A time–dependent FEM-BEM coupling method for fluid–structure interaction in 3d

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Abstract

We consider the well-posedness and a priori error estimates of a 3d FEM-BEM coupling method for fluid-structure interaction in the time domain. For an elastic body immersed in a fluid, the exterior linear wave equation for the fluid is reduced to an integral equation on the boundary involving the Poincaré-Steklov operator. The resulting problem is solved using a Galerkin boundary element method in the time domain, coupled to a finite element method for the Lamé equation inside the elastic body. Based on ideas from the time-independent coupling formulation, we obtain an a priori error estimate and discuss the implementation of the proposed method. Numerical experiments illustrate the performance of our scheme for model problems.

Keywords: Fluid-structure interaction; FEM-BEM coupling; space-time methods; a priori error estimate; wave equation.

1. Introduction

Coupled finite and boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of elliptic interface and contact problems, particularly in unbounded domains [19, 29]. On the other hand, much of the current interest in boundary element procedures focuses on hyperbolic problems in the time domain, both on Galerkin methods and convolution quadrature [2, 9, 14, 25, 28].

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To compute the scattering of time-dependent waves by a bounded, penetrable obstacle, the coupling of time domain finite elements (FEM) and boundary elements (BEM) becomes relevant. The recent mathematical analysis of FEM-BEM coupling in the time domain was initiated in [1], coupling discontinuous finite elements to Galerkin boundary elements for the 3d wave equation. A general analysis of the coupling between different discretizations for acoustic wave equations was provided in [6], with a focus on convolution quadrature. Since then, FEM-BEM coupling for convolution quadrature methods has been applied in a variety of applications in 2 dimensions, such as fluid-structure and fluidthermoelastic problems, as well as nonlinear elastic problems involving piezoelectric scatterers [22, 23, 24, 27]. For time-dependent Galerkin methods and their application to 3d problems, on the other hand, much less is known. In addition to [1], previous related work includes the energy-based formulations investigated by Aimi and collaborators for wave-wave coupling in 3d multidomains and layered media [3, 4].

In this article we study a simple space-time Galerkin FEM-BEM coupling method for fluid-structure interaction, describing the transient scattering of waves in an inviscid homogeneous fluid by an elastic obstacle. Based on ideas from time–independent coupling formulations [7, 10] and the analysis in the frequency domain [22], we present a basic a priori error estimate for a space-time Galerkin approximation in anisotropic Sobolev spaces [5]. We discuss in detail the numerical implemention of our proposed coupling method in 3d. Numerical experiments for model problems illustrate the performance of the scheme.

To describe the results of this article in more detail, recall the equations for an elastic body submersed in a fluid. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded Lipschitz domain, and \( \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega} \). The elastic deformation \( u \) in \( \Omega \) is described by the Lamé operator \( \Delta^* u = \mu \Delta u + (\lambda + \mu) \nabla (\text{div} u) \), with Lamé constants \( \mu \geq 0 \) and \( \lambda \), such that \( 3\lambda + 2\mu \geq 0 \). The deformation is coupled to the wave equation in \( \Omega^c \), leading to the coupled interface problem

\[
\frac{\partial^2 v}{\partial t^2} - c^2 \Delta v = 0, \quad (x, t) \in \Omega^c \times (0, \infty), \tag{1a}
\]

\[
\rho_1 \frac{\partial^2 u}{\partial t^2} - \Delta^* u = 0, \quad (x, t) \in \Omega \times (0, \infty), \tag{1b}
\]

\[
\rho_2 \dot{\sigma}(u) \cdot n + \frac{\partial \nu}{\partial t} n = -\frac{\partial \nu^{inc}}{\partial t} n \quad \text{on } \Gamma \times (0, \infty), \tag{1c}
\]

\[
\frac{\partial u}{\partial t} n + \frac{\partial \nu}{\partial n} = -\frac{\partial \nu^{inc}}{\partial n} \quad \text{on } \Gamma \times (0, \infty), \tag{1d}
\]

\[
v(x, 0) = \dot{v}(x, 0) = 0 \quad \text{in } \Omega^c, \tag{1e}
\]

\[
u(x, 0) = \dot{u}(x, 0) = 0 \quad \text{in } \Omega. \tag{1f}
\]

Here \( n \) denotes the outward-pointing unit normal vector on \( \partial \Omega \), and we choose units in which \( c = \rho_1 = \rho_2 = 1 \). The stress is given in terms of the deformation as
\[ \sigma(u) = (\lambda \text{div} u) I + 2 \mu \varepsilon(u), \quad \varepsilon(u) = \frac{1}{2}((\nabla u) + (\nabla u)^T), \] with \( I \) the identity matrix.

