\mathsf{id} + \mathsf{id})$

$$
\begin{array}{lcl}
\langle \mathsf{exp} \rangle & \Rightarrow & -\langle \mathsf{exp} \rangle \\
 & \Rightarrow & -(\langle \mathsf{exp} \rangle) \\
 & \Rightarrow & -(\langle \mathsf{exp} \rangle + \langle \mathsf{exp} \rangle) \\
 & \Rightarrow & -(\mathsf{id} + \langle \mathsf{exp} \rangle) \\
 & \Rightarrow & -(\mathsf{id} + \mathsf{id})
\end{array}
$$

$-(\mathsf{id}+\mathsf{id})$ is a sentential form, a sentence, and a leftmost sentential form.

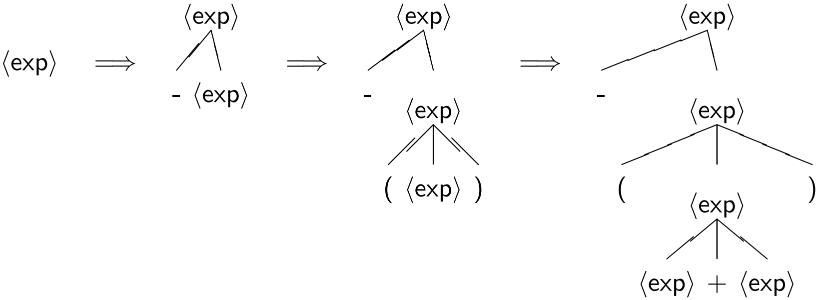
Rightmost derivation of $-(\mathsf{id}+\mathsf{id})$

$$\begin{array}{rcl}
\langle \mathsf{exp} \rangle & \Rightarrow & -\langle \mathsf{exp} \rangle \\
 & \Rightarrow & -(\langle \mathsf{exp} \rangle) \\
 & \Rightarrow & -(\langle \mathsf{exp} \rangle+\langle \mathsf{exp} \rangle) \\
 & \Rightarrow & -(\langle \mathsf{exp} \rangle+\mathsf{id}) \\
 & \Rightarrow & -(\mathsf{id}+\mathsf{id})
\end{array}$$

As it happens, $-(\mathsf{id}+\mathsf{id})$ is also a rightmost sentential form.

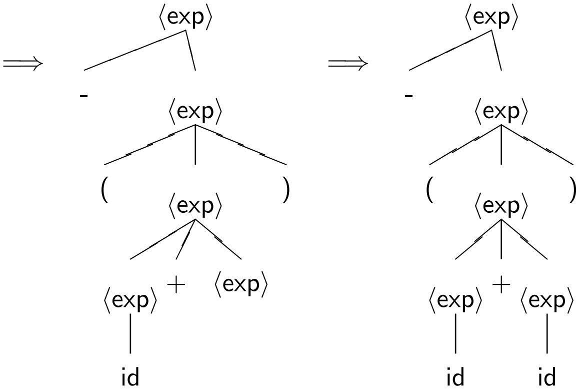
Parse trees and derivations. . .

Parse trees remember how a sentence was produced.



. . . just a bit more

The parse tree on the far right represents both leftmost and rightmost derivations given previously.



Different grammars

Two different grammars can define the same language $\mathbb{L}$.

Call two grammars equivalent when they describe the same language.

However, equivalent grammars can define different parse trees.

Two grammars

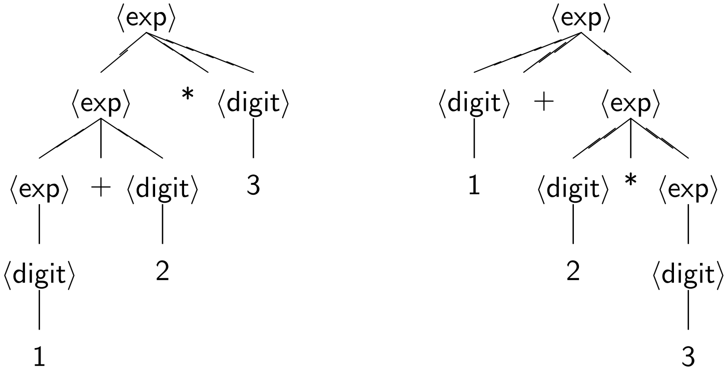
Grammar 1:

$$\begin{array}{rcl} \langle\text{exp}\rangle & ::= & \langle\text{digit}\rangle \mid \langle\text{exp}\rangle{+}\langle\text{digit}\rangle \mid \langle\text{exp}\rangle{*}\langle\text{digit}\rangle \\ \langle\text{digit}\rangle & ::= & 1 \mid 2 \mid 3 \end{array}$$

Grammar 2:

$$\begin{array}{rcl} \langle\text{exp}\rangle & ::= & \langle\text{digit}\rangle \mid \langle\text{digit}\rangle{+}\langle\text{exp}\rangle \mid \langle\text{digit}\rangle{*}\langle\text{exp}\rangle \\ \langle\text{digit}\rangle & ::= & 1 \mid 2 \mid 3 \end{array}$$

Different parse trees



Different parse trees

This is important, because different parse trees may induce different intuitive meanings.

Programs are not just syntax: we write a program because we give it meaning.

That meaning can be influenced by how we parse the string.

- The tree on the left intuitively means 9.
- The tree on the right intuitively means 7.