News About
A Recent Application of Parametricity

Janis Voigtländer

Technische Universität Dresden

ISS-AiPL'09

Functional Programming in Haskell

A standard function:

$$
\begin{aligned}
&\texttt{map} :: (\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta] \\
&\texttt{map}\ f\ [\,] \qquad\quad = [\,] \\
&\texttt{map}\ f\ (a : as) = (f\ a) : (\texttt{map}\ f\ as)
\end{aligned}
$$

Some invocations:

$$
\begin{array}{lll}
\texttt{map succ}\ [1,2,3] & = [2,3,4] & \text{— } \alpha, \beta \mapsto \mathsf{Int}, \mathsf{Int} \\
\texttt{map not}\ \ [\mathsf{True}, \mathsf{False}] & = [\mathsf{False}, \mathsf{True}] & \text{— } \alpha, \beta \mapsto \mathsf{Bool}, \mathsf{Bool} \\
\texttt{map even}\ [1,2,3] & = [\mathsf{False}, \mathsf{True}, \mathsf{False}] & \text{— } \alpha, \beta \mapsto \mathsf{Int}, \mathsf{Bool} \\
\texttt{map not}\ \ [1,2,3] & \lightning\ \text{rejected at compile-time} &
\end{array}
$$

Another Example

$$\begin{array}{l} \texttt{reverse} :: [\alpha] \to [\alpha] \\ \texttt{reverse}\ [\,] \qquad\quad = [\,] \\ \texttt{reverse}\ (a : as) = (\texttt{reverse}\ as) \mathbin{+\!\!+} [a] \end{array}$$

For every choice of f and l:

$$\texttt{reverse}\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{reverse}\ l)$$

Provable by induction.

Or as a "free theorem" [Wadler, FPCA'89].

Another Example

$$\begin{aligned} \texttt{reverse} &:: [\alpha] \to [\alpha] \\ \texttt{tail} &:: [\alpha] \to [\alpha] \\ \texttt{g} &:: [\alpha] \to [\alpha] \end{aligned}$$

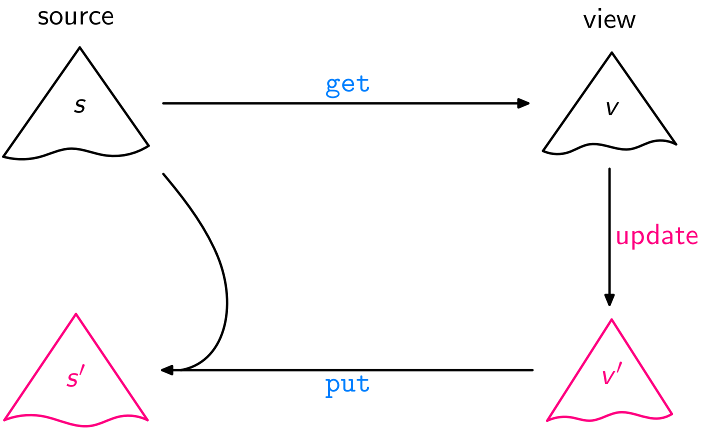
For every choice of f and l:

$$\begin{aligned} \texttt{reverse}\ (\texttt{map}\ f\ \ l) &= \texttt{map}\ f\ (\texttt{reverse}\ l) \\ \texttt{tail}\ (\texttt{map}\ f\ \ l) &= \texttt{map}\ f\ (\texttt{tail}\ l) \\ \texttt{g}\ (\texttt{map}\ f\ \ l) &= \texttt{map}\ f\ (\texttt{g}\ l) \end{aligned}$$

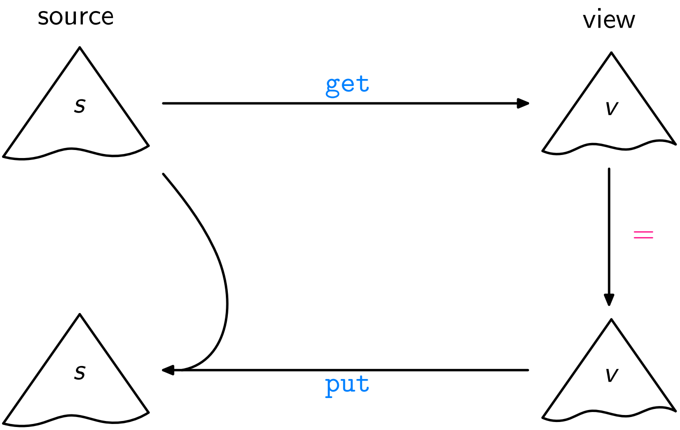
Some Applications

- Short Cut Fusion [Gill et al., FPCA'93]
- The Dual of Short Cut Fusion [Svenningsson, ICFP'02]
- Circular Short Cut Fusion [Fernandes et al., Haskell'07]
- . . .
- Knuth's 0-1-principle and the like [Day et al., Haskell'99], [V., POPL'08]
- Bidirectionalisation [V., POPL'09]
- Reasoning about invariants for monadic programs [V., ICFP'09]