To solve this interface problem numerically, we use the Poincaré-Steklov operator for the exterior wave equation to reformulate it as a coupled domain / boundary integral equation in \( \Omega \) and \( \Gamma \). The Poincaré-Steklov operator is expressed in terms of layer potentials for the wave equation, as known for time-independent symmetric FEM-BEM coupling methods. The resulting space-time weak formulation is approximated using finite elements in \( \Omega \) and Galerkin boundary elements on \( \Gamma \), based on tensor products of piecewise polynomial functions on a quasi-uniform mesh in space and a uniform mesh in time. Our a priori estimates assure convergence. We discuss a numerical implementation in detail and study the numerical performance of the method.

The article is organized as follows: Section 2 reformulates the coupled problem (1) as a domain / boundary integral equation in \( \Omega \) and \( \Gamma \) and discusses its discretization and well-posedness. It provides the basis for the derivation of an a priori error estimate in Section 3. Numerical examples and related algorithmic considerations are the content of Section 4. Two appendices discuss boundary integral operators for the wave equation and the appropriate space-time anisotropic Sobolev spaces, as well as detailed algorithmic aspects of the proposed scheme.

Notation: To simplify notation, we will write \( f \lesssim g \), if there exists a constant \( C > 0 \) independent of the arguments of the functions \( f \) and \( g \) such that \( f \leq C g \). We will write \( f \lesssim_{\sigma} g \), if \( C \) may depend on \( \sigma \). We also denote time derivatives by a dot, e.g. \( \dot{v} = \frac{\partial}{\partial t} v \).

## 2. Weak formulation and FEM-BEM coupling

Recall that the fluid-structure interaction problem (1) is well-posed [11]:

**Theorem 1.** Let \( s \geq 0 \) and assume that \( v^{\text{inc}} \in H^{s+\sigma}(\mathbb{R}^+,H^{\frac{1}{2}}(\Gamma)) \), \( \partial_n v^{\text{inc}} \in H^{s+\sigma}(\mathbb{R}^+,H^{-\frac{1}{2}}(\Gamma)) \). Then the system (1) admits a unique solution \( (u,v) \in H^1(\Omega^c) \times H^{s}(\Gamma) \), which depends continuously on the data.

In this section we reformulate problem (1) as a coupled domain / boundary integral equation and propose a finite element / boundary element coupling method for its numerical solution.

The key ingredient to reduce equation (1a) in \( \Omega^c \) to \( \Gamma \) is the retarded Poincaré-Steklov operator \( \mathcal{S} \) on \( \Gamma \), defined as \( \mathcal{S}v|_{\Gamma} = \partial_n v^{\text{inc}} \) for a solution \( v \) of (1a). Using \( \mathcal{S} \), equation (1d) becomes:

\[ -\frac{\partial u}{\partial t} n - \mathcal{S}v|_{\Gamma} = \frac{\partial v^{\text{inc}}}{\partial n} \quad \text{on } \Gamma. \]

To compute \( \mathcal{S} \) we use the following formulation in terms of boundary integral operators [19], as reviewed in Appendix A:

\[ \mathcal{S}v|_{\Gamma} = Wv|_{\Gamma} + (K^T - \frac{1}{2})V^{-1}(K - \frac{1}{2})v|_{\Gamma}. \]
In terms of $\phi = v|_\Gamma$ and an auxiliary variable $\lambda = V^{-1}(K - \frac{1}{2})\phi$, (2) becomes

$$-\frac{\partial n}{\partial t}n - W\phi + (K^T - \frac{1}{2})\lambda = \frac{\partial v_{inc}}{\partial n}.$$  

Problem (1) is therefore equivalent to the following system on $\Omega$ and $\Gamma$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta X_u = 0, \quad (x, t) \in \Omega \times (0, \infty),$$  

$$(\tilde{\sigma}(u) \cdot n + \frac{\partial n}{\partial t}n = -\frac{\partial v_{inc}}{\partial n}n \quad \text{on} \Gamma \times (0, \infty),$$  

$$-\frac{\partial n}{\partial t}n - W\phi + (K^T - \frac{1}{2})\lambda = \frac{\partial v_{inc}}{\partial n} \quad \text{on} \Gamma \times (0, \infty),$$  

$$(\frac{1}{2} - K)\phi + V\lambda = 0 \quad \text{on} \Gamma \times (0, \infty),$$  

$$\phi(x, 0) = \phi(x, 0) = \lambda(x, 0) = 0 \quad \text{on} \Gamma,$$  

$$u(x, 0) = \bar{u}(x, 0) = 0 \quad \text{in} \Omega.$$  

The solution $v$ in $\Omega^c$ is recovered from the representation formula $v = D\phi - S\lambda$, using the single and double layer potentials from Appendix A.

