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The Lieb-Thirring inequality

Let $H = -\Delta + V$ (The Schrödinger operator), where $-\Delta$ is the Laplace operator and $V \in L^2_{loc}(\mathbb{R}^d)$. Then $D(H) = C_0^2(\mathbb{R}^d) \overset{dense}{\subseteq} L^2(\mathbb{R}^d)$ and maps to $L^2(\mathbb{R}^d)$, and H is essentially selfadjoint.

Thm 1 (Lieb-Thirring inequality):

If the negative part of V, $V_- = \min\{V, 0\}$, satisfies $V_- \in L^{\frac{d}{2}+1}(\mathbb{R}^d)$, then H is bounded below and the min-max values, $\mu_n := \inf\{\sup\{\langle Hx, x\rangle \mid x \in L, \|x\| = 1\} \mid L \subseteq D(H), \dim L = n\}$ satisfy the Lieb-Thirring ineq.:

$$\sum [\mu_n]_- \geq -\tilde{C}_d \int |V_-(x)|^{\frac{d}{2}+1} dx =: -C_d, \quad \tilde{C}_d \text{ constant}$$

depending on the dimension d.

First we need to recall some general things about selfadjoint operators to proove the Theorem.

Let H be a selfadjoint operator, then the spectrum is:

$$\sigma(H) = (\{\text{eigenvalues of finite multiplicity}\} \cup \operatorname{ess}\sigma(H)) \subseteq \mathbb{R}$$

The $\operatorname{ess}\sigma(H)$ can be descriped by following Theorem which wont be prooved here.

②

Thm 2: $\lambda \in \mathrm{ess}\,\sigma(H) \iff \exists$ sequence of approximating eigenvectors $\{x_n\}_{n=1}^{\infty} \subseteq D(H)$, with $x_n \perp x_m$ $\forall m \neq n$, $\|x_n\| = 1$ $\forall n$, s.t. $\|Hx_n - \lambda x_n\| \searrow 0$, $n \to \infty$.

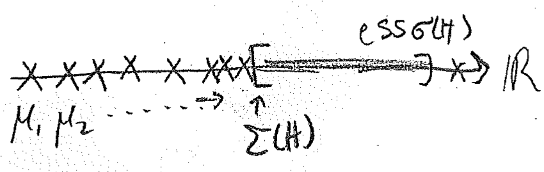
And another Theorem which wont be proved is:

Thm 3: If H is lowerbounded by a constant C and $\Sigma(H) := \inf \mathrm{ess}\,\sigma(H) > -\infty$, then either

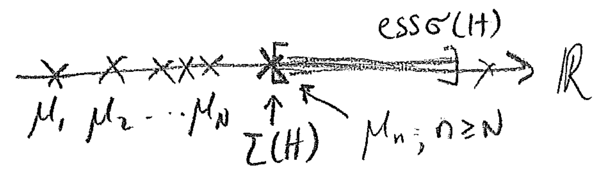
(i) $\mu_n \nearrow \Sigma(H)$, $n \to \infty$ and $\sigma(H) \cap [-C, \Sigma(H)) = \bigcup_{j=1}^{\infty} \{\mu_j\}$

or (ii) $\exists N$ s.t. $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N < \Sigma(H)$, $\mu_n = \Sigma(H)$ $\forall n \geq N$ and $\sigma(H) \cap [-C, \Sigma(H)) = \{\mu_1, \mu_2, \dots \mu_N\}$.

So $\sigma(H)$ must look like



or



Then for $\mu_n < \Sigma(H)$ we have $\mu_n \in \sigma(H)/\mathrm{ess}\,\sigma(H)$ and hence μ_n is an eigen value of H. So either

(i) $\{\mu_n\}_{n=1}^{\infty} = \{\lambda_n\}_{n=1}^{\infty}$, λ_n eigenvalues of H or

(ii) $\{\mu_n\}_{n=1}^{N} = \{\lambda_n\}_{n=1}^{N}$, — " —

So if we proove Thm 1, $H = -\Delta + V$ will be lower bounded and then we can use Thm 3 to show that $\Sigma(H) \geq 0$:

Because if $0 > \Sigma(H) \overset{\text{Thm 3}}{\geq} \mu_n$, $n = 1, 2, \dots$, then $\sum_{n=1}^{\infty} [\mu_n]_- = \sum_{n=1}^{\infty} \mu_n = -\infty$, and then $-C \overset{\text{Thm 1}}{\leq} \sum_{n=1}^{\infty} [\mu_n]_- = -\infty$ ↯.