Bidirectional Transformation

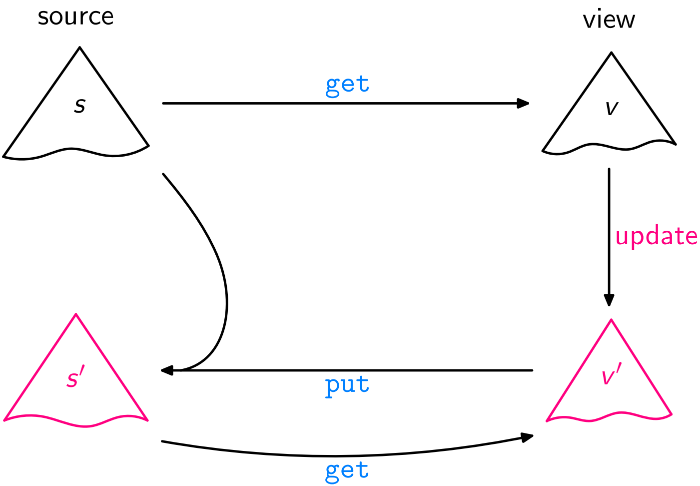


Bidirectional Transformation



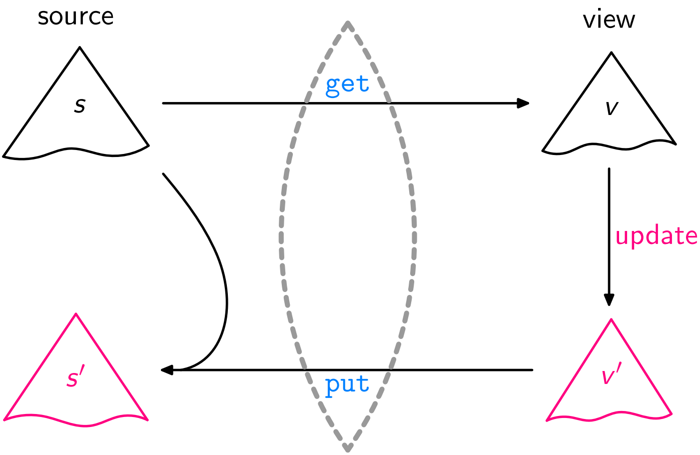
Acceptability / GetPut

Bidirectional Transformation



Consistency / PutGet

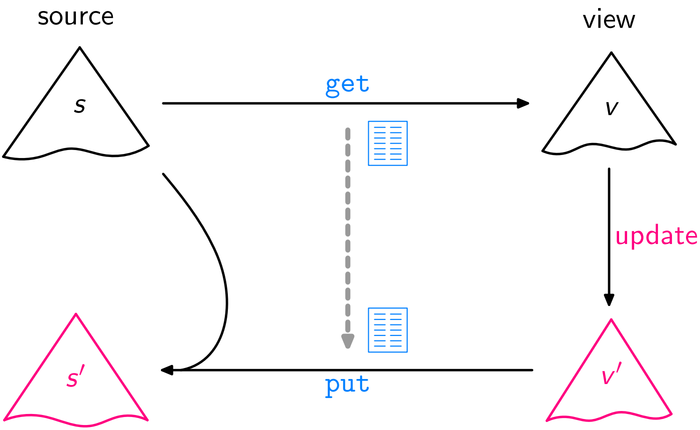
Bidirectional Transformation



Lenses, DSLs

[Foster et al., ACM TOPLAS'07, ...]

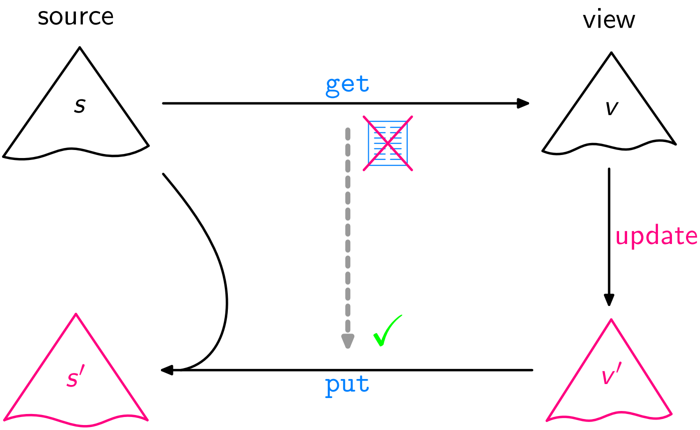
Bidirectional Transformation



Syntactic Bidirectionalisation

[Matsuda et al., ICFP'07]

Bidirectional Transformation



Semantic Bidirectionalisation

[V., POPL'09]

Semantic Bidirectionalisation

Aim: Write a higher-order function `bff`† such that any `get` and `bff get` satisfy GetPut, PutGet, ...