To derive the weak formulation of (3), recall Betti’s formula, with $\sigma > 0$:

$$(\tilde{\sigma}(u) \cdot n, w|_\Gamma)_{\Gamma \times \mathbb{R}^3, \sigma} = (\tilde{\sigma}(u), \varepsilon(w))_{\Omega \times \mathbb{R}^3} + (\Delta X_u, w)_{\Omega \times \mathbb{R}^3}.$$  

For a smooth solution $(u, \phi, \lambda)$ of (3), Betti’s formula, (3a) and (3b) lead to

$$(\tilde{\sigma}(u), \varepsilon(\tilde{w}))_{\Omega \times \mathbb{R}^3} + (\tilde{u}, \tilde{w})_{\Omega \times \mathbb{R}^3} + (\phi, \tilde{w}|_\Gamma \cdot n)_{\Gamma \times \mathbb{R}^3} + (\tilde{\phi}, \tilde{w}|_\Gamma \cdot n)_{\Gamma \times \mathbb{R}^3} = -(\tilde{v}_{inc}, \tilde{w}|_\Gamma \cdot n)_{\Omega \times \mathbb{R}^3}.$$  

The first two terms on the left hand side define a bilinear form

$$a(u, \tilde{w}) := (\tilde{\sigma}(u), \varepsilon(\tilde{w}))_{\Omega \times \mathbb{R}^3} + (\tilde{u}, \tilde{w})_{\Omega \times \mathbb{R}^3}.$$  

Adding (4) and the weak formulations of (3c) and (3d), one obtains a weak formulation of (3) in $X := H_0^1(\mathbb{R}^3, H^1(\Omega))^3 \times H^1_0(\mathbb{R}^3, H^{1/2}(\Gamma)) \times H^1_0(\mathbb{R}^3, H^{-1/2}(\Gamma))$: Find $(u, \phi, \lambda) \in \tilde{X}$ such that for all $(w, v, m) \in \tilde{X}$

$$a(u, \tilde{w}) + (\phi, \tilde{w}|_\Gamma \cdot n)_{\Gamma \times \mathbb{R}^3} - (\tilde{u}, \tilde{w})_{\Gamma \times \mathbb{R}^3} - (W\phi, \tilde{w})_{\Gamma \times \mathbb{R}^3}$$  

$$-((\frac{1}{2} - K^T)\lambda, \tilde{w})_{\Gamma \times \mathbb{R}^3} + ((\phi, \tilde{w}|_\Gamma \cdot n)_{\Gamma \times \mathbb{R}^3} + (V\lambda, \tilde{w})_{\Gamma \times \mathbb{R}^3}$$  

$$= -(\tilde{v}_{inc}, \tilde{w}|_\Gamma \cdot n)_{\Omega \times \mathbb{R}^3} + (\tilde{\phi}, \tilde{w}|_\Gamma \cdot n)_{\Omega \times \mathbb{R}^3}.$$  

Let $Z_h(\Delta t) = V_h^{1,2}(\Omega)^3 \subset H^2_0(\mathbb{R}^3, H^1(\Omega))^3$, $Y_h(\Delta t) = V_h^{1,2} \subset H^1_0(\mathbb{R}^3, H^{1/2}(\Gamma))$, $X_h(\Delta t) = V_h^{0,1} \subset H^1_0(\mathbb{R}^3, H^{-1/2}(\Gamma))$ be the conforming discretization spaces from Appendix A in $\Omega$, resp. $\Gamma$, based on tensor products of piecewise polynomial functions on a quasi-uniform mesh in space and a uniform mesh in time. Let $\tilde{X}_h(\Delta t) := Z_h(\Delta t) \times Y_h(\Delta t) \times X_h(\Delta t)$.

Then the discrete formulation reads:

1. \[ \frac{\partial u}{\partial t} - \Delta u = 0, \quad (x, t) \in \Omega \times (0, \infty), \]
2. \[ (\tilde{\sigma}(u) \cdot n + \frac{\partial n}{\partial t}n, w|_\Gamma)_{\Gamma \times \mathbb{R}^3} = (\tilde{\sigma}(u), \varepsilon(w))_{\Omega \times \mathbb{R}^3} + (\Delta u, w)_{\Omega \times \mathbb{R}^3}, \]
3. \[ (\frac{1}{2} \nabla \cdot \tilde{u}, w|_\Gamma)_{\Gamma \times \mathbb{R}^3} = 0, \]
4. \[ \tilde{u}(x, 0) = \tilde{u}(x, 0) = 0 \quad \text{in} \Omega. \]