③ So $\Sigma(H) \geq 0$ and that means all $\mu_n < 0$ will be smaller than $\Sigma(H)$ and hence eigenvalues of H. So the Lieb-Thirring inequality is an estimate of the negative eigenvalues. ((i) $\sum_{n=1}^{\infty}[\mu_n]_- = \sum_{j=1}^{\infty}\mu_j$ or (ii) $\sum_{n=1}^{\infty}[\mu_n]_- = \sum_{j=1}^{N}\mu_j, \mu_j < 0$)

Now, to proove Thm 1, we will first show that it's encought to proove

$$(*) \quad \sum_{j=1}^{N}\langle\phi_j, H\phi_j\rangle \geq -C, \quad \forall \text{ finite orthonormal families } \{\phi_j\}_{j=1}^{N} \subseteq C_0^2(\mathbb{R}^d).$$

If $(*)$ holds, then H must be lower bounded, since for all $\phi \in C_0^\infty(\mathbb{R}^d)$, $\|\phi\| = 1$ $\langle\phi, H\phi\rangle \geq -C > -\infty$ ($N = 1$ in $(*)$) and the definition of lower bounded is:

$$m(H) = \inf\{\mathrm{Re}\langle H\phi, \phi\rangle \mid \phi \in D(H) = C_0^2(\mathbb{R}^d), \|\phi\| = 1\} > -\infty$$

From $(*)$ we can also conclude tha $\Sigma(H) \geq 0$:
For by Thm 2, since $\Sigma(H) \in \mathrm{ess}\,\sigma(H)$, there exists orthonormal seq. $\{\phi_n\}_{n=1}^{\infty}$ s.t. $\|H\phi_n - \Sigma(H)\phi_n\| \searrow 0$, $n \to \infty$.

$$\Rightarrow \langle\phi_n, H\phi_n\rangle = \underbrace{\langle\phi_n, H\phi_n - \Sigma(H)\phi_n\rangle}_{\to 0,\, n\to\infty} + \Sigma(H)\underbrace{\langle\phi_n, \phi_n\rangle}_{=1} \to \Sigma(H), \; n \to \infty.$$

$\Rightarrow \sum_{j=1}^{N}\langle\phi_n, H\phi_n\rangle \to \sum_{j=1}^{N}\Sigma(H) = \Sigma(H)\cdot N$. This goes to $-\infty$ if $\Sigma(H) < 0$ and we let $N \to \infty$. So $\Sigma(H) < 0$ would contradict the fact that $\sum_{j=1}^{N}\langle\phi_n, H\phi_n\rangle \geq -C$.

Since then $\Sigma(H) \geq 0$ all $\mu_n < 0 \leq \Sigma(H)$ are eigen values and because H is self adjoint, then

④ $\sum_{n=1}^{\infty}[\mu_n]_- = \sum_{n=1}^{N}[\lambda_n]_- = \sum_{j=1}^{N}\lambda_j$, λ_j negative eigenvalue of H,

$$= \sum_{j=1}^{N}\langle\phi_j, H\phi_j\rangle, \qquad N = \#\{\mu_n < 0\} \overset{(vi)}{\subseteq} \overset{(i)}{\mathbb{N}\cup\{\infty\}}$$

where $\{\phi_j\}_{j=1}^{\tilde N}$ is an orthonormal family of eigenvectors in $D(H)$ s.t. $H\phi_j = \lambda_j\phi_j$. Hence $\sum_{n=1}^{\infty}[\mu_n]_- \geq -C$.

Now to show $(*)$ we will split the proof into the steps:

$$\sum_{j=1}^{N}\langle\phi_j, H\phi_j\rangle \overset{(1)}{\geq} \sum_{j=1}^{N}\int_{\mathbb{R}^d}|\nabla\phi_j(x)|^2dx - \underbrace{\sum_{j=1}^{N}\int_{\mathbb{R}^d}|V_-(x)||\phi_j(x)|^2dx}_{=:V}$$

$$\overset{(2)}{\geq} \int_{\mathbb{R}^d}\int_0^{\infty}\Big[\underbrace{\Big(\sum_{j=1}^{N}|\phi_j(x)|\Big)^{1/2}}_{A:=} - \Big(\sum_{j=1}^{N}|\phi_j^{e,-}(x)|^2\Big)^{1/2}\Big]_+^2 de\,dx - V$$

$$\overset{(3)}{\geq} \int_{\mathbb{R}^d}\int_0^{\infty}\Big[A - (2\pi)^{-\frac{d}{2}}V_d(1)^{1/2}e^{d/4}\Big]_+^2 de\,dx - V$$

$$\overset{(4)}{\geq} \frac{(2\pi)^2 d^2 V_d(1)^{-\frac{2}{d}}}{(d+2)(d+4)}\int_{\mathbb{R}^d}A^{2+\frac{4}{d}}dx - V$$