Examples:

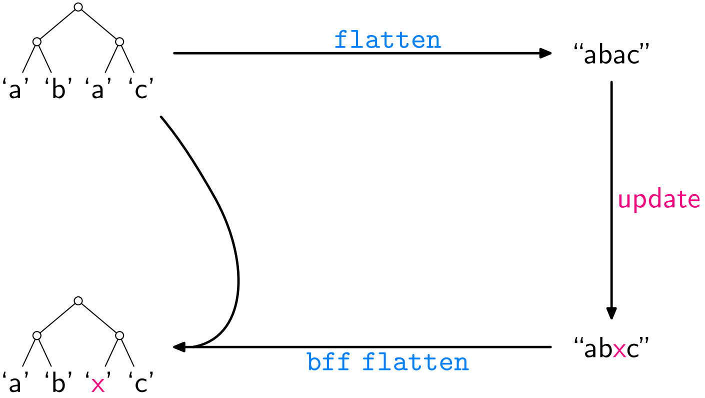
$$
\begin{tikzcd}
\text{“abc”} \arrow[r, "\texttt{tail}"] \arrow[d, bend left] & \text{“bc”} \arrow[d, "\text{update}"] \\
\text{“ade”} & \text{“de”} \arrow[l, "\texttt{bff tail}"]
\end{tikzcd}
$$

† “Bidirectionalization for free!”

Semantic Bidirectionalisation

Aim: Write a higher-order function `bff`[†] such that any `get` and `bff get` satisfy GetPut, PutGet, ...

Examples:

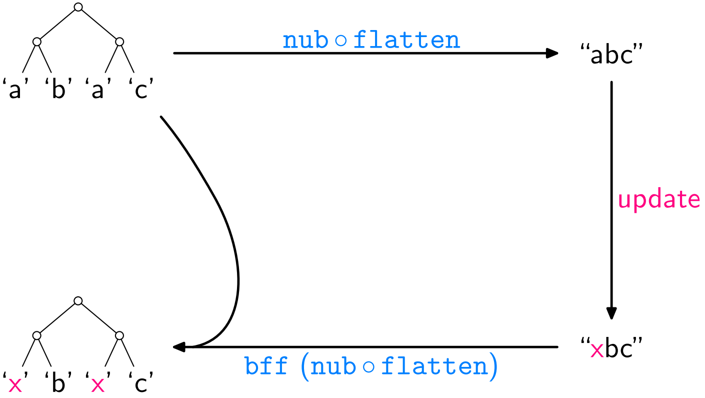


[†] “Bidirectionalization for free!”

Semantic Bidirectionalisation

Aim: Write a higher-order function `bff`† such that any `get` and `bff get` satisfy GetPut, PutGet, ...

Examples:



† "Bidirectionalization for free!"

Analysing Specific Instances

Assume we are given some

$$\texttt{get} :: [\alpha] \to [\alpha]$$

How can we, or `bff`, analyse it without access to its source code?

Idea: How about applying `get` to some input?

Like:

$$\texttt{get}\ [0..n] = \begin{cases} [1..n] & \text{if } \texttt{get} = \texttt{tail} \\ [n..0] & \text{if } \texttt{get} = \texttt{reverse} \\ [0..(\texttt{min}\ 4\ n)] & \text{if } \texttt{get} = \texttt{take}\ 5 \\ & \vdots \end{cases}$$

Then transfer the gained insights to source lists other than $[0..n]$!

Using a Free Theorem

For every

$$\mathtt{g} :: [\alpha] \to [\alpha]$$

we have

$$\mathtt{map}\ f\ (\mathtt{g}\ l) \quad = \quad \mathtt{g}\ (\mathtt{map}\ f\ l)$$

for arbitrary f and l.

Given an arbitrary list s of length $n + 1$, set $\mathtt{g} = \mathtt{get}$, $l = [0..n]$, $f = (s\,!!)$, leading to:

$$\begin{aligned} \mathtt{map}\ (s\,!!)\ (\mathtt{get}\ [0..n]) \quad &= \quad \mathtt{get}\ (\underbrace{\mathtt{map}\ (s\,!!)\ [0..n]}_{s}) \\ &= \quad \mathtt{get}\ s \end{aligned}$$

Using a Free Theorem

For every

$$\texttt{g} :: [\alpha] \to [\alpha]$$

we have

$$\texttt{map}\ f\ (\texttt{g}\ l) \quad = \quad \texttt{g}\ (\texttt{map}\ f\ l)$$

for arbitrary f and l.

Given an arbitrary list s of length $n + 1$,

$$\texttt{get}\ s \quad = \quad \texttt{map}\ (s\,!!)\ (\texttt{get}\ [0..n])$$

for every $\texttt{get} :: [\alpha] \to [\alpha]$.

The Constant-Complement Approach [Bancilhon & Spyratos, ACM TODS'81]

In general, given

$$\mathtt{get} :: S \to V$$

define a V^C and

$$\mathtt{compl} :: S \to V^C$$

such that

$$\lambda s \to (\mathtt{get}\ s, \mathtt{compl}\ s)$$

is injective and has an inverse

$$\mathtt{inv} :: (V, V^C) \to S$$

Then:

$$\begin{aligned} &\mathtt{put} :: S \to V \to S \\ &\mathtt{put}\ s\ v' = \mathtt{inv}\ (v', \mathtt{compl}\ s) \end{aligned}$$

Important: `compl` should "collapse" as much as possible.