The solution $u$ in $\Omega^c$ is recovered from the representation formula $u = D\phi - S\lambda$, using the single and double layer potentials from Appendix A.
Find \((u, \phi, \lambda) \in \tilde{X}_{h, (\Delta t)}\) such that for all \((w, \dot{w}, m) \in \tilde{X}_{h, (\Delta t)}\)

\[
a(u, \dot{w}) + \left\langle \hat{\phi}, w \right\rangle_{\Gamma} - \left\langle \hat{u}, n, \dot{w} \right\rangle_{\Gamma} - (W \phi, \dot{w})_{\Gamma} + \left( T \right) \phi, \dot{m} \right\rangle_{\Gamma} + (V \lambda, \dot{m})_{\Gamma} = 0 \tag{6}
\]

In order to prove the well-posedness of the discrete formulation, we show the equivalence to a coercive formulation.

**Proposition 2.** Let \((u, \phi, \lambda) \in \tilde{X}_{h, (\Delta t)}\) be a solution to the weak formulation (6). Then with

\[
v = D\phi - S\lambda, \tag{7}
\]

\((u, v) \in Z_{h, (\Delta t)} \times H^{1}_{0}(\mathbb{R}^{+}, H^{1}(\mathbb{R}^{3} \setminus \Gamma))\) satisfies for all \((w, \dot{w}, m) \in \tilde{X}_{h, (\Delta t)}\):

\[
a(u, w) + \left\langle \hat{\phi}, w \right\rangle_{\Gamma} - \left\langle \hat{u}, n, w \right\rangle_{\Gamma} = 0 \tag{8a}
\]

\[-\Delta v + \dot{v} = 0 \quad \text{in} \quad \mathbb{R}^{3} \setminus \Gamma \tag{8b}
\]

\[
[v]_{\Gamma} \in Y^{0}_{h, (\Delta t)}, \quad [\partial_{n} v] \in X^{0}_{h, (\Delta t)} \tag{8c}
\]

\[
-\left\langle \hat{u}, n, w \right\rangle_{\Gamma} - \left\langle \partial_{n}^{+} v, w \right\rangle_{\Gamma} + \left( \frac{\partial v}{\partial m}, w \right)_{\Gamma} \tag{8d}
\]

\[
\left\langle v^{+}, m \right\rangle_{\Gamma} = 0 \quad \forall m \in X^{0}_{h, (\Delta t)} \tag{8e}
\]

where \([v]_{\Gamma}, \left[ \partial_{n} v \right]\) denote the jump of \(v\), resp. \(\partial_{n} v\) across \(\Gamma\).

Conversely, if \((u, \phi, \lambda) = (u, [v]_{\Gamma}, [\partial_{n} v]) \in \tilde{X}_{h, (\Delta t)}\) satisfies (8) then the weak formulation (6) and (7) hold.

**Proof.** Let \((u, \phi, \lambda) \in \tilde{X}_{h, (\Delta t)}\) fulfill the weak formulation (6). Setting \(v = D\phi - S\lambda\), the wave equation (8b) holds outside \(\Gamma\). Going onto the boundary with the jump relations

\[
v^{+} |_{\Gamma} = (D\phi)^{+} |_{\Gamma} - (S\lambda)^{+} |_{\Gamma} = (\frac{1}{2}I + K)\phi - V\lambda, \tag{8a}
\]

\[
v^{-} |_{\Gamma} = (D\phi)^{-} |_{\Gamma} - (S\lambda)^{-} |_{\Gamma} = (-\frac{1}{2}I + K)\phi - V\lambda, \tag{8b}
\]

we obtain

\[
[v]_{\Gamma} = v^{+} |_{\Gamma} - v^{-} |_{\Gamma} = \phi \in Y_{h, (\Delta t)}, \tag{9}
\]

and therefore the first assertion in (8c). Now inserting it in (6) yields (8a).

The jump relations give for \(v\) in (7):

\[
\partial_{n}^{+} v = \partial_{n}^{+} (D\phi) - \partial_{n}^{+} (S\lambda) = W\phi - K\lambda + \frac{1}{2}I\lambda, \tag{8a}
\]

\[
\partial_{n}^{-} v = \partial_{n}^{-} (D\phi) - \partial_{n}^{-} (S\lambda) = W\phi - K\lambda - \frac{1}{2}I\lambda. \tag{8b}
\]

Hence

\[
[\partial_{n} v] = \partial_{n}^{+} v - \partial_{n}^{-} v = \lambda, \tag{8c}
\]

and thus \([\partial_{n} v] \in X_{h, (\Delta t)}\), which establishes (8c). From (6) with

\[
-\partial_{n}^{+} v = -W\phi + K\lambda - \frac{1}{2}I\lambda, \tag{8d}
\]

\[
-\partial_{n}^{+} v = -W\phi + K\lambda + \frac{1}{2}I\lambda, \tag{8e}
\]

5
(8d) holds. Similarly for
\[-v|_{\partial \Omega} = (\frac{1}{2} I - K) \phi + V \lambda ,\]
(8e) holds. Altogether (8) holds.

