F71SB2 STATISTICS II

Problem sheet 4 - Solutions

(1) $P(0.7 \leq X < 0.9) = P(X \leq 0.9) - P(X \leq 0.7)$

$$= 0.9 - 0.7 = \underline{0.2}$$

$$P(0.35 < X < 0.55) = P(X \leq 0.55) - P(X \leq 0.35)$$

$$= 0.55 - 0.35 = \underline{0.2}$$

Comment: Both probabilities equal to 0.2. For the $U(0,1)$ distribution such probabilities only depend on the length of the interval in which X lies.
Also notice that for continuous r.v.s it does not matter if inequalities are strict or not!

(2) $X \sim Exp(2)$

i) The CDF is given by $F_X(x) = P(X \leq x) = 1 - e^{-2x}$

ii) $P(X < 1) = F_X(1) = 1 - e^{-2} = \underline{0.865}$

$$P(1 \leq X < 2) = F_X(2) - F_X(1) = (1 - e^{-4}) - (1 - e^{-2})$$

$$= e^{-2} - e^{-4} = \underline{0.117}$$

$$P(X \geq 2) = 1 - F_X(2) = e^{-4} = \underline{0.018}.$$

(3) $T \sim Exp(0.6)$

i) $P(T > 3 / T > 1) = P(T > 2)$ [lack-of-memory property]

$$= 1 - (1 - e^{-0.6 \times 2}) = e^{-1.2} = \underline{0.3012}$$

[Notice that you can also do this by using first principles on conditional probability.]

ii) The probability that one component will last for more than a year is given by

$$P(T>1) = e^{-0.6} = 0.549$$

Therefore, in a sample of 100 components, the expected number surviving for longer than 1 year is

$$100 \times P(T>1) = \underline{54.9}$$

[Notice that you can express the number of components that last for more than a year as a Binomial(100, p) random variable, with $p = P(T>1) = 0.549$. The answer then is given by the expected value of this distribution.]

(4)

i) $P(Z>z) = P(X>z \cap Y>z)$

$= P(X>z) \times P(Y>z)$ [since X, Y independent]

$= \{P(X>z)\}^2$ [since X, Y have same distribution]

$= \{1 - F_X(z)\}^2$

$= \{1-(1-e^{-\lambda z})\}^2 = e^{-2\lambda z}$

Therefore, the CDF of Z is given by

$$F_Z(z) = P(Z \leq z) = \underline{1 - e^{-2\lambda z}}$$

ii) From i) we can recognize the CDF of Z as the CDF of the Exp(2λ) distribution. Therefore Z also has the exponential distribution with parameter 2λ.

iii) If $Z = \min(X_1, X_2, \dots, X_n)$ with $X_1, \dots, X_n$ iid $Exp(\lambda)$,

$$P(Z > z) = P(X_1 > z, X_2 > z, \dots, X_n > z)$$
$$= P(X_1 > z) \times P(X_2 > z) \times \dots \times P(X_n > z)$$
$$= e^{-\lambda z} \times e^{-\lambda z} \times \dots \times e^{-\lambda z} = \underline{\underline{e^{-n\lambda z}}}$$

It follows that

$$F_Z(z) = 1 - e^{-n\lambda z}$$

so that $\underline{\underline{Z \sim Exp(n\lambda)}}$

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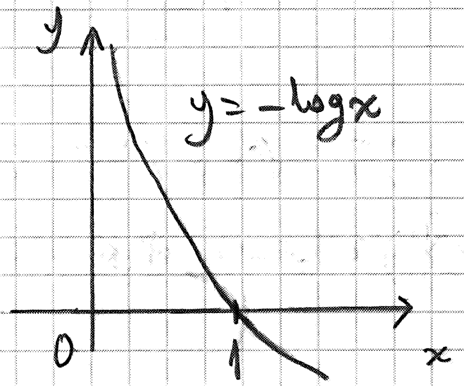
i) Range of X: $\{x : 0 < x < 1\}$

Range of Y: $\{y : y > 0\}$



ii) If $Y = y$, then $-\log x = y \Rightarrow e^{-\log x} = e^y$

$$\Rightarrow \frac{1}{x} = e^y \Rightarrow \underline{\underline{x = e^{-y}}}$$

Now, $Y \le y \Rightarrow -\log X \le y$

$$\Rightarrow X \ge e^{-y}$$

and therefore

$$P(Y \le y) = P(X \ge e^{-y}) = 1 - P(X \le e^{-y})$$

$= 1 - F_X(e^{-y})$ where F_X is the CDF of the $U(0,1)$ distribution

$$= \underline{\underline{1 - e^{-y}}}.$$

iii) From ii) the CDF of Y is that of the Exp(1) distribution. Therefore $Y \sim \text{Exp}(1)$.

⑥ i) $X, Y \sim$ Uniform $(0,1)$ and independent. $Z = \max\{X, Y\}$.