The Constant-Complement Approach

For our setting,

$$\texttt{get} :: [\alpha] \rightarrow [\alpha]\,,$$

what should be V^C and

$$\texttt{compl} :: [\alpha] \rightarrow V^C \quad ???$$

To make

$$\lambda s \rightarrow (\texttt{get}\ s, \texttt{compl}\ s)$$

injective, need to record information discarded by `get`.

Candidates:

1. length of the source list
2. discarded list elements

For the moment, be maximally conservative.

The Complement Function

$$
\begin{aligned}
&\textbf{type}\ \mathsf{IntMap}\ \alpha = [(\mathsf{Int}, \alpha)] \\[1ex]
&\texttt{compl} :: [\alpha] \to (\mathsf{Int}, \mathsf{IntMap}\ \alpha) \\
&\texttt{compl}\ s = \textbf{let}\ \begin{aligned}[t] n &= (\texttt{length}\ s) - 1 \\ t &= [0..n] \\ g &= \texttt{zip}\ t\ s \\ g' &= \texttt{filter}\ (\lambda(i, \_) \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ g \end{aligned} \\
&\qquad\quad\ \ \textbf{in}\ \ (n+1, g')
\end{aligned}
$$

For example:

$$
\begin{array}{lcl}
\texttt{get} = \texttt{tail} & \rightsquigarrow & \texttt{compl}\ \text{“abcde”} = (5, [(0, \text{'a'})]) \\
\texttt{get} = \texttt{take}\ 3 & \rightsquigarrow & \texttt{compl}\ \text{“abcde”} = (5, [(3, \text{'d'}), (4, \text{'e'})]) \\
\texttt{get} = \texttt{reverse} & \rightsquigarrow & \texttt{compl}\ \text{“abcde”} = (5, [\,])
\end{array}
$$

An Inverse of $\lambda s \to (\texttt{get}\ s, \texttt{compl}\ s)$

$$
\begin{array}{l}
\texttt{inv} :: ([\alpha], (\mathsf{Int}, \mathsf{IntMap}\ \alpha)) \to [\alpha] \\
\texttt{inv}\ (v', (n+1, g')) = \mathbf{let}\ \begin{array}[t]{l} t\ \ = [0..n] \\ h\ \ = \texttt{assoc}^{\dagger}\ (\texttt{get}\ t)\ v' \\ h' = h + g' \end{array} \\
\qquad\qquad\qquad\qquad\ \ \mathbf{in}\ \ \texttt{seq}\ h\ (\texttt{map}\ (\lambda i \to \texttt{fromJust}\ (\texttt{lookup}\ i\ h'))\ t)
\end{array}
$$

For example:

$\texttt{get} = \texttt{tail}$ $\rightsquigarrow$ $\texttt{inv}\ (\text{"bcde"}, (5, [(0, \text{'a'})])) = \text{"abcde"}$

$\texttt{get} = \texttt{take}\ 3$ $\rightsquigarrow$ $\texttt{inv}\ (\text{"xyz"}, (5, [(3, \text{'d'}), (4, \text{'e'})])) = \text{"xyzde"}$

To prove formally:

- $\texttt{inv}\ (\texttt{get}\ s, \texttt{compl}\ s) = s$
- if $\texttt{inv}\ (v, c)$ defined, then $\texttt{get}\ (\texttt{inv}\ (v, c)) = v$
- if $\texttt{inv}\ (v, c)$ defined, then $\texttt{compl}\ (\texttt{inv}\ (v, c)) = c$

† Can be thought of as `zip` for the moment.

Altogether:

$$\textbf{type}\ \mathsf{IntMap}\ \alpha = [(\mathsf{Int}, \alpha)]$$

$$\begin{array}{l}
\texttt{compl} :: [\alpha] \to (\mathsf{Int}, \mathsf{IntMap}\ \alpha)\\
\texttt{compl}\ s = \textbf{let}\ \begin{array}[t]{ll} n & = (\texttt{length}\ s) - 1\\ t & = [0..n]\\ g & = \texttt{zip}\ t\ s\\ g' & = \texttt{filter}\ (\lambda(i, _) \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ g \end{array}\\
\qquad\qquad\ \textbf{in}\ \ (n+1, g')
\end{array}$$

$$\begin{array}{l}
\texttt{inv} :: ([\alpha], (\mathsf{Int}, \mathsf{IntMap}\ \alpha)) \to [\alpha]\\
\texttt{inv}\ (v', (n+1, g')) = \textbf{let}\ \begin{array}[t]{ll} t & = [0..n]\\ h & = \texttt{assoc}\ (\texttt{get}\ t)\ v'\\ h' & = h \mathbin{+\!\!+} g' \end{array}\\
\qquad\qquad\qquad\qquad\ \textbf{in}\ \ \texttt{seq}\ h\ (\texttt{map}\ (\lambda i \to \texttt{fromJust}\ (\texttt{lookup}\ i\ h'))\ t)
\end{array}$$

$$\begin{array}{l}
\texttt{put} :: [\alpha] \to [\alpha] \to [\alpha]\\
\texttt{put}\ s\ v' = \texttt{inv}\ (v', \texttt{compl}\ s)
\end{array}$$