To show the converse direction, we define \((u, \phi, \lambda) := (u[\Gamma], [\partial_n v], [\partial_n v]) \in Z_{h,(\Delta t)} \times Y_{h,(\Delta t)} \times X_{h,(\Delta t)}\), where \(u\) and \(v\) fulfill (8). Since \(v\) satisfies the wave equation (8b), we get (7) from the representation formula:
\[v = D[v|_{\Gamma}] - S[\partial_n v] = D\phi - S\lambda .\]
Combining (8d) with the jump relations for \(\partial_n^* v\) and \(v|_{\Gamma}\), as well as setting \(\phi\) in the equation (8a), we obtain (6).

**Proposition 3.** Let
\[\tilde{Z}_{h,(\Delta t)} = \{v \in H^1(\mathbb{R}^3; \mathbb{R}^3) : [v|_{\Gamma}] \in Y_{h,(\Delta t)} , (v|_{\Gamma}, v|_{\Gamma}, v|_{\Gamma}, v|_{\Gamma}) = 0 \forall m \in X_{h,(\Delta t)}\} .\]
Then Problem (8) is equivalent to:
Find \((u, v) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)}\) such that
\[A((u, v), (w, w)) = f((w, w)) \quad \forall (w, w) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)} ,\]
where
\[A((u, v), (w, w)) := (\tilde{\sigma}(u), \varepsilon(w))_{\Omega \times \mathbb{R}^3, \sigma} + (\tilde{\sigma}(u), \varepsilon(w))_{\Omega \times \mathbb{R}^3} + (\tilde{\sigma}(v), \varepsilon(w))_{\Omega \times \mathbb{R}^3} + (\tilde{\sigma}(v), \varepsilon(w))_{\Omega \times \mathbb{R}^3} + (\tilde{\sigma}(w), \varepsilon(v))_{\Omega \times \mathbb{R}^3, \sigma}\]
and
\[f((w, w)) := -(\delta^{inc}(w|_{\Gamma}) - n)_{\Gamma \times \mathbb{R}^3, \sigma} + \frac{\partial_n}{\partial_n} (\delta^{inc}(w|_{\Gamma}))_{\Gamma \times \mathbb{R}^3, \sigma} .\]
**Proof.** First, we assume that (8) holds with \((u, \phi, \lambda) \in Z_{h,(\Delta t)} \times Y_{h,(\Delta t)} \times X_{h,(\Delta t)}\). Since (8c) and (8e) hold, we know that \((u, v) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)}\). Now for all \(w \in \tilde{Z}_{h,(\Delta t)}\) using the second relation in (8c) and \(\tilde{\sigma}(\partial_n v), w|_{\Gamma} = 0\), Green’s formula and (8b) lead to
\[-(\partial_n^* v, [w|_{\Gamma}])_{\Gamma \times \mathbb{R}^3, \sigma} = (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} - (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} \]
\[= (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} - (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} + (\tilde{\sigma}(v), w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} \]
\[= (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} - (\partial_n^* v, w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} \]
\[= (\nabla v, \nabla w)_{\Omega \times \mathbb{R}^3, \sigma} + (\Delta v, w)_{\Omega \times \mathbb{R}^3, \sigma} + (\nabla v, \nabla w)_{\Omega \times \mathbb{R}^3, \sigma} + (\Delta v, w)_{\Omega \times \mathbb{R}^3, \sigma} \]
\[= (\nabla v, \nabla w)_{\Omega \times \mathbb{R}^3, \sigma} + (\tilde{\sigma}(v), w|_{\Gamma})_{\Gamma \times \mathbb{R}^3, \sigma} .\]
Therefore testing (8d) with \([w|_{\Gamma}]\) for \(w \in \tilde{Z}_{h,(\Delta t)}\),
\[-(u|_{\Gamma} \cdot n + \partial_n^* v + \partial_n v^{inc}, [w|_{\Gamma}])_{\Gamma \times \mathbb{R}^3, \sigma} = 0 ,\]
we get for all $w \in \tilde{Z}_{h,\cdot}(\Delta t)$

$$-(\bar{u}|_{\Gamma} \cdot n, [w|_{\Gamma}])_{\Gamma \times \Omega} + (\nabla v, \nabla w)_{\Omega \times \Omega} - (\bar{v}, w)_{\Gamma \times \Omega} = \langle \partial_n v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega, \sigma}. \tag{11}$$

Adding up (11) and (8a) yields $\mathcal{A}((u, v), (w, w)) = f((w, w))$.

