Cramer-Rao lower bound: an example

Suppose that $X = (X)$, a single observation from $Bin(m, p)$, where $m$ is known. The pmf is given by

$$f(x; p) = \binom{m}{x} p^x (1-p)^{m-x} \quad \text{where} \quad x = 0, 1, \ldots, m.$$  

Note that the range of $X$ depends on $m$, but not on the unknown parameter $p$. Also, the sample size is $n = 1$.

Cramer-Rao lower bound

Since the range of $X$ does not depend on the unknown parameter $p$ which we wish to estimate, we can proceed to compute and use the Cramer-Rao lower bound for unbiased estimators:

$$\log f(x; p) = \binom{m}{x} \log \binom{m}{x} + x \log p + (m-x) \log (1-p)$$

$$\frac{\partial}{\partial p} \log f(x; p) = \frac{x}{p} - \frac{m-x}{1-p} = \frac{x - mp}{p(1-p)}$$

$$\left( \frac{\partial}{\partial p} \log f(x; p) \right)^2 = \frac{(x - mp)^2}{p^2(1-p)^2}.$$

Thus,

$$E \left( \left( \frac{\partial}{\partial p} \log f(X; p) \right)^2 \right) = \frac{E(X - mp)^2}{p^2(1-p)^2} = \frac{Var(X)}{p^2(1-p)^2} = \frac{m}{p(1-p)}.$$  

It follows that for any unbiased estimator, $g(X)$, for $p$, we have

$$Var(g(X)) \geq \frac{1}{\frac{m}{p(1-p)}} = \frac{p(1-p)}{m}.$$  

Alternatively, we can compute the Cramer-Rao lower bound as follows:

$$\frac{\partial^2}{\partial p^2} \log f(x; p) = \frac{\partial}{\partial p} \left( \frac{\partial}{\partial p} \log f(x; p) \right) = \frac{x}{p^2} \frac{m-x}{1-p} = \frac{-x}{p^2} \frac{(m-x)}{(1-p)^2}.$$  

Thus,

$$E \left( \frac{\partial^2}{\partial p^2} \log f(X; p) \right) = -\frac{E(X)^2}{p^2} - \frac{(m - E(X))}{(1-p)^2} = -\frac{mp}{p^2} - \frac{(m-mp)}{(1-p)^2} = \frac{-m}{p(1-p)}.$$  

It follows that the Cramer-Rao lower bound is given by

$$\frac{1}{-nE \left( \frac{\partial^2}{\partial p^2} \log f(X; p) \right)} = \frac{1}{-1 \cdot \frac{m}{p(1-p)}} = \frac{p(1-p)}{m}$$  

as above.
Comparing estimators

Consider the estimator \( g_1(X) = \frac{X}{m} \).

\[
E(g_1(X)) = \frac{E(X)}{m} = \frac{mp}{m} = p
\]

so \( g_1(X) \) is an unbiased estimator of \( p \). Is it the most efficient unbiased estimator for \( p \)?

To answer this question, we compute the variance of \( g_1 \) and compare it to the Cramer-Rao lower bound which we calculated above.

\[
Var(g_1(X)) = Var\left(\frac{X}{m}\right) = \frac{Var(X)}{m^2} = mp(1-p) = \frac{p(1-p)}{m}
\]

Since \( Var(g_1) \) equals the Cramer-Rao lower bound, we can conclude that \( g_1(X) \) is the most efficient unbiased estimator for \( p \).

Now consider the estimator \( g_2(X) = \frac{X+1}{m+2} \).

\[
E(g_2(X)) = \frac{E(X) + 1}{m + 2} = \frac{mp + 1}{m + 2} \neq p \quad \text{(except when } p = 1/2).\]

So \( g_2 \) is a biased estimator with

\[
\text{bias}(g_2) = E(g_2(X)) - p = \frac{mp + 1}{m + 2} - p = \frac{1 - 2p}{m + 2}.
\]

To compare the performance of \( g_2 \) with the performance of \( g_1 \), we must first compute the mean square error of \( g_2 \):

\[
MSE(g_2) = Var(g_2) + \text{bias}(g_2)^2 = \frac{mp(1-p)}{(m+2)^2} + \frac{(1 - 2p)^2}{(m+2)^2} = \frac{1}{(m+2)^2} (1 + (m-4)p - (m-4)p^2).
\]

We can compare the (relative) efficiency of \( g_1 \) and \( g_2 \) by comparing the graphs of \( MSE(g_1) \) (which is just the variance of \( g_1 \)) and \( MSE(g_2) \) as functions of \( p \).

Exercise: Fix \( m = 10 \) and sketch the graphs of \( MSE(g_1) \) and \( MSE(g_2) \) as functions of \( p \). Also, determine the values of \( p \) for which \( g_2 \) is more efficient than \( g_1 \).