Exercise sheet 6

Exercise class week 22 + 23

Calculus of pseudodifferential operators

Recall that

$$S^m := \left\{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}_0^n \exists C_{\alpha\beta} : \left| \partial_x^\beta \partial_\xi^\alpha a(x,\xi) \right| \leq C_{\alpha\beta} \left(1 + |\xi|\right)^{m-|\alpha|} \right\}.$$

Furthermore set $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m$ and $S^\infty := \bigcup_{m \in \mathbb{R}} S^m$.

Exercise 19:

a) (Borel's Lemma) Let $(a_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{C}$, $(\varepsilon_j)_{j \in \mathbb{N}_0} \subseteq (0, \infty)$, $\varepsilon_j \searrow 0$ sufficiently fast and $\eta \in C_c^\infty(\mathbb{R})$, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$.
Show that $f(x) := \sum_{j=0}^\infty \frac{a_j}{j!} x^j \eta\left(\frac{x}{\varepsilon_j}\right)$ is smooth and $(\partial_x^j f)(0) = a_j \ \forall j$.

Hint: For any $x \neq 0$, $f(x)$ is defined by a finite sum.

b) (asymptotic summation) Let $a_j \in S^{m_j}$, $m_j \searrow -\infty, \varepsilon_j$ as above and $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x) = 1$ for $|x| \geq 2$, $\varphi(x) = 0$ for $|x| \leq 1$.
Show that

$$\sum_{j=0}^\infty a_j(x,\xi)\varphi(\varepsilon_j \xi) =: a(x,\xi) \in S^{m_0} \text{ and}$$

$$a - \sum_{j=0}^k a_j \in S^{m_{k+1}} \quad \forall k \in \mathbb{N}_0. \qquad (*)$$

Notation: We write $a \sim \sum_{j=0}^\infty a_j$ if ($*$) holds. Observe that $a \sim 0$ if and only if $a \in S^{-\infty}$. Using b), we define an associative product $\# : S^\infty/S^{-\infty} \times S^\infty/S^{-\infty} \to S^\infty/S^{-\infty}$ via

$$(a + S^{-\infty})\#(b + S^{-\infty}) \sim \sum_{|\alpha| \geq 0} \frac{i^\alpha}{\alpha!} D_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi).$$

Exercise 20:

a) Let $a \in S^m$, $b \in S^{\tilde{m}}$, $m, \tilde{m} \in \mathbb{R} \cup \{-\infty\}$. Show that $a + b \in S^{\max\{m,\tilde{m}\}}$, $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$, $ab \in S^{m+\tilde{m}}$. Conclude that if $R_i \in S^{-\infty}$:
$(a + R_1 + S^{-\infty})\#(b + R_2 + S^{-\infty}) = (a + S^{-\infty})\#(b + S^{-\infty}) \in S^{m+\tilde{m}}/S^{-\infty}$ is independent of R_i. The $\#$-product is thus well-defined and we write $a\#b := (a + S^{-\infty})\#(b + S^{-\infty})$. If $|a(x,\xi)| \geq C\langle\xi\rangle^m$, show that $a^{-1} \in S^{-m}$ and $r := a\#a^{-1} - 1,\ \tilde{r} := a^{-1}\#a - 1 \in S^{-1}/S^{-\infty}$.

b) (Neumann series) Let $b \in S^{-1}$, $b^{\#n} = \underbrace{b\#\ldots\#b}_{n \text{ factors}}$. Show that

$$\left(1 + b + b^{\#2} + \cdots + b^{\#N}\right)\#(1 - b) \sim 1 - b^{\#(N+1)}$$

and hence that

$$\left(1+\sum_{j=1}^{\infty} b^{\#j}\right)\#(1-b)\sim 1.$$

c) Continuing a), let $s \sim \sum\limits_{j=1}^{\infty} r^{\#j},\ \tilde{s} \sim \sum\limits_{j=1}^{\infty} \tilde{r}^{\#j},\ b = a^{-1}\#(1+s),\ \tilde{b} = (1+\tilde{s})\#a^{-1}$. Show that $\tilde{b}\#a \sim a\#b \sim 1$ and therefore $\tilde{b} \sim \tilde{b}\#a\#b \sim b$, or $a\#b-1 \sim b\#a-1 \sim 0$.

Interpretation: Given $a \in S^m$ elliptic ($|a| \geq C\langle\xi\rangle^m$), we have found an inverse $b \in S^{-m}$ of a up to a negligible remainder $\in S^{-\infty}$.

Exercise 21:
a) Let $a \in S^m$. Recall

$$\begin{aligned}\operatorname{op}(a)f(x) &:= \int \mathrm{d}\xi a(x,\xi)\hat{f}(\xi)e^{ix\xi}\\ &= \int \mathrm{d}\xi L_\xi^N\left[a(x,\xi)\hat{f}(\xi)\right]e^{ix\xi}\end{aligned}$$

for $L_\xi := \left(1+|x|^2\right)^{-1}(1-\Delta_\xi)$ and $N \in \mathbb{N}$ from Exercise 17.
Write $\operatorname{op}(a)f(x) = (k(x,\cdot)*f)(x)$, i.e. $k(x,\cdot) = \mathcal{F}_{\xi\to\cdot}a(x,\xi)$ is a distribution $\in S'(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$. Show that $k(x,\cdot)|_{\mathbb{R}^n\setminus\{0\}} \in L^\infty_{loc}(\mathbb{R}^n\setminus B_\varepsilon(0))$ for all $\varepsilon > 0$ and all $x \in \mathbb{R}^n$. Furthermore $|k(x,z)| \leq C_N|z|^{-N}$ for $|z| \geq 1,\ N \in \mathbb{N}$ uniformly for all $x \in \mathbb{R}^n$.

Hint: Use that $\partial_\xi^\alpha a(x,\xi) \in L^1_\xi(\mathbb{R}^n)$ for all $\alpha \geq m+n+1$ and $\left|\mathcal{F}^{-1}_{\xi\to z}\partial_\xi^\alpha a(x,\xi)\right| = |z^\alpha k(x,z)|$.

b) Define the adjoint of $a(x,D)$ with respect to the L^2-scalar product:$\forall f,g \in S(\mathbb{R}^d)$: $\langle a(x,D)^*f,g\rangle_{L^2} := \langle f,a(x,D)g\rangle_{L^2}$. Integrate by parts as in Exercise 17 to show, that $a(x,D)^* : S \to S$ and conclude that $a(x,D)$ extends by duality, $\forall f \in S, \forall g \in S'$: $\langle f,a(x,D)g\rangle := \langle a(x,D)^*f,g\rangle$, to an operator on S'.

c) Let $a \in S^{-\infty}$. Show that $a(x,D), a(x,D)^* : \mathcal{E}' \to S$ and therefore by duality, also $a(x,D), a(x,D)^* : S' \to C^\infty$.

Hint: Integrate by parts yet again.