Conversely, assume (10) holds. Then (11) still holds. Testing with a function $w \in \tilde{Z}_{h,\cdot}(\Delta t)$, with compact support in $\mathbb{R}^3 \setminus \Gamma$, yields

$$(\bar{v}, w)_{\mathbb{R}^3 \times \Omega} + (\nabla v, \nabla w)_{\mathbb{R}^3 \times \Omega} - (\bar{u}|_{\Gamma} \cdot n, [w|_{\Gamma}])_{\Gamma \times \Omega} = \langle \partial_n v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega, \sigma}.$$Using integration by parts:

$$\langle \partial_n^+ v, w|_{\Gamma} \rangle_{\Gamma \times \Omega} - (\Delta v, w)_{\mathbb{R}^3 \times \Omega} = \langle \partial_n^+ v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega, \sigma}.$$Since $\nu$ satisfies the wave equation on the support of $w$ in $\mathbb{R}^3 \setminus \Gamma$,

$$(-\Delta v + \bar{v}, w)_{\mathbb{R}^3 \times \Omega} = 0.$$Equation (8b) follows. Next

$$\langle \partial_n^- v, w|_{\Gamma} \rangle_{\Gamma \times \Omega} - (\partial_n^+ v, w|_{\Gamma} \rangle_{\Gamma \times \Omega} = \langle \partial_n v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega, \sigma}.$$Hence, for all $w \in \tilde{Z}_{h,\Delta}$

$$-\langle \partial_n^+ v + \bar{u}|_{\Gamma} \cdot n + \partial_n v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega} = 0.$$Choosing $\langle [w|_{\Gamma}] \rangle = 0$, we get the second relation in (8c) because

$$\langle [\partial_n v], w|_{\Gamma} \rangle_{\Gamma \times \Omega} = 0.$$Second, choose $w \in \tilde{Z}_{h,\cdot}(\Delta t)$ such that $w|_{\Gamma} = 0$ yields

$$-\langle \partial_n v + \bar{u}|_{\Gamma} \cdot n + \partial_n v^{inc}, [w|_{\Gamma}] \rangle_{\Gamma \times \Omega}.$$and hence (8d), since $[w|_{\Gamma}] \in Y_{h,\cdot}(\Delta t)$. From the definition of $\tilde{Z}_{h,\cdot}(\Delta t)$ we already get (8e) and (8c).

Finally, we obtain the equation (8a) from the remaining terms in (10).

An analogous assertion to Propositions 2 and 3 holds for the continuous problem, instead of the finite element discretization.

We now aim to prove coercivity of the problem for $(u, v)$ in a suitable norm, defined as:

$$\| (u, v) \|^2 = \langle \bar{\sigma}(u), \varepsilon(u) \rangle_{\Omega \times \Omega} + \langle \bar{u}, \bar{v} \rangle_{\Gamma \times \Omega} + (\nabla v, \nabla v)_{\Omega \times \Omega} - (\bar{v}, w)_{\Gamma \times \Omega}.$$
Note that
\[ A((u, v), (\hat{u}, \hat{v})) = (\hat{\sigma}(u), \varepsilon(\hat{u}))_{\Omega \times \mathbb{R}^+} + (\hat{u}, \hat{u})_{\Omega \times \mathbb{R}^+} + (\nabla u, \nabla \hat{v})_{\mathbb{R}^3 \times \mathbb{R}^+} \]
\[ + (\hat{v}, \hat{v})_{\mathbb{R}^3 \times \mathbb{R}^+} + \varepsilon(\hat{u}) \cdot n \cdot [\varepsilon(\hat{v})]_{\mathbb{R}^3 \times \mathbb{R}^+} - \frac{\partial [\gamma^2 v]}{\partial t}, \hat{u}[\Gamma \cdot n]_{\mathbb{R}^3 \times \mathbb{R}^+} \]
\[ = (\hat{\sigma}(u), \varepsilon(\hat{u}))_{\Omega \times \mathbb{R}^+} + (\hat{u}, \hat{u})_{\Omega \times \mathbb{R}^+} + (\nabla u, \nabla \hat{v})_{\mathbb{R}^3 \times \mathbb{R}^+} + (\hat{v}, \hat{v})_{\mathbb{R}^3 \times \mathbb{R}^+} \]
\[ = \int_0^\infty \int_\Omega \hat{\sigma}(u) : \varepsilon(\hat{u}) dx e^{-2\sigma t} dt + \int_0^\infty \int_\Omega \hat{u} u dx e^{-2\sigma t} dt \]
\[ + \int_0^\infty \int_{\mathbb{R}^3 \setminus \Omega} \nabla v \nabla \hat{v} dx e^{-2\sigma t} dt + \int_0^\infty \int_{\mathbb{R}^3 \setminus \Omega} \hat{v} \hat{v} dx e^{-2\sigma t} dt \]
\[ = \frac{1}{2} \partial_t (\int_\Omega \hat{\sigma}(u) : \varepsilon(u) dx) e^{-2\sigma t} dt + \frac{1}{2} \partial_t (\int_{\mathbb{R}^3 \setminus \Omega} \hat{u} u dx) e^{-2\sigma t} dt \]
\[ + \frac{1}{2} \partial_t (\int_{\mathbb{R}^3 \setminus \Omega} \nabla v \nabla \hat{v} dx) e^{-2\sigma t} dt + \frac{1}{2} \partial_t (\int_{\mathbb{R}^3 \setminus \Omega} \hat{v} \hat{v} dx) e^{-2\sigma t} dt . \]