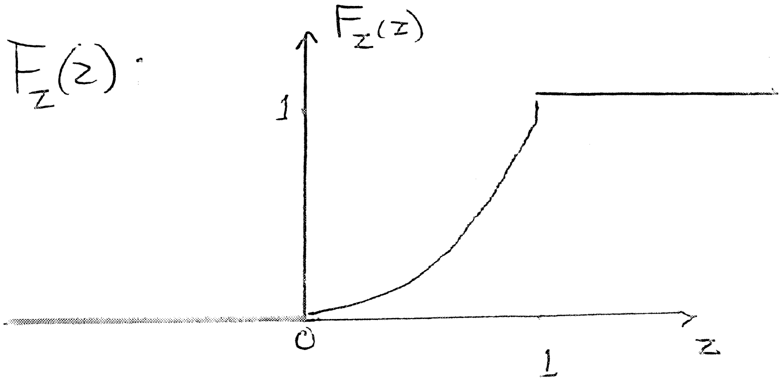
$$F_Z(z) = P(Z \le z) = P(X \le z, Y \le z)$$

$$= P(X \le z) \times P(Y \le z) \quad \left(\text{since } X, Y \text{ independent}\right)$$

$$= P(X \le z)^2 \quad \left(\text{since } X \,\&\, Y \text{ have same distribution}\right)$$

$$= \begin{cases} 0 & \text{if } z \le 0 \\ z^2 & \text{if } 0 < z < 1 \\ 1 & \text{if } z \ge 1 \end{cases}$$

Graph of $F_Z(z)$:



ii) $P(Z > 0.5) = 1 - F_Z(0.5) = 1 - \left(\frac{1}{2}\right)^2$

iii) Repeating the argument of i) for $X_1, X_2, \ldots, X_n$ i.i.d. Uniform $(0,1)$ and $Z = \max\{X_1, X_2, \ldots, X_n\}$ we can show that

$$F_Z(z) = \begin{cases} 0, & z \le 0 \\ z^n, & 0 < z \le 1 \\ 1, & z \ge 1 \end{cases}$$

It follows that $P(\max\{X_1, \ldots, X_n\} > 0.5) = 1 - F_Z(0.5)$ $= 1 - \frac{1}{2^n}$. As n gets large, this probability approaches 1. ("Take enough independent samples and one of them is bound to be $> \frac{1}{2}$")

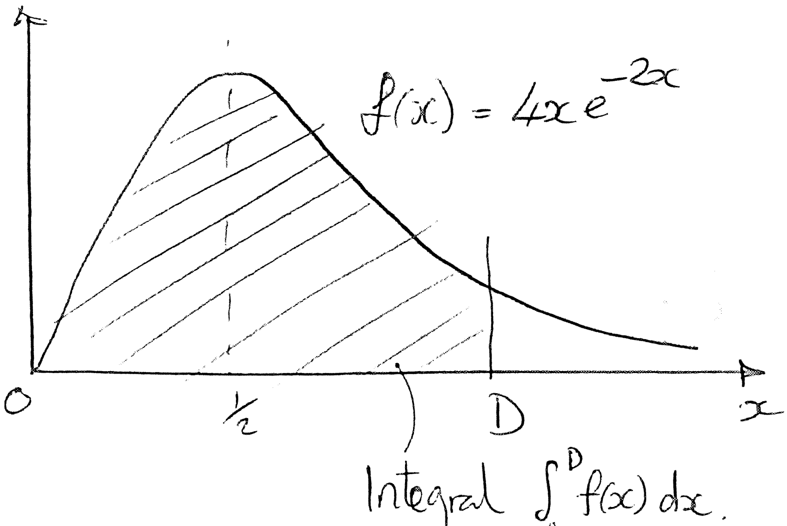
(7) i) $f(x) = 4x e^{-2x}, \quad x > 0.$

To sketch the graph note that $f(0) = 0$, and $f(x) \to 0$ as $x \to +\infty$. (See "Hint" in part ii)).

Also $f'(x) = 4e^{-2x} - 8x e^{-2x} = 4e^{-2x}(1-2x)$

so that $f'(x) = 0$ when $x = \frac{1}{2}$. (This must a maximum)

Graph of $f(x)$:



Now $\int_0^D f(x)\,dx = \int_0^D 4xe^{-2x}\,dx$

$$= \left[4x \frac{e^{-2x}}{-2} \right]_0^D - \int_0^D 4 \frac{e^{-2x}}{-2}\,dx$$

$$= -2De^{-2D} + \int_0^D 2e^{-2x}\,dx$$

$$= -2De^{-2D} + \left[\frac{2e^{-2x}}{-2} \right]_0^D$$

$$= -2De^{-2D} - e^{-2D} + 1 \quad \text{as required.}$$

ii) Now $\int_0^{\infty} f(x)dx$ is the limit of $\int_0^D f(x)dx$ as D approaches ∞. From i) we have

$$\int_0^D f(x)dx = 1 - 2De^{-2D} - e^{-2D} \longrightarrow 1 \quad \text{as} \quad D \to \infty.$$

(Terms involving D $\to 0$ – see Hint.)

iii) $P(X \leq 1) = \int_0^1 f(x)\,dx = 1 - 2e^{-2} - e^{-2}$

$= 1 - 3e^{-2}$

$= 0.594$