"Fusion"

Inlining `compl` and `inv` into `put`:

$$
\begin{array}{l}
\mathtt{put}\ s\ v' = \mathbf{let}\ \begin{array}[t]{lcl} n & = & (\mathtt{length}\ s) - 1 \\ t & = & [0..n] \\ g & = & \mathtt{zip}\ t\ s \\ g' & = & \mathtt{filter}\ (\lambda(i, \_) \rightarrow \mathtt{notElem}\ i\ (\mathtt{get}\ t))\ g \\ h & = & \mathtt{assoc}\ (\mathtt{get}\ t)\ v' \\ h' & = & h +\!\!+\ g' \end{array} \\
\qquad\qquad\quad \mathbf{in}\ \ \mathtt{seq}\ h\ (\mathtt{map}\ (\lambda i \rightarrow \mathtt{fromJust}\ (\mathtt{lookup}\ i\ h'))\ t)
\end{array}
$$

$$
\begin{array}{lll}
\mathtt{assoc}\ [\,]\quad\ \ [\,] & = & [\,] \\
\mathtt{assoc}\ (i : is)\ (b : bs) & = & \mathbf{let}\ m = \mathtt{assoc}\ is\ bs \\
 & & \mathbf{in}\ \ \mathbf{case}\ \mathtt{lookup}\ i\ m\ \mathbf{of} \\
 & & \qquad \begin{array}{ll} \mathrm{Nothing} & \rightarrow (i, b) : m \\ \mathrm{Just}\ c \mid b == c & \rightarrow m \end{array}
\end{array}
$$

“Fusion”

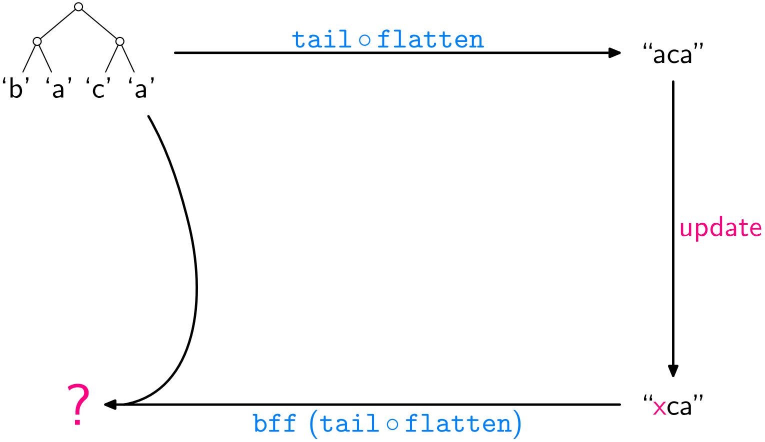
Inlining `compl` and `inv` into `put`:

$$
\begin{aligned}
\texttt{bff}\ \mathit{get}\ s\ v' = \textbf{let}\ & n\ = (\texttt{length}\ s) - 1\\
& t\ = [0..n]\\
& g\ = \texttt{zip}\ t\ s\\
& g' = \texttt{filter}\ (\lambda(i, \_) \rightarrow \texttt{notElem}\ i\ (\mathit{get}\ \ t))\ g\\
& h\ = \texttt{assoc}\ (\mathit{get}\ \ t)\ v'\\
& h' = h \mathbin{+\!\!+} g'\\
\textbf{in}\ \ & \texttt{seq}\ h\ (\texttt{map}\ (\lambda i \rightarrow \texttt{fromJust}\ (\texttt{lookup}\ i\ h'))\ t)
\end{aligned}
$$