Using integration by parts in time, the zero initial condition and \(\sigma > 0\), we obtain the coercivity estimate:
\[ A((u, v), (\hat{u}, \hat{v})) = \frac{1}{2} \int_0^\infty (\int_\Omega \hat{\sigma}(u) : \varepsilon(\hat{u}) dx) \partial_t (e^{-2\sigma t}) dt - \frac{1}{2} \int_0^\infty (\int_{\mathbb{R}^3 \setminus \Omega} \hat{u} u dx) \partial_t (e^{-2\sigma t}) dt \]
\[ - \frac{1}{2} \int_0^\infty (\int_{\mathbb{R}^3 \setminus \Omega} \nabla v \nabla \hat{v} dx) \partial_t (e^{-2\sigma t}) dt - \frac{1}{2} \int_0^\infty (\int_{\mathbb{R}^3 \setminus \Omega} \hat{v} \hat{v} dx) \partial_t (e^{-2\sigma t}) dt \]
\[ = \sigma \| (u, v) \|^2 \gtrsim \| u \|^2_{0, 1, \Omega} + \| v \|^2_{0, 1, \Omega\varepsilon} . \]

This implies, in particular, uniqueness of solutions to (5) and (6).

**3. A priori error estimate**

We state an a priori error estimate:

**Theorem 4.** Let \((u, \phi, \lambda) \in \bar{X}\) satisfy (5) and \((u_h, \phi_h, \lambda_h) \in \bar{X}_h,\Delta t\) satisfy (6). Then
\[ \| u - u_h \|^2_{0, 1, \Omega} + \| \phi - \phi_h \|^2_{0, 1, 2, \Gamma} + \| \lambda - \lambda_h \|^2_{0, -1, 2, \Gamma} \lesssim \]
\[ \inf_{(w_h, \psi_h, \mu_h) \in \bar{X}_h,\Delta t} \left( 1 + \frac{1}{(\Delta t)^2} \right) \| u - w_h \|^2_{1, 1, \Omega} \]
\[ + \| \phi - \psi_h \|^2_{1, 1, 2, \Gamma} + \left( 1 + \frac{1}{(\Delta t)^2} \right) \| \lambda - \mu_h \|^2_{1, -1, 2, \Gamma} . \]

**Proof.** Let \((u, \phi, \lambda) \in \bar{X}\) satisfy (5) and \((u_h, \phi_h, \lambda_h) \in \bar{X}_h,\Delta t\) satisfy (6). Then for all \((\hat{w}, \hat{\phi}, \hat{\lambda}) \in \bar{X}_h,\Delta t\)
\[ \| u - u_h \|^2_{0, 1, \Omega} + \| \phi - \phi_h \|^2_{0, 1, 2, \Gamma} + \| \lambda - \lambda_h \|^2_{0, -1, 2, \Gamma} \]
\[ \lesssim \| u - \hat{w} \|^2_{0, 1, \Omega} + \| \phi - \hat{\phi} \|^2_{0, 1, 1, \Omega} + \| \lambda - \hat{\lambda} \|^2_{0, 1, 2, \Gamma} + \| \phi - \phi_h \|^2_{0, 1, 2, \Gamma} \]
\[ + \| \lambda - \lambda_h \|^2_{0, -1, 2, \Gamma} + \| \hat{\lambda} - \lambda_h \|^2_{0, -1, 2, \Gamma} . \]
We therefore focus on estimates for $\| \mathbf{w} - \mathbf{u}_h \|_{0,1,\Omega}^2 + \| \phi - \phi_h \|_{0,1/2,\Gamma}^2 + \| \lambda - \lambda_h \|_{0,-1/2,\Gamma}^2$.

