F71SB2 Statistics II Handout 1(b)

Some results about infinite series of positive numbers

Sometimes in probability calculations involving experiments, where the outcome space is countably infinite (i.e. outcomes can be put in one-to-one correspondence with the natural numbers) we will encounter infinite series of positive terms. These look like:

$a_1 + a_2 + a_3 + a_4 + \ldots..$ (*infinite number of terms*)

Given such a series we can define the n^{th} partial sum (or the sum to n terms) as

$S_n \quad = \quad a_1 + a_2 + a_3 \ldots + a_n$

If we were to plot S_n against n we would get something that looked like:

Note that $S_{\text{n}} = S_{n\text{-}1} + a_n$, so thinking of this as a staircase, the size of the n^{th} step is a_n. An interesting question is whether it is possible to construct a roof (parallel to the horizontal axis) such that the staircase never reaches the roof. If we can do this we say that the series *converges*. The height of the lowest such roof can be thought of as the sum of the entire series (or the sum to infinity, S_∞ of the series).

Some examples:

a) The case where a_n = constant. For example:

$$1 + 1+ 1 + 1 + 1 + \ldots..$$

In this case $S_n = n$. It is clear that the staircase is going to crash (eventually) through any ceiling of finite height so that S_∞ does not exist.

b) *A geometric series*. In this case a_n looks like $ar^{n\text{-}1}$. That is $a_1 = a > 0$, and $a_n = a_{n\text{-}1}.r$ where r>0 is known as the common ratio. We can derive a formula for the sum to n terms as follows

First note that

$$S_n = a + ar + ar^2 + .. + ar^{n\text{-}1}. \qquad (1)$$

Now if we multiply S_n by r we obtain

$$rS_n = ar + ar^2 + ar^3 + .. + ar^n \qquad (2)$$

Subtracting (2) from (1) we obtain

$S_n - rS_n = (a + ar + ar^2 + .. + ar^{n-1}) - (ar + ar^2 + ar^3 + .. + ar^n) = a - ar^n$.

It follows that $(1-r)S_n = a(1 - r^n)$, so that (so long as $r \neq 1$), we get

$$S_n = \frac{a(1-r^n)}{1-r}$$

Suppose now that $|r| < 1$. Then as n gets very large r^n gets closer and closer to zero, and we see that the numerator of the above expression approaches $a(1 - 0) = a$. It follows that

$$S_\infty = \frac{a}{1-r} \qquad \text{(sum to infinity of a geometric series)}$$

c) <u>A series that doesn't converge</u> is the following (with $a_n = 1/(n+1)$, $n = 1, 2, 3, \ldots$.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+....$$

To see this we note that:

- The first term is 1/2
- The next 2 terms are ≥ 1/4, so that their sum is ≥ 1/2
- The next 4 terms are ≥ 1/8 so that their sum ≥ 1/2
- The next 8 terms are ≥ 1/16 so that their sum ≥ 1/2

Now continuing in this way we see that we can group all the terms in the series into an infinite number of groups such that the sum of terms within any group is at least 1/2. Therefore we make S_n as large as we like just by taking n large enough.

When we look at expectations of random variables we shall encounter this series again.

d) Consider the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+....+\frac{1}{n\times(n+1)}+..... \quad = \quad \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+.....$$

See if you can show that $S_n = 1-\frac{1}{n+1}$. (Hint: $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$).

Hence show that $S_\infty = 1$. This series will arise when we consider *Polya's urn*.