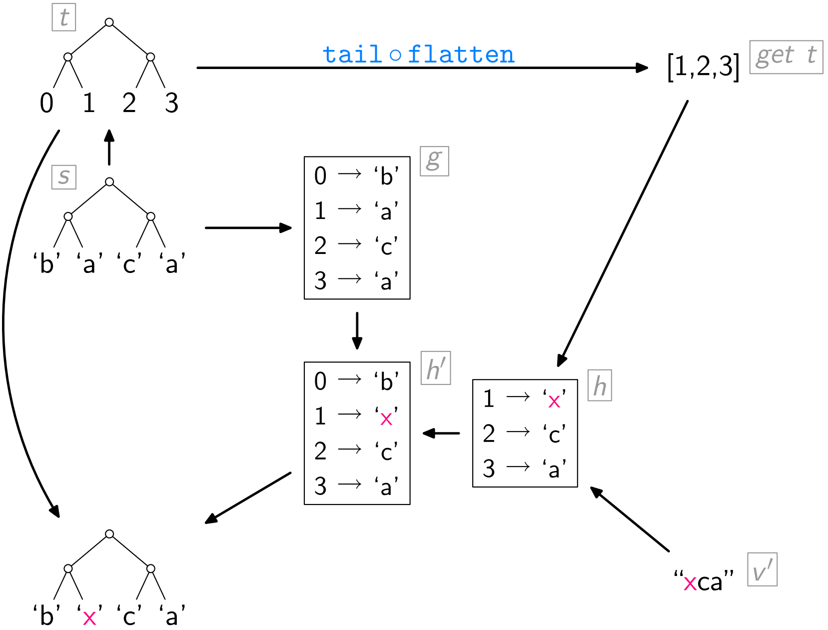
$$
\begin{aligned}
&\texttt{assoc}\ [\,]\qquad\ \ [\,] &&= [\,]\\
&\texttt{assoc}\ (i : is)\ (b : bs) &&= \textbf{let}\ m = \texttt{assoc}\ is\ bs\\
& &&\quad \textbf{in}\ \ \textbf{case}\ \texttt{lookup}\ i\ m\ \textbf{of}\\
& &&\qquad \text{Nothing} \qquad\qquad\ \rightarrow (i, b) : m\\
& &&\qquad \text{Just}\ c\ \mid\ b == c \rightarrow m
\end{aligned}
$$

Actual code only slightly more elaborate!

The Resulting Bidirectionalisation Method in Action



The Resulting Bidirectionalisation Method in Action



Extending the Technique

Major Problem:

- Shape-affecting updates lead to failure.
- For example, `bff tail` "abcde" "xyz" ...

Analysis as to Why:

- Our approach to making

$$\lambda s \rightarrow (\texttt{get}\ s, \texttt{compl}\ s)$$

  injective was to record, via `compl`, the following information:

  1. length of the source list
  2. discarded list elements

- Being maximally conservative this way often does not "collapse enough".

- For example:

  $\texttt{get} = \texttt{tail} \quad \rightsquigarrow \quad$ `put` "abcde" "xyz" fails precisely because $\texttt{compl}$ "abcde" $= (5, [(0, \text{'a'})])$

Assuming Shape-Injectivity

So assume there is a function

$$\texttt{shapeInv} :: \mathsf{Int} \to \mathsf{Int}$$

with, for every source list s,

$$\texttt{length}\ s = \texttt{shapeInv}\ (\texttt{length}\ (\texttt{get}\ s))$$

Then:

$$
\begin{array}{l}
\texttt{compl} :: [\alpha] \to \qquad \mathsf{IntMap}\ \alpha \\
\texttt{compl}\ s = \textbf{let}\ \begin{array}{lcl} n & = & (\texttt{length}\ s) - 1 \\ t & = & [0..n] \\ g & = & \texttt{zip}\ t\ s \\ g' & = & \texttt{filter}\ (\lambda(i, \_) \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ g \end{array} \\
\qquad\qquad\ \textbf{in} \qquad\quad g'
\end{array}
$$

Assuming Shape-Injectivity

$$
\begin{array}{l}
\texttt{inv} :: ([\alpha], \quad \mathrm{IntMap}\ \alpha\ ) \to [\alpha] \\
\texttt{inv}\ (v', \quad g'\ ) = \textbf{let}\ \begin{array}[t]{ll} n & = (\texttt{shapeInv}\ (\texttt{length}\ v')) - 1 \\ t & = [0..n] \\ h & = \texttt{assoc}\ (\texttt{get}\ t)\ v' \\ h' & = h +\!\!+\ g' \end{array} \\
\qquad\qquad\qquad\quad \textbf{in}\ \ \texttt{seq}\ h\ (\texttt{map}\ (\lambda i \to \texttt{fromJust}\ (\texttt{lookup}\ i\ h'))\ t)
\end{array}
$$

But how to obtain `shapeInv` ???

One possibility: provided by user.

Another possibility: determined statically (dependent types?).

Just for experimentation:

$$
\begin{array}{l}
\texttt{shapeInv} :: \mathrm{Int} \to \mathrm{Int} \\
\texttt{shapeInv}\ l = \texttt{head}\ [n+1 \mid n \leftarrow [0..],\ (\texttt{length}\ (\texttt{get}\ [0..n])) == l]
\end{array}
$$

Not Quite There, Yet

Works quite nicely in some cases:

$\texttt{get} = \texttt{tail} \quad \rightsquigarrow \quad \texttt{put}$ "abcde" "xyz" = "axyz", using
$\texttt{compl}$ "abcde" = [(0, 'a')]

But not so in others:

$\texttt{get} = \texttt{init} \quad \rightsquigarrow \quad \texttt{put}$ "abcde" "xyz" fails, because
$\texttt{compl}$ "abcde" = [(4, 'e')]

The problem: by keeping indices around, `compl` still does not "collapse enough".

Note: even without these indices, $\lambda s \rightarrow (\texttt{get}\ s, \texttt{compl}\ s)$ would be injective.

