Statistics II
Problem sheet 5 - Solutions

1. i) We need to have

$$\int_0^4 f(x)\,dx = 1 \Rightarrow \int_0^4 c x^{-\frac{1}{2}}\,dx = 1$$

$$\Rightarrow c\left[\frac{x^{1/2}}{1/2}\right]_0^4 = 1 \Rightarrow c(2\cdot 2) = 1$$

$$\Rightarrow c = \frac{1}{4}$$

ii) The CDF is given by:

$$F(x) = P(X \le x) = \int_0^x f(t)\,dt$$

$$= \frac{1}{4}\int_0^x t^{-1/2}\,dt = \frac{1}{4}\left[\frac{t^{1/2}}{1/2}\right]_0^x$$

$$= \frac{1}{4}\left(2x^{1/2}\right) = \frac{1}{2}\sqrt{x}$$

Hence,

$$P\left(X < \frac{1}{4}\right) = F(1/4) = \frac{1}{2}\sqrt{1/4} = \frac{1}{4}$$

iii)

$$E(X) = \int_0^4 x f(x)\,dx = \int_0^4 x\frac{1}{4}x^{-1/2}\,dx$$

$$= \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{1}{4}\left[\frac{x^{3/2}}{3/2}\right]_0^4$$

$$= \frac{1}{4}\left(\frac{2}{3}4^{3/2}\right) = \frac{1}{4}\cdot\frac{2}{3}\cdot\cancel{4}\cdot 4^{1/2}$$

$$= \frac{4}{3} = 1.333$$

For the variance we also need to evaluate $E(X^2)$:

$$E(X^2) = \int_0^4 x^2 f(x)\,dx = \int_0^4 \frac{1}{4} x^{3/2}\,dx = \frac{1}{4}\left[\frac{x^{5/2}}{5/2}\right]_0^4$$

$$= \frac{1}{4}\left(\frac{2}{5} 4^{5/2}\right) = \frac{1}{\cancel{4}} \cdot \frac{2}{5} \cdot \cancel{4}^{2}\, 4^{1/2}$$

$$= \frac{16}{5}$$

Thus:

$$\text{var}(X) = E(X^2) - E^2(X) = \frac{16}{5} - \left(\frac{4}{3}\right)^2$$

$$= \frac{16}{5} - \frac{16}{9} = \frac{64}{45}$$

$$= 1.422$$

2. i) For these data we have:

$$\sum_{i=1}^{10} x_i = 4.29 \quad , \quad \bar{x} = \frac{1}{10}\sum x_i = 0.429$$

For method of moments estimate we need

$$E(X) = \bar{x} \Rightarrow \frac{1}{\hat{\lambda}} = \bar{x} \Rightarrow \hat{\lambda} = 2.331$$

ii) $y = 1/3\, x \Rightarrow x = 3y = g^{-1}(y)$

The range of Y is $(0, \infty)$ and function g is increasing. Therefore we have:

$$f_Y(y) = f_X\{g^{-1}(y)\}\left|\frac{d}{dy} g^{-1}(y)\right| = \lambda \cdot e^{-\lambda(3y)} \cdot 3$$

$$= (3\lambda)\, e^{-(3\lambda)y}$$

This is the pdf of an $Exp(3\lambda)$ r.v. Thus $Y \sim Exp(3\lambda)$

3. $X \sim \text{Uniform}(0,1)$. $Y = g(X) = X^3$.

$$y = g(x) = x^3 \Rightarrow x = \sqrt[3]{y} = g^{-1}(y)$$

The range of Y is $(0,1)$.

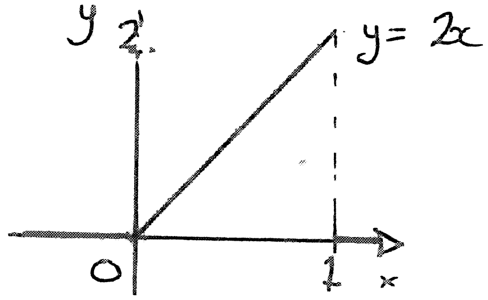
The function g is increasing. Therefore we use the formula

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$$

$$= 1 \cdot \frac{d}{dy}y^{1/3}$$

$$= \frac{1}{3}y^{-2/3}$$

4. i) $f_X(x) = 2x, \quad 0 < x < 1$.



