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Date: August 27, 2015.
For a representation $\pi$ of a connected Lie group $G$ on a topological vector space $E$ we defined in [1] a vector subspace $E^\omega$ of $E$ of analytic vectors. Further we equipped $E^\omega$ with an inductive limit topology. We called a representation $(\pi, E)$ analytic if $E = E^\omega$ as topological vector spaces.

Some mistakes in the paper have been pointed out by Helge Glöckner (see [2]). For a representation $(\pi, E)$ and a closed $G$-invariant subspace $F$ of $E$ we asserted in Lemma 3.6 (i) that $F^\omega = E^\omega \cap F$ as a topological space. Based on that we further asserted in Lemma 3.6 (ii) that the inclusion $E^\omega / F^\omega \to (E/F)^\omega$ is continuous and in Lemma 3.11 that if $(\pi, E)$ is analytic then so is the restriction to $F$. However, there is a gap in the proof of the first assertion, and presently it is not clear to us whether the above statements are then true in this generality (for unitary representations $(\pi, E)$ they are straightforward). Our proof does give the following weaker version of the two lemmas:

**Lemma 1.** Let $(\pi, E)$ be a representation, and let $F \subset E$ be a closed invariant subspace. Then

(i) $F^\omega = E^\omega \cap F$ as vector spaces and with continuous inclusion $F^\omega \to E^\omega$.

(ii) $E^\omega / E^\omega \cap F \subset (E/F)^\omega$ continuously.

(iii) If $(\pi, E)$ is an analytic representation, then $\pi$ induces an analytic representation on $E/F$.

Indeed, for (iii) note that if $E$ is analytic, $E/F = E^\omega / E^\omega \cap F \subset (E/F)^\omega$ continuously by (ii), and $(E/F)^\omega \subset E/F$ continuously.

Further we asserted in Proposition 3.7 a general completeness property of the functor which associates $E^\omega$ to $E$. However, there is a gap in the proof, which asserts that $v_i \to v$ in the topology of $E^\omega$. As statements in this generality are not needed for the main result, we can leave out the proposition (together with Remark 3.8).

Attached to $G$ we introduced a certain analytic convolution algebra $\mathcal{A}(G)$. A central theme of the paper is the relation of analytic representations of $G$ to algebra representations of $\mathcal{A}(G)$ on $E$: $\mathcal{A}(G) \times E \to E$. In Proposition 4.2 (ii) we claimed that the bilinear map $\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G)$ is continuous. However, the proof shows only separate continuity. For a similar reason we need to weaken Proposition 4.6 to:

**Proposition 2.** Let $(\pi, E)$ be an $F$-representation. The assignment

$$(f, v) \mapsto \Pi(f)v := \int_G f(g)\pi(g)v \, dg$$

is continuous.
defines a continuous bilinear map

\[ A_n(G) \times E \to E_n \]

for every \( n \in \mathbb{N} \), and a separately continuous map

\[ A(G) \times E \to E^\omega \]

(with convergence of the defining integral in \( E^\omega \)). Moreover, if \((\pi, E)\) is a Banach representation, then the latter bilinear map is continuous.

**Proof.** The first statement is proved in the article, and thus only the statement for \( \pi \) a Banach representation remains to be proved. We repeat the first part of the proof, now with \( p \) denoting the fixed norm of \( E \). The constants \( c, C \) such that

\[ p(\pi(g)v) \leq Ce^{\epsilon d(g)}p(v) \quad (g \in G, v \in E) \]

and \( N, C_1 \) such that

\[ C_1 := \int_G e^{(c-N)d(g)} \, dg < \infty, \]

are then all fixed, and so is \( \epsilon = 1/(CC_1) \).

Let \( n \in \mathbb{N} \) and an open 0-neighborhood \( W_n \subset E_n \) be given. We may assume

\[ W_n = \{ v \in E_n \mid p(\pi(K_n)v) < \epsilon_n \} \]

with \( K_n \subset GV_n \) compact and \( \epsilon_n > 0 \). Let

\[ O_n := \{ f \in \mathcal{O}(V_n G) \mid \sup_{z \in K_n, g \in G} |f(z^{-1}g)|e^{Nd(g)} < \epsilon \epsilon_n \} \subset A_n(G). \]

The computation in the given proof shows that if \( f \in O_n \) and \( p(v) < 1 \) then \( \Pi(f)v \in W_n \). The asserted bi-continuity of \( A(G) \times E \to E^\omega \) follows. \( \square \)

As a consequence we obtain as in Example 4.10(a), but only for Banach representations \((\pi, E)\), that \( E^\omega \) is \( A(G) \)-tempered. In particular \( A(G) \) need not itself be \( A(G) \)-tempered, and we need to replace Lemma 5.1(i) by the following weaker version:

**Lemma 3.** \( V^{\min} \) is an analytic globalization of \( V \) and it carries an algebra action

\[ (f, v) \mapsto \Pi(f)v, \quad A(G) \times V^{\min} \to V^{\min}, \]

of \( A(G) \), which is separately continuous.