Eliminating Indices

$$
\begin{array}{l}
\texttt{compl} :: [\alpha] \to [\quad \alpha\ ] \\
\texttt{compl}\ s = \textbf{let}\ \begin{array}[t]{lcl} n & = & (\texttt{length}\ s) - 1 \\ t & = & [0..n] \\ g & = & \texttt{zip}\ t\ s \\ g' & = & \texttt{filter}\ (\lambda(i, \_) \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ g \end{array} \\
\qquad\qquad \textbf{in}\ \ \texttt{map snd}\ g'
\end{array}
$$

$$
\begin{array}{l}
\texttt{inv} :: ([\alpha], [\quad \alpha\ ]) \to [\alpha] \\
\texttt{inv}\ (v', c\ ) = \textbf{let}\ \begin{array}[t]{lcl} n & = & (\texttt{shapeInv}\ (\texttt{length}\ v')) - 1 \\ t & = & [0..n] \\ h & = & \texttt{assoc}\ (\texttt{get}\ t)\ v' \\ g' & = & \texttt{zip}\ (\texttt{filter}\ (\lambda i \to \texttt{notElem}\ i\ (\texttt{get}\ t))\ t)\ c \\ h' & = & h \mathbin{+\!\!+} g' \end{array} \\
\qquad\qquad\quad \textbf{in}\ \ \texttt{seq}\ h\ (\texttt{map}\ (\lambda i \to \texttt{fromJust}\ (\texttt{lookup}\ i\ \ h'))\ t)
\end{array}
$$

Now:

$$\texttt{get} = \texttt{init} \quad \leadsto \quad \texttt{put}\ \text{“abcde”}\ \ \text{“xyz”} = \text{“xyze”}$$

More Examples

Let $\texttt{get} = \texttt{sieve}$ with:

$$\begin{aligned}
&\texttt{sieve} :: [\alpha] \to [\alpha] \\
&\texttt{sieve}\ (a : b : cs) = b : (\texttt{sieve}\ cs) \\
&\texttt{sieve}\ _ \qquad\qquad\quad = [\,]
\end{aligned}$$

Then:

$$\begin{aligned}
\texttt{put}\ [1..8]\ [2, -4, 6, 8] &= [1, 2, 3, -4, 5, 6, 7, 8] \\
\texttt{put}\ [1..8]\ [2, -4, 6] &= [1, 2, 3, -4, 5, 6] \\
\texttt{put}\ [1..8]\ [2, -4, 6, 8, 10, 12] &= [1, 2, 3, -4, 5, 6, 7, 8, \bot, 10, \bot, 12]
\end{aligned}$$

However:

$$\texttt{put}\ [1..8]\ [0, 2, -4, 6, 8] = [1, 0, 3, 2, 5, -4, 7, 6, \bot, 8]$$

Whereas we might have preferred:

$$\texttt{put}\ [1..8]\ [0, 2, -4, 6, 8] = [\bot, 0, 1, 2, 3, -4, 5, 6, 7, 8]$$

Conclusion

Types:

- constrain the behaviour of programs
- thus lead to interesting theorems about programs
- enable lightweight, semantic analysis methods

On the practical side:

- efficiency-improving program transformations
- applications in specific domains (more out there?)

Bidirectionalisation in particular:

- hot topic (databases, models community, ...)
- need a way to inject/exploit "user knowledge"

On the programming language side:

- push towards full programming languages
- aim for exploiting more expressive type systems

References I

F. Bancilhon and N. Spyratos.
Update semantics of relational views.
ACM Transactions on Database Systems, 6(3):557–575, 1981.

N.A. Day, J. Launchbury, and J. Lewis.
Logical abstractions in Haskell.
In *Haskell Workshop, Proceedings*. Technical Report UU-CS-1999-28, Utrecht University, 1999.

J.N. Foster, M.B. Greenwald, J.T. Moore, B.C. Pierce, and A. Schmitt.
Combinators for bidirectional tree transformations: A linguistic approach to the view-update problem.
ACM Transactions on Programming Languages and Systems, 29(3):17, 2007.

References II

J.P. Fernandes, A. Pardo, and J. Saraiva.
A shortcut fusion rule for circular program calculation.
In *Haskell Workshop, Proceedings*, pages 95–106. ACM Press, 2007.

A. Gill, J. Launchbury, and S.L. Peyton Jones.
A short cut to deforestation.
In *Functional Programming Languages and Computer Architecture, Proceedings*, pages 223–232. ACM Press, 1993.

P. Johann and J. Voigtländer.
Free theorems in the presence of seq.
In *Principles of Programming Languages, Proceedings*, pages 99–110. ACM Press, 2004.

References III

- P. Johann and J. Voigtländer.
  A family of syntactic logical relations for the semantics of Haskell-like languages.
  Information and Computation, 207(2):341–368, 2009.

- K. Matsuda, Z. Hu, K. Nakano, M. Hamana, and M. Takeichi.
  Bidirectionalization transformation based on automatic derivation of view complement functions.
  In *International Conference on Functional Programming, Proceedings*, pages 47–58. ACM Press, 2007.

- F. Stenger and J. Voigtländer.
  Parametricity for Haskell with imprecise error semantics.
  In *Typed Lambda Calculi and Applications, Proceedings*, volume 5608 of *LNCS*, pages 294–308. Springer-Verlag, 2009.

References IV

J. Svenningsson.
Shortcut fusion for accumulating parameters & zip-like functions.
In *International Conference on Functional Programming, Proceedings*, pages 124–132. ACM Press, 2002.

J. Voigtländer.
Much ado about two: A pearl on parallel prefix computation.
In *Principles of Programming Languages, Proceedings*, pages 29–35. ACM Press, 2008.

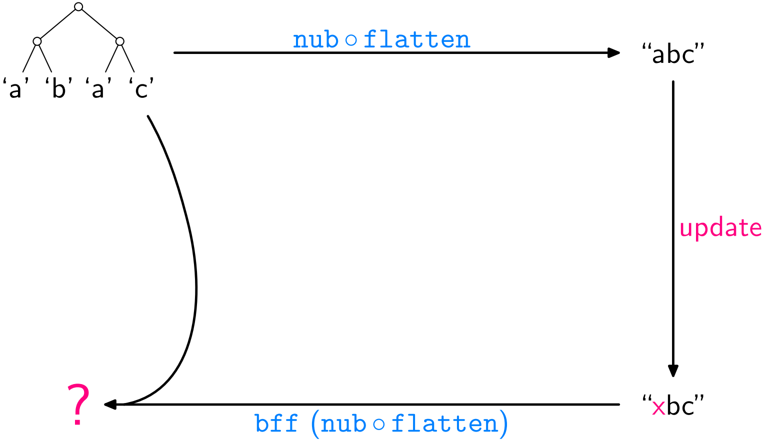
J. Voigtländer.
Bidirectionalization for free!
In *Principles of Programming Languages, Proceedings*, pages 165–176. ACM Press, 2009.

References V

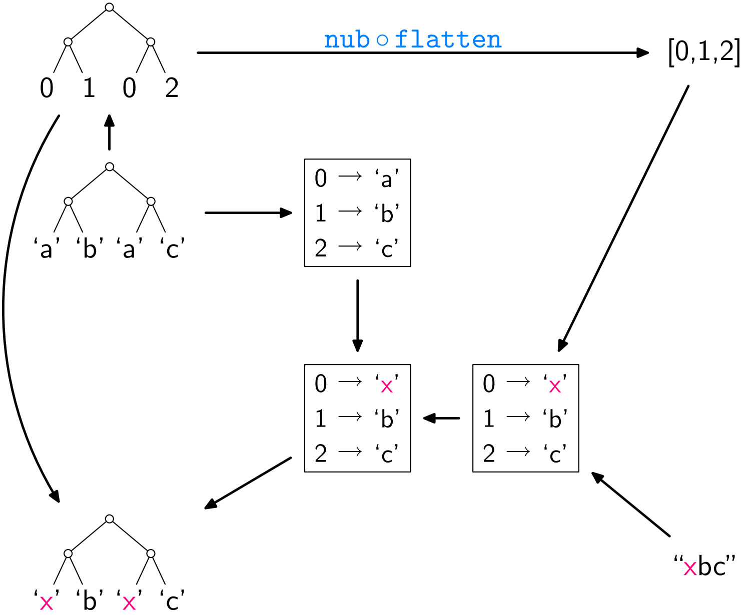
J. Voigtländer.
Free theorems involving type constructor classes.
In *International Conference on Functional Programming, Proceedings*. ACM Press, 2009.

P. Wadler.
Theorems for free!
In *Functional Programming Languages and Computer Architecture, Proceedings*, pages 347–359. ACM Press, 1989.

Another Interesting Example (involving Eq type class)



Another Interesting Example (involving Eq type class)



Why g (map f l) = map f (g l), intuitively

- g :: $[\alpha] \to [\alpha]$ must work uniformly for every instantiation of α.
- The output list can only contain elements from the input list l.
- Which, and in which order/multiplicity, can only be decided based on l.
- The only means for this decision is to inspect the length of l.
- The lists (map f l) and l always have equal length.
- g always chooses "the same" elements from (map f l) for output as it does from l, except that in the former case it outputs their images under f.
- g (map f l) is equivalent to map f (g l).
- That is what we wanted to prove!

Revising Free Theorems

[Wadler, FPCA'89] : for every $\texttt{g} :: (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$,

$$\texttt{g}\ p\ (\texttt{map}\ f\ l) \;=\; \texttt{map}\ f\ (\texttt{g}\ (p \circ f)\ l)$$

- if f strict ($f\ \bot = \bot$).

[Johann & V., POPL'04] : in presence of **seq**, if additionally:

- $p \neq \bot$,
- f total ($\forall x \neq \bot.\ f\ x \neq \bot$).

[Johann & V., I&C'09] : taking finite failures into account

[Stenger & V., TLCA'09] : taking imprecise error semantics into account

⋮