A mountain pass solution in cylinder buckling

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From: Rhodes & Walker '80 Thin walled structures
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Part 1 Experimental Evidence
  ▶ Axially localized solution
  ▶ Buckle/failure load

Part 2 Model

Part 3 Post-Buckle
  ▶ Homoclinic solution
    (with G. Hunt (Bath) and A. Champneys (Bristol))

Part 4 Failure load for cylinder
  ▶ Mountain pass
Part I: Experimental results

Typical end shortening vs load plot:

△ Post-buckle minimum load
△ Post-buckle plateau in load
- Localized buckled solution

- Translation invariant

- Well defined circumferential wave number $s$

- 2 forms of solution: **Symmetric** & **Cross Symmetric**.
Collection of experimental results:

- Donnell (steel): $Y = 1.0275X^{-0.17596}$
- Donnell (brass): $Y = 0.51114X^{-0.087858}$
- Bridget et al: $y = 0.37343x^{-0.0050409}$
- Ballerstedt & Wagner: $y = 4.2246x^{-0.3745}$

Linear prediction $\lambda_{fail}/\lambda_{cr} = 1$
Questions:

1. Can we compute post-buckle solution and loads?

2. Can we predict the load at which cylinder buckles?
Part II: A Model

von Kármán-Donnell equations:

\[
\begin{align*}
\kappa^2 \Delta^2 w + \lambda w_{xx} - \rho \phi_{xx} - 2G(w, \phi) &= 0 \\
\Delta^2 \phi + \rho w_{xx} + G(w, w) &= 0.
\end{align*}
\]

where

\[
G(u, v) = \frac{1}{2} u_{xx} v_{yy} + \frac{1}{2} u_{yy} v_{xx} - u_{xy} v_{xy}
\]

\[
\kappa^2 = \frac{t^2}{12(1 - \nu^2)}, \quad \lambda = \frac{P}{2\pi R E t}, \quad \rho = \frac{1}{R}
\]

\[(x, y) \in \Omega = [-L, L] \times [0, 2\pi R).\]

Assumptions:

▷ Thin, isotropic shell
▷ Elastic buckle and curvature not too large
▷ No pre-buckle
▷ Normals stay normal, plane stress
   and small angle approximation for strain tensor.
\[ \kappa^2 \Delta^2 w + \lambda w_{xx} - \rho \phi_{xx} - 2G(w, \phi) = 0 \]
\[ \Delta^2 \phi + \rho w_{xx} + G(w, w) = 0. \]

Stored energy:

\[ E(w) = \frac{Et}{2} \int_\Omega \left[ \kappa^2 \Delta w^2 + \Delta \phi^2 \right] dx dy, \]

Constraint is the average axial end-shortening associated with deflection \( w \)

\[ S(w) = \frac{1}{4\pi R} \int_\Omega w_x^2 dx dy. \]

- Solutions of vKD equations are stationary points of Total Potential \( F_\lambda(w) = E(w) - \lambda S(w). \)
- Solutions also stationary points of \( E(w) \) under constant \( S(w) \).
Part III: Post-buckle paths:

○ Dynamic Analogy:
  ▶ Seek localized buckle solutions as homoclinic solution
  ▶ BCs ($L = \infty$): $w, \phi + \text{derivatives} \to 0$ as $x \to \pm \infty$.
  ▶ Seek solution in subspace of circumferential wave number.
  ▶ Discretize by Galerkin circumferentially have large system of ODEs in axial direction.

○ Use numerical continuation
Test Results for Yamaki Shell:

\[ L = 160.9\, (mm) \quad R = 100\, (mm) \quad t = 0.247\, (mm) \]

\[ E = 5.56\, (GPa) \quad \nu = 0.3 \]

For this shell: \( L/2\pi R \approx 0.25 \) … not very long

Number of circumferential waves \( s = 11 \).
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<table>
<thead>
<tr>
<th>s=11</th>
<th>( \frac{\lambda_{\text{min}}}{\lambda_{\text{cl}}} )</th>
<th>( W_{\text{min}} )</th>
<th>( W_{\text{max}} )</th>
<th>Rel. Error: ( \approx 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment</td>
<td>0.24</td>
<td>-0.9</td>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>Homoclinic</td>
<td>0.242</td>
<td>-0.866</td>
<td>1.966</td>
<td></td>
</tr>
</tbody>
</table>
Cellular buckling
Cellular buckling

\[ n=13, \, r=0.08 \]
Cellular buckling

\[ n=13, \rho=0.08 \]
Post-buckle & Homoclinics

- Given circumferential wave number $s$ get good agreement with post-buckle regime ... 