$$\overset{(5)}{\geq} \frac{(2\pi)^2 d^2 V_d(1)^{-\frac{2}{d}}}{(d+2)(d+4)}\int_{\mathbb{R}^d}A^{2+\frac{4}{d}}dx - \Big(\int_{\mathbb{R}^d}|V_-(x)|^{\frac{d+2}{2}}dx\Big)^{\frac{2}{d+2}}\Big(\int_{\mathbb{R}^d}A^{2+\frac{4}{d}}dx\Big)^{\frac{d}{d+2}}$$

$$\overset{(6)}{\geq} \underbrace{-\frac{2(2\pi)^{-d}V_d(1)}{(d+2)}\Big(\frac{d+4}{d}\Big)^{\frac{d}{2}}}_{=:-\tilde C_d}\int|V_-(x)|^{\frac{d+2}{2}}dx$$

$$= -C_d$$

(5)

$$\Delta = \nabla \cdot \nabla$$

(1) $\sum_{j=1}^{N} \langle \phi_j, H\phi_j \rangle = \sum_{j=1}^{N} \left(\langle \phi_j, -\Delta\phi_j \rangle + \langle \phi_j, V\phi_j \rangle \right) = \sum_{j=1}^{N} \langle \nabla\phi_j, \nabla\phi_j \rangle + \sum \langle \phi_j, V\phi_j \rangle$

$$= \sum_{j=1}^{N} \int_{\mathbb{R}^d} |\nabla\phi_j(x)|^2 dx + \sum_{j=1} \int_{\mathbb{R}^d} V(x) |\phi_j(x)|^2 dx$$

$V(x) \geq -|V_-(x)|$: $\geq \sum_{j=1}^{N} \int_{\mathbb{R}^d} |\nabla\phi_j(x)|^2 dx - \sum_{j=1}^{N} \int |V_-(x)| |\phi_j(x)|^2 dx$

(2) We will use the Fourier transform of $f \in L^1(\mathbb{R}^d)$:

$$\hat{f}(p) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(p) e^{-ipx} dx,$$

since it extends to a unitary map on $L^2(\mathbb{R}^d)$.

For $e>0$ we write $\phi = \phi^{e,+} + \phi^{e,-}$, where

$$\hat{\phi}^{e,+}(p) = \begin{cases} \hat{\phi}(p), & p^2 > e \\ 0, & p^2 \leq e \end{cases}, \quad \hat{\phi}^{e,-}(p) = \begin{cases} 0, & p^2 > e \\ \hat{\phi}(p), & p^2 \leq e \end{cases}$$

Then we have:

$$\sum \int_{\mathbb{R}^d} |\nabla\phi_j(x)|^2 dx = \sum \|\nabla\phi_j\|^2 \overset{\text{unitary}}{=} \sum \|\widehat{\nabla\phi_j}\|^2 = \sum \int |\widehat{\nabla\phi_j}(p)|^2 dp$$

$$= \sum \int \left| (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \nabla\phi_j(x) e^{-ipx} dx \right|^2 dp$$

$\phi \in C_0^2(\mathbb{R}^d)$: $= \sum \int \left| -(2\pi)^{-\frac{d}{2}} \int \phi_j(x) (-ip e^{-ipx}) dx \right|^2 dp$

$$= \sum \int |p\hat{\phi}_j(p)|^2 dp = \sum \int \left(|\hat{\phi}_j(p)|^2 \int_0^{p^2} de \right) dp$$

$$= \sum \int \left(\int_0^{p^2} |\hat{\phi}_j(p)|^2 de \right) dp = \sum \int \left(\int_0^{\infty} |\hat{\phi}_j^{e,+}(p)|^2 de \right) dp$$

$$= \sum \int_0^{\infty} \|\hat{\phi}_j^{e,+}\|^2 de = \sum \int_0^{\infty} \|\phi_j^{e,+}\|^2 de$$

$$= \int_0^{\infty} \int_{\mathbb{R}^d} \sum_{j=1}^{N} |\phi_j^{e,+}(x)|^2 dx\, de \geq \int_0^{\infty} \int_{\mathbb{R}^d} \left[\left(\sum_{j=1}^{N} |\phi_j(x)|^2 \right)^{1/2} - \left(\sum |\phi_j^{e,-}(x)|^2 \right)^{1/2} \right]_+^2 dx\, de$$

Δ-ineq in $\mathbb{C}^N$:

$\|x+y\| \leq \|x\| + \|y\|$, $\|x\| = \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2}$

$\Rightarrow \|x\|^2 \geq [\|x+y\| - \|y\|]^2$

(3) $$\sum_{j=1}^{N} |\phi_j^{e,-}(x)|^2 = \sum_{j=1}^{N} \left|(2\pi)^{-\frac{d}{2}} \int e^{ipx} \hat{\phi}_j^{e,-}(p)\, dp\right|^2$$

$$= (2\pi)^{-d} \sum_{j=1}^{N} \left|\int e^{ipx} \hat{\phi}_j(p) \mathbb{1}_{(0,e)}(p^2)\, dp\right|^2$$