Since the density increases as x increases, larger values of x in the range $(0,1)$ are more likely than smaller ones. Therefore it is intuitively plausible that the mean of X is greater than 0.5.

ii) $$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\,2x\,dx = \int_0^1 2x^2\,dx$$

$$= \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}.$$

$$E(X^2) = \int_0^1 x^2\,2x\,dx = \int_0^1 2x^3\,dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{2}$$

$$\text{var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

5. i) Clearly, $f_Y(y) > 0$ for $y > 0$.

We also need to show that $\int_0^\infty f_Y(y)\,dy = 1$.

Using integration by parts:

$$\int_0^\infty f_Y(y)\,dy = \int_0^\infty \frac{1}{2} y^2 e^{-y}\,dy = \frac{1}{2}\int_0^\infty y^2 e^{-y}\,dy \quad (*)$$

Now,

$$\int_0^\infty y^2 e^{-y}\,dy = \lim_{w\to\infty} \int_0^w y^2 e^{-y}\,dy$$

$$= \lim_{w\to\infty} \left\{ \left[-y^2 e^{-y}\right]_0^w - \int_0^w -2y e^{-y}\,dy \right\} \quad \left[\begin{array}{c}\text{Use IBP} \\ \text{again}\end{array}\right]$$

$$= \lim_{w\to\infty} \left\{ \left(-w^2 e^{-w}\right) + \left[-2y e^{-y}\right]_0^w - \int_0^w -2 e^{-y}\,dy \right\}$$

$$= \lim_{w\to\infty} \left\{ -w^2 e^{-w} - 2w e^{-w} + 2\left[-e^{-y}\right]_0^w \right\}$$

$$= \lim_{w\to\infty} \left\{ -w^2 e^{-w} - 2w e^{-w} + 2\left(-e^{-w} + 1\right) \right\}$$

$$= 2 \quad [\text{Use hint in Q7, P.S. 4}]. \quad (**)$$

Then from $(*)$ we have that

$$\int_0^\infty f_Y(y)\,dy = 1.$$

ii) $E(Y) = \int_0^{\infty} y f_Y(y)\, dy = \int_0^{\infty} \frac{1}{2} y^3 e^{-y}\, dy$

$$= \frac{1}{2} \left\{ \lim_{w\to\infty} \left[-y^3 e^{-y}\right]_0^w + 3\int_0^{\infty} y^2 e^{-y}\, dy \right\}$$

$$= \frac{3}{2} \int_0^{\infty} y^2 e^{-y}\, dy \qquad (***)$$

The integral above has been calculated in (**).

Thus:

$$E(Y) = \frac{3}{2} \cdot 2 = \underline{3}$$

Also:

$$E(Y^2) = \int_0^{\infty} y^2 f_Y(y)\, dy = \int_0^{\infty} \frac{1}{2} y^4 e^{-y}\, dy$$

$$= \frac{1}{2} \left\{ \lim_{w\to\infty} \left[-y^4 e^{-y}\right]_0^w + 4\int_0^{\infty} y^3 e^{-y}\, dy \right\}$$

$= \ldots$ [working as above with IBP]

$$= \frac{1}{2}(4 \times 6) = 12$$

So:

$$\mathrm{var}(Y) = E(Y^2) - \{E(Y)\}^2 = 12 - 3^2$$

$$= \underline{3}$$

iii) First notice that $E(X_i) = \mathrm{var}(X_i) = 1$, $i = 1,2,3$.

Thus:

$$E(Z) = E(X_1 + X_2 + X_3) = 3E(X_1) = 3$$

Same as for Y.

$$\mathrm{var}(Z) = \mathrm{var}(X_1 + X_2 + X_3) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \mathrm{var}(X_3) \quad \text{(independence)}$$

$$= \underline{3}$$