Summer 2006

**Actuarial Mathematics and Statistics**
**Statistics 6**
**Tutorial Sheet 1**

1. Let $\underline{X}$ be a random sample from an exponential distribution with parameter $\lambda > 0$. By first noting that $\sum_{i=1}^{n} X_i$ has a gamma $\Gamma(n, \lambda)$ distribution, show that

   $2n\lambda\bar{X} \sim \chi^2_{2n}$, and hence construct an equal tailed $100(1-\alpha)\%$ CI for $\lambda$.

   If $n = 15$ and $\bar{X} = 13.2$, calculate the 95% CI for $\lambda$.

   **Hint:** Use moment generating functions to determine the distribution of $2n\lambda\bar{X}$.

2. Let $\underline{X}$ be a random sample of size $n$ from the uniform distribution $U(0, \theta)$. Recall that the MLE is given by $\hat{\theta} = X_{(n)}$ (i.e. the $n^{th}$ order statistic). Show that $\frac{1}{\theta}X_{(n)} \sim \beta(n, 1)$. Hence construct the equal-tailed $100(1-\alpha)\%$ CI for $\theta$.

   **Note:** You do not need tables of percentage points to do this. The density for $\beta(n, 1)$ is easy to work with, and the calculations can be done by hand.

   If $n = 10$ and $X_{(10)} = 2.5$, calculate the 95% CI for $\theta$.

3. Let $\underline{X}$ be a random sample of size $n$ from the normal distribution $N(\mu, \sigma^2)$. Develop 95% CI's for $\sigma^2$ using the *large sample normal approximation* for $\chi^2_{n-1}$ and using the approximation given in the statistical tables.

   For $n = 101$ and $S^2 = 1$, compute each CI and compare your answers with the 'exact' CI.

4. Let $X$ be a single observation from a Poisson distribution with parameter $\lambda > 0$. If $X = 7$, compute the equal-tailed 95% CI for $\lambda$.

   **Note:** You need to use tables or the function ppois() in R.

5. Let $\underline{X}$ be a random sample of size $n$ from a Poisson distribution with parameter $\lambda > 0$. By considering the distribution of $\sum_{i=1}^{n} X_i$, construct the equal-tailed 95% CI for $\lambda$. Construct approximate CI's using two (different) normal approximations.

   If $n = 10$ and $\sum_{i=1}^{n} X_i = 20$, calculate each of these three CI's.

   **Note:** You may need to use the function ppois() in R.

**Selected Answers**

1.

$$\left( \frac{\chi^2_{2n}(100(1-\frac{\alpha}{2}))}{2n\bar{X}}, \frac{\chi^2_{2n}(100\frac{\alpha}{2})}{2n\bar{X}} \right), \qquad \text{and} \qquad (0.042, 0.119)$$

2.

$$\left( \frac{X_{(n)}}{(1-\frac{\alpha}{2})^{1/n}}, \frac{X_{(n)}}{(\frac{\alpha}{2}^{1/n}} \right) \qquad \text{and}(2.51, 3.62)$$

3. Use $\chi^2_{n-1} \sim N((n-1), 2(n-1))$ to obtain the (approximate) 95% CI:

$$\left( \frac{(n-1)S^2}{(n-1)+1.96\sqrt{2(n-1)}}, \frac{(n-1)S^2}{(n-1)-1.96\sqrt{2(n-1)}} \right) \qquad \text{and} \qquad (0.783, 1.384).$$

Use $\sqrt{2\chi^2_{n-1}} \sim N(\sqrt{2n-3}, 1)$ to obtain the (approximate) 95% CI:

$$\left( \frac{2(n-1)S^2}{(\sqrt{2n-3}+1.96)^2}, \frac{2(n-1)S^2}{(\sqrt{2n-3}-1.96)^2} \right) \qquad \text{and} \qquad (0.775, 1.356).$$

Finally, the 'exact' 95% CI is given by:

$$\left( \frac{(n-1)S^2}{\chi^2_{n-1}(2.5)}, \frac{(n-1)S^2}{\chi^2_{n-1}(97.5)} \right) \qquad \text{and} \qquad (0.772, 1.347).$$

4. $(2.8, 14.4)$

5. Use $\sum_{i=1}^n X_i \sim N(n\lambda, n\lambda)$ to obtain the (approximate) 95% CI:

$$(\frac{1}{2}(2\bar{X}+\frac{(1.96)^2}{n}) - \sqrt{(2\bar{X}+\frac{(1.96)^2}{n})^2 - 4\bar{X}^2}, \frac{1}{2}(2\bar{X}+\frac{(1.96)^2}{n}) + \sqrt{(2\bar{X}+\frac{(1.96)^2}{n})^2 - 4\bar{X})}$$

$$\text{and} \qquad (1.29, 3.09).$$

Use $\sum_{i=1}^n X_i \sim N(n\lambda, n\lambda)$ again, but this time use the estimate $n\bar{X}$ for the variance. The 95% (approximate) CI is then

$$(\bar{X} - 1.96\sqrt{\frac{\bar{X}}{n}}, \bar{X} + 1.96\sqrt{\frac{\bar{X}}{n}}) \qquad \text{and} \qquad (1.22, 2.88).$$

The 'exact' 95% CI for the data is given by $(1.22, 3.09)$.