Bessel's ineq.: For $\{\phi_j\}_{j=1}^N$ ON then $\sum_{j=1}^{N} |\langle x, \phi_j\rangle|^2 \leq \|x\|^2$

$$\leq (2\pi)^{-d} \int |e^{ipx} \mathbb{1}_{(0,e)}(p^2)|^2\, dp$$

$$= (2\pi)^{-d} \int \mathbb{1}_{(0,e)}(p^2)\, dp = (2\pi)^{-d} V_d(e^{1/2}) = (2\pi)^{-d} V_d(1)(e^{1/2})^d$$

where $V_d(r)$ is the volume of the ball of radius r in d dimensions. (3) follows since $\left(\sum_{j=1}^{N} |\phi_j(x)|\right)^{1/2} \geq \left(\sum |\phi_j^{e,-}(x)|^2\right)^{1/2} \geq 0$

(4) Let $A := \left(\sum |\phi_j(x)|^2\right)^{1/2}$, $B := (2\pi)^{-\frac{d}{2}} V_d(1)^{1/2}$.

Notice that $A - Be^{d/4} \geq 0 \iff \left(\frac{A}{B}\right)^{4/d} \geq e$. Then we have

$$\int_0^\infty [A - Be^{d/4}]_+^2\, de = \int_0^{\left(\frac{A}{B}\right)^{4/d}} (A - Be^{d/4})^2\, de = \int_0^{\left(\frac{A}{B}\right)^{4/d}} (A^2 + B^2 e^{d/2} - 2ABe^{d/4})\, de$$

$$= \left[A^2 e - B^2\left(\frac{1}{\frac{d}{2}+1}\right) e^{\frac{d}{2}+1} - 2AB \frac{1}{\frac{d}{4}+1} e^{\frac{d}{4}+1}\right]_0^{\left(\frac{A}{B}\right)^{\frac{4}{d}}}$$

$$= A^2 \left(\frac{A}{B}\right)^{4/d} + B^2\left(\frac{2}{d+2}\right)\left(\frac{A}{B}\right)^{2+\frac{4}{d}} - AB\left(\frac{8}{d+4}\right)\left(\frac{A}{B}\right)^{1+\frac{4}{d}}$$

$$= A^2\left(\frac{A}{B}\right)^{\frac{4}{d}}\left(1 + \left(\frac{2}{d+2}\right) - \left(\frac{8}{d+4}\right)\right) = A^2 \left(\frac{A}{B}\right)^{\frac{4}{d}} \left(\frac{d^2}{(d+2)(d+4)}\right)$$

$$= (2\pi)^2 V_d(1)^{-\frac{2}{d}} \frac{d^2}{(d+2)(d+4)} A^{2+\frac{4}{d}}$$

(5) $$\sum_{j=1}^{N} \int_{\mathbb{R}^d} |V_-(x)| |\phi_j(x)|^2\, dx = \int_{\mathbb{R}^d} \left|V_-(x) \sum_{j=1}^{N} |\phi_j(x)|^2\right| dx = \|V_- \sum |\phi_j|^2\|_1$$

Hölder's ineq.: $\|fg\|_1 \leq \|f\|_p \|g\|_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\leq \|V_-\|_{\frac{d+2}{2}} \left\|\sum |\phi_j|^2\right\|_{\frac{d+2}{d}}$$

$p = \frac{d+2}{2}$, $q = \frac{d+2}{d}$

$$= \left(\int |V_-(x)|^{\frac{d+2}{2}} dx\right)^{\frac{2}{d+2}} \left(\int \left(\sum |\phi_j(x)|^2\right)^{\frac{d+2}{d}} dx\right)^{\frac{d}{d+2}}$$

(6) A function $\mathbb{R}_+ \ni x \mapsto ax - bx^{\frac{d}{d+2}}$, $a, b > 0$,

has the minimal value $-\frac{2}{d+2}\left(\frac{d}{d+2}\right)^{d/2} a^{-\frac{d}{2}} b^{\frac{d+2}{2}}$.

For $a = \frac{(2\pi)^2 d^2 V_d(1)^{-\frac{2}{d}}}{(d+2)(d+4)}$, $b = \left(\int |V_-(x)|^{\frac{d+2}{2}} dx\right)^{\frac{2}{d+2}}$, $x = \int_{\mathbb{R}^d} \left(\sum_{j=1}^{N} |\phi_j(x)|^2\right)^{\frac{d+2}{d}} dx$

we get the inequality (6).