Now using coercivity and Galerkin orthogonality

$$
\| \mathbf{w} - \mathbf{u}_h \|_{0,1,\Omega}^2 + \| \phi - \phi_h \|_{0,1/2,\Gamma}^2 + \| \lambda - \lambda_h \|_{0,-1/2,\Gamma}^2 \\
\leq \| \mathbf{w} - \mathbf{u}_h \|_{0,1,\Omega}^2 + \| \hat{\mathbf{r}} - v_h \|_{0,1,\Gamma}^2
$$

By definition of the single and double layer potentials $S$ and $D$, the functions $v = D\phi - S\lambda$, $v_h = D\phi_h - S\lambda_h$ and $\hat{\mathbf{r}} := D\hat{\phi} - S\hat{\lambda}$ all satisfy the wave equation in $\mathbb{R}^3 \setminus \Gamma$. Further, from Proposition 2, $[v_h]_{\Gamma}, [\hat{\mathbf{r}}]_{\Gamma} \in Y^0_{h,\Delta t}$, $[\partial_t v_h], [\partial_t \hat{\mathbf{r}}] \in X^0_{h,\Delta t}$ and $\langle \gamma^{-1} v_h, m \rangle_{\Gamma \times \mathbb{R}^+} = 0$, $\langle \gamma^{-1} \hat{\mathbf{r}}, m \rangle_{\Gamma \times \mathbb{R}^+} = 0$. Using the definition of $A$ and Green’s theorem, we find

$$
\| \mathbf{w} - \mathbf{u}_h \|_{0,1,\Omega}^2 + \| \phi - \phi_h \|_{0,1/2,\Gamma}^2 + \| \lambda - \lambda_h \|_{0,-1/2,\Gamma}^2 \\
\leq \int_0^\infty e^{-2\sigma t} \left\{ \int_{\Omega} \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) \cdot \varepsilon(\partial_t (\tilde{\mathbf{w}} - \mathbf{u}_h)) dx + \int_{\Omega} \partial_t^2(\tilde{\mathbf{w}} - \mathbf{u}) \partial_t(\tilde{\mathbf{w}} - \mathbf{u}_h) dx \\
+ \int_{\mathbb{R}^3 \setminus \Gamma} \nabla (\tilde{\mathbf{r}} - v) \nabla (\partial_t (\tilde{\mathbf{r}} - v_h)) dx + \int_{\mathbb{R}^3 \setminus \Gamma} \partial_t^2 (\tilde{\mathbf{r}} - v) \partial_t (\tilde{\mathbf{r}} - v_h) dx \\
- \int_{\Gamma} (\partial_t (\tilde{\mathbf{w}} - \mathbf{u}) |_{\Gamma} \cdot n [\partial_t (\tilde{\mathbf{r}} - v_h)]_{\Gamma} ds_x + \int_{\Gamma} [\partial_t (\tilde{\mathbf{r}} - v)]_{\Gamma} [\partial_t (\tilde{\mathbf{w}} - \mathbf{u}_h)]_{\Gamma} ds_x \right\} dt
$$

$$
= \int_0^\infty e^{-2\sigma t} \left\{ \int_{\Omega} \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) : \varepsilon (\partial_t (\tilde{\mathbf{w}} - \mathbf{u}_h)) dx + \int_{\Omega} \partial_t^2 (\tilde{\mathbf{w}} - \mathbf{u}) \partial_t (\tilde{\mathbf{w}} - \mathbf{u}_h) dx \\
+ \int_{\Gamma} \partial_t^2 (\tilde{\mathbf{r}} - v) [\partial_t (\tilde{\mathbf{r}} - v_h)]_{\Gamma} ds_x - \int_{\Gamma} \partial_t (\tilde{\mathbf{w}} - \mathbf{u}) [\partial_t (\tilde{\mathbf{r}} - v_h)]_{\Gamma} ds_x \\
+ \int_{\Gamma} [\partial_t (\tilde{\mathbf{r}} - v)]_{\Gamma} [\partial_t (\tilde{\mathbf{w}} - \mathbf{u}_h)]_{\Gamma} ds_x \right\} dt
$$
\[
\begin{align*}
&= \int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\omega} - \mathbf{u}) : \varepsilon(\partial_t(\tilde{\omega} - \mathbf{u}_h)) \, dx + \int_\Omega \partial_t^2(\tilde{\omega} - \mathbf{u}) \partial_t(\tilde{\omega} - \mathbf{u}_h) \, dx \\
&\quad + \int_\Gamma (W(\tilde{\phi} - \phi) - (K^T - \frac{1}{2} I)(\tilde{\lambda} - \lambda))(\partial_t(\tilde{\phi} - \phi_h)) \, ds_x \\
&\quad - \int_\Gamma \partial_t(\tilde{\omega} - \mathbf{u}) |_\Gamma \cdot n(\partial_t(\tilde{\phi} - \phi_h)) \, ds