The main result of the paper Theorem 5.7, has two statements concerning a Harish-Chandra module \( V \) with a globalization \( E \):

1. If \( E \) is analytic \( A(G) \)-tempered then \( E = V^{\min} \)
(2) If $E$ is an $F$-globalization then $E^\omega = V^{\min}$.

The proof, which relied on Lemma 3.11 and Proposition 4.6 respectively, needs to be corrected. The proof of (1) if $V$ is irreducible needs no modification. For the general case it can be adjusted as follows.

Like in the paper, it suffices to consider an exact sequence of Harish-Chandra modules $0 \to V_1 \to V \to V_2 \to 0$, where both $V_1$ and $V_2$ have unique analytic $\mathcal{A}(G)$-tempered globalizations. We show that the same holds for $V$.

Let $E_1$ be the closure of $V_1$ in $E$ and $E_2 = E/E_1$. By Lemma 1(iii), $E_2$ is an analytic $\mathcal{A}(G)$-tempered globalization of $V_2$, so that by assumption $E_2 = V^{\min}_2 = \mathcal{A}(G)V_2$ as topological vector spaces.

In a first step we prove that $E_1 = V^{\min}_1 = \mathcal{A}(G)V_1$ as vector spaces. For that we note first that $E_1$ is $\mathcal{A}(G)$-tempered and that $V^{\min}_1 \subset E_1$ continuously. Next, by Proposition 5.3 (which holds for any $\mathcal{A}(G)$-tempered representation), we may embed $E_1 \subset F_1$ continuously into a Banach globalization of $F_1$ of $V_1$. Moreover, the proof shows that the embedding is compatible with the action by $\mathcal{A}(G)$. It follows that $E_1^\omega \subset F_1^\omega$ continuously and as $\mathcal{A}(G)$-modules. Further note that since $E$ is analytic, from Lemma 1(i) we also obtain $E_1^\omega = E^\omega \cap E_1 = E_1$ as vector spaces. Hence $V^{\min}_1 \subset E_1 \subset F_1^\omega$. By assumption $V_1$ has unique $\mathcal{A}(G)$-tempered globalization, hence $F_1^\omega \simeq V^{\min}_1$. Therefore $V^{\min}_1 \subset E_1 \subset F_1^\omega \simeq V^{\min}_1$. As these maps respect the structure as $\mathcal{A}(G)$-module, the inclusion is also surjective: $V^{\min}_1 = E_1$.

Being an inductive limit, $E_1 = F_1^\omega$ is an ultrabornological space, and $V^{\min}_1$ is webbed (see the reference in the proof of Proposition 4.6). We conclude from the open mapping theorem that $V^{\min}_1 = E_1$ also as topological vector spaces.

With Lemma 5.2 we now have a diagram of topological vector spaces

$$
\begin{array}{cccccc}
0 & \to & V^{\min}_1 & \to & V^{\min} & \to & V^{\min}_2 & \to & 0 \\
0 & \to & E_1 & \to & E & \to & E_2 & \to & 0 \\
\end{array}
$$

where the vertical arrow in the middle signifies the continuous inclusion $V^{\min} = \mathcal{A}(G)V \subset E$, and where the rows are exact. The five-lemma implies $V^{\min} = E$ as a vector space, and as in the article we conclude from [DS79] that this is then a topological identity.

Finally, for (2) we recall from Corollary 3.5 that $(E^{\infty})^\omega = E^\omega$. The Casselman-Wallach smooth globalization theorem asserts the existence of a Banach globalization $F$ of $V$ such that $F^\infty = E^\infty$ and therefore $F^\omega = E^\omega$. In particular, $E^\omega$ is $\mathcal{A}(G)$-tempered by Proposition 2. Now (1) applies.
Acknowledgement. The authors wish to thank Helge Glöckner for pointing out the discussed mistakes.

References


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