- Finite shell length in experiments: but infinite homoclinic approximation works well.

- Determination of circumferential wave number next project ??
Part IV : Mountain Pass Solution

Let \( w_1 \neq w_2 \) be two vectors in a space \( X \). Define

\[
\Gamma = \{ \gamma \in C([0, 1], X) \mid \gamma(0) = w_1, \gamma(1) = w_2 \},
\]

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} F(\gamma(t)).
\]

If \( c > \max\{ F(w_1), F(w_2) \} \) and \( F \) satisfies \((PS)_c\), then \( c \) is a critical value of \( F \).
Mountain Pass

**MP1.** We show that $w_1 = 0$ is a local minimizer: there are $\varrho, \alpha > 0$ such that $F_\lambda(w) \geq \alpha$ for all $w$ with $\|w\|_X = \varrho$;

**MP2.** If domain is large enough, then there exists $w_2$ with $\|w_2\|_X > \varrho$ and $F_\lambda(w_2) \leq 0$. Based on Yoshimura diamond pattern.

**MP3.** Given a sequence of paths $\gamma_n$ that approximates the infimum in defn, we extract a (Palais-Smale) sequence of points $w_n \in \gamma_n$, each close to the maximum along $\gamma_n$, and show that this sequence converges in an appropriate manner.
Phase 1 — Initial discrete path. Take $P$ points:

$$z_j = w_1 + \frac{j}{P}(w_2 - w_1), \ j \in \{0, 1, \ldots, P\}$$

Phase 2 — Main loop:

(a) find $m$: $\forall j \ F(z_m) \geq F(z_j)$, interpolate,

(b) compute $\nabla F(z_m)$,

(c) deform the path: $\delta > 0$ (small) $z_m^{\text{new}} = z_m - \delta \nabla F(z_m)$,

(d) STOP when $F$ increases.

Phase 3 — Infinite loop: re-distribute points on path
Numerical Solutions

\[ \Omega = \left( -100, 100 \right) \times \left( -100, 100 \right), \ \Delta x = \Delta y = 0.5, \ \lambda = 1.1 \]

Found using different choices of \( w_2 \).

Min energy solution \( \equiv \) Single Dimple
Steepest Descent:

(a) $F_\lambda \approx 5$  (b) $F_\lambda \approx -15$  (c) $F_\lambda \approx -5 \times 10^4$

$\Omega = (-200, 200) \times (-115, 115)$, $\Delta x = \Delta y = 0.5$, $\lambda = 1.1$
Interpretation of MP?

Have found the mountain pass energy for the perfect cylinder – how does this give a handle on an imperfect “real” cylinder?

Consider the minimum mountain-pass energy: \( V = \inf \omega_2 F_\lambda \).

In order to leave the basin of attraction of \( \omega_1 \), the surplus energy should exceed \( V(\lambda) \).
Imperfections and MP

Suppose stored energy from being under load can be transferred to overcome the mountain pass. Rescale MP energy $V(\lambda)$ by elastic strain energy stored in cylinder of length $L$:

$$\alpha = \frac{1}{2\pi \sqrt{3(1 - \nu^2)}} \frac{t}{L} \frac{V(\lambda)}{\lambda^2}.$$
Imperfections and MP

1. The general trend of the constant-$\alpha$ curves is very similar to the trend of the experimental data;

2. The $\alpha = 1$ curve, which indicates the load at which the mountain-pass energy equals the stored energy in the prebuckled cylinder, appears to be a lower bound to the data.
Nasa knockdown

- Donnell (steel) fit
- Donnell (brass)
- Bridget et al
- Ballerstedt & Wagner
- Vmp fit $a=1$
- Vmp fit $a=0.1$
- Nasa Rec $L=2\pi R$
Other single dimples ...

- Single dimples are seen in the high-speed camera images of Esslinger.
- Some “worst imperfections” by Deml and Wunderlich, Deml, Wunderlich and Albertin are single dimples.
- Single dimples are seen in finite element simuations (eg Schweizerhof)
Summary ...

- Axially localized solutions: found as homoclinic orbit
- Computations of post-buckle paths and cellular buckling
- Mountain pass solutions
  - Elements for proof
  - Numerical algorithm
  - Solutions
- From MP solutions seems can get a lower bound on the buckling load.
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