**a-Posteriori** Error Estimate for the Partition of Unity Method for Transient Heat Diffusion Problems

*M. Iqbal*¹, H. Gimperlein², M.S. Mohamed¹, O. Lagrouche¹

¹Institute for Infrastructure and Environment, Heriot Watt University, Edinburgh EH14 4AS
²Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot Watt University, Edinburgh EH14 4AS

*mi130@hw.ac.uk

**ABSTRACT**

This paper investigates an error estimate for the partition of unity method (PUM) for the solution of time dependent diffusion problems. The PUM is used to enrich the finite element space to solve a transient diffusion problem. We consider a problem for which we know the exact solution. We compute actual error in the solution and its derivatives and compare this error to a mathematically rigorous, computable error estimate. Results are computed using different values of time steps and enrichment functions. It is confirmed that at each time step the error estimate decrease similar to the actual error as the number of enrichment functions is increased.

**Key Words:** Partition of Unity Method; Transient Heat Diffusion; a-Posteriori; Error Estimate

**1. Introduction**

In recent decades various generalized and meshless numerical methods have been introduced to solve complex engineering problems with ease and sufficient accuracy. One of the important subclass of these methods is the Partition of Unity Method (PUM). This method was introduced by Melenk and Babuska [1], who developed the mathematical background of this method. The Partition of Unity Method has gained increasing importance for solution of complex engineering and scientific problems. The fact that PUM uses coarse meshes to approximate the finite element solution of complex problems, it is necessary to address the accuracy and reliability of this method.

This paper deals with the accuracy of PUM for transient heat diffusion problems. Shadi *et al* [2] used PUM to solve two dimension transient heat diffusion problem having an exact solution. The authors showed the effectiveness of PUM compared to FEM. A full description of previous work related to heat transfer problems can be found in their work. In our present work we used the PUM to solve a transient diffusion problem and calculate the actual errors in the solution. We define mathematically rigorous computable error indicators and compare the actual errors in the solution with these error indicators. For an acceptable numerical solution, the actual errors in the solution should be less than these error indicators. We define the governing PDE for transient diffusion problem in the next section with appropriate initial and boundary conditions and its transformation to weak form. In Section 3, we define the error estimates followed by results of numerical analysis in Section 4. Section 5 includes some concluding remarks.

**2. Boundary value problem and weak form**

Given an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma$ and a given time interval $[0, T]$, we are interested to solve the following transient diffusion equation

$$\frac{\partial u}{\partial t} - \lambda \nabla^2 u = f(t, \mathbf{x}), \quad (t, \mathbf{x}) \in [0, T] \times \Omega$$

(1)
where \( x = (x, y)^T \) are the spatial coordinates, \( t \) is the time variable, \( \lambda \) is the diffusion coefficient and \( f(t, x) \) represents the effects of internal sources/sinks. We consider an initial condition and Robin type boundary condition

\[
u(t = 0, x) = u_0(x), \quad (x) \in \Omega, \quad \frac{\partial u}{\partial n} + hu = g(t, x), \quad (x, t) \in [0, T] \times \Gamma
\]  

where \( u_0(x) \) is a prescribed initial field, \( \mathbf{n} \) is the outward unit normal on the boundary \( \Gamma \) and \( g \) is a given boundary function. To solve equation (1)-(2) numerically, the time interval is divided into \( N_t \) subintervals \([t_n, t_{n+1}]\) with duration \( \Delta t = t_{n+1} - t_n \) for \( n = 0, 1, \ldots N_t \) and then discretized it using an implicit scheme

\[
\frac{u^{n+1} - u^n}{\Delta t} - \lambda \nabla^2 u^{n+1} = f(t_{n+1}, x)
\]

This can be rearranged as

\[
-\nabla^2 u^{n+1} + ku^{n+1} = F
\]

where \( F \) and \( k \) are defined as

\[
F = k\left(\delta f(t_{n+1}, x) + u^n\right), \quad k = \frac{1}{\lambda \Delta t}
\]

To solve equation (4) with the finite element method we first multiply the equation with a weighting function, \( W \), and then integrate over \( \Omega \)

\[
-\int_{\Omega} \nabla W \cdot \nabla u^{n+1} d\Omega + \int_{\Omega} W ku^{n+1} d\Omega = \int_{\Omega} W F d\Omega
\]

The final weak form of our problem which will be solved using the PUM is given by

\[
\int_{\Omega} (\nabla W \cdot \nabla u^{n+1} + W k u^{n+1}) d\Omega + \int_{\Gamma} W ku^{n+1} d\Gamma = \int_{\Omega} W F d\Omega + \int_{\Gamma} W g d\Gamma
\]

To solve the weak form (6) with PUFEM, the nodal values \( u_i \) are written as a combination of enrichment functions. We considered the following sum of global exponential functions to enrich the solution space

\[
F_{enr} = G_1, G_2, G_3, \ldots, G_Q
\]

where

\[
G_q = \frac{e^{-(\frac{y}{R_c})^q} - e^{-(\frac{x}{x_c})^q}}{1 - e^{-(\frac{x}{x_c})}}, \quad q = 1, 2, \ldots, Q
\]

with \( R_c :=|x - x_c|\) being the distance from the function control point \( x_c \) to \( x \). The constants \( R_c = \sqrt{\frac{14}{1.195}} \) and \( C = \sqrt{\frac{1}{1.195}} \) control the shape of enrichment function \( G_q \).

3. Definition of Error Estimates

In this section we define the actual errors in the solution and the error estimates. We use the PUM to find the actual errors in the solution and will compare these to the error estimates.

Let define

\[
u(x, t) = \frac{t - t_n}{t_{n+1} - t_n} u^{r+1}(x) + \frac{t_{n+1} - t}{t_{n+1} - t_n} u^n(x)
\]

and

\[
\tilde{u}(x, t) = u(x, t_{n+1}), \quad \tilde{f}(x, t) = f(x, t_{n+1}), \quad t \in [t_n, t_{n+1}]
\]

Let \( U \) and \( u \) be the exact and numerical solution of the equations (1) - (2) then

\[
\int_{\Omega} (U - u)^2 + \lambda \int_{t_n}^{t_{n+1}} \int_{\Omega} \Delta(U - u)^2 \leq \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 + \eta_5^2
\]
The LHS of expression (9) indicates the actual errors in the method and RHS indicates the error estimates comprising of five error indicators $\eta_1^2$ to $\eta_5^2$, where

$$
\eta_1^2 = \| U_o - u_o \|^2_{L^2(\Omega)}, \quad \eta_2^2 = \int_{t_n}^{t_{n+1}} \| (f - \hat{\partial}_t u + \lambda \Delta u) \|^2_{H^{-1}(\Omega)} \quad (10)
$$

$$
\eta_3^2 = \int_{t_n}^{t_{n+1}} \| (f - \hat{f}) \|^2_{H^{-1}(\Omega)}, \quad \eta_4^2 = \lambda \int_{t_n}^{t_{n+1}} \| \nabla (u - \hat{u}) \|^2_{L^2(\Omega)}, \quad \eta_5^2 = \lambda \sum_{\text{edges}} \| \frac{\partial \hat{u}}{\partial n} \|^2_{L^2(E)}
$$

For our case we calculate the values of $\eta_2^2$, $\eta_4^2$ and $\eta_5^2$ only. $\eta_1^2$ and $\eta_3^2$ are zero in our case as $\eta_1^2$ is based on the initial condition and the function $f$ in $\eta_3^2$ is computed using exact solution.

4. Numerical Results

The PUM approach is used to solve a transient heat diffusion problem given by equations (1) - (2). We consider a square domain $\Omega$ defined by $(x \in \Omega; \ 0 \leq x \leq 2)$ with reaction term $f(t, x)$, the boundary function $g$ and the initial condition $u_0(x)$ are chosen such that the exact solution is given by

$$
U(x, t) = x^{20} y^{20} (2 - x)^{20} (2 - y)^{20} (1 - e^{-\lambda t}) \quad (11)
$$

where $x = (x, y)^T$ are the spatial coordinated and $t$ is the time variable. We use a course mesh of only 25 elements using different number of enrichment functions $Q = 2, 3, \ldots, 6$. For all the analyses the heat diffusion coefficient is assumed to be $\lambda = 0.1$ kg m$^{-1}$ K$s^{-2}$, and the convection heat transfer coefficient $h = 1$ kg K$s^{-2}$. Figure (1) shows comparison of temperature distribution using exact analytical solution and numerical solution using PUFE.M.

To quantify the error in the solution, we compute the actual errors in the solution and the error indicators and compare these values as defined by expression (9). When compared to the exact solution, the LHS of expression (9) calculates the actual error in the numerical solution and its derivatives. For an acceptable numerical solution, these errors should be less than the summation of error indicators $\eta_1^2 + \eta_2^2 + \ldots \eta_5^2$.

Figures (2)- (4) show the comparison of these quantities for different values of time steps. In graphs the term 'Error' represents the actual error in the method whereas 'ET' is the summation of $\eta_2^2$, $\eta_4^2$ and $\eta_5^2$. The enrichment functions 'Q' are plotted on abscissa while the errors on the ordinate; both in logarithmic scale. Figure (2) shows the results for $\Delta t = 0.0001$, Figure (3) for $\Delta t = 0.001$ while Figure (4) for $\Delta t = 0.01$. All the figures show that the actual errors in the solution are well below the summation of $\eta_2^2$, $\eta_4^2$ and $\eta_5^2$. The actual error in the solution decreases with increasing number of enrichment functions. The error estimate also follows similar decreasing pattern as we increase the number of enrichment functions.

5. Conclusions

We used the PUFE.M to solve the time dependent diffusion equation. We calculated the errors in the solution and its derivatives and compared these errors with the error estimate. Based on the analysis, we can draw the following conclusions:

- Results of the actual error are well below the defined error estimate.
- With increasing number of enrichment functions, the actual error and the error estimate show a similar decrease.
- Results are obtained with three different values of time step $\Delta t$, and for each value of $\Delta t$, the results are within the range of error estimate, which shows the effectiveness of the method for time dependent diffusion problem.

References


Figure 1: Temperature distribution for $\Delta t = 0.01$ and $t = 10$ s. (a) Finite element mesh (b) Analytical result (c) Numerical result obtained using PUFEM with $Q = 6$

Figure 2: Comparison of actual error and error estimate for $\Delta t = 0.0001$ (a) at 50th time step, (b) at 100th time step

Figure 3: Comparison of actual error and error estimate for $\Delta t = 0.001$ (a) at 50th time step, (b) at 100th time step

Figure 4: Comparison of actual error and error estimate for $\Delta t = 0.01$ (a) at 50th time step, (b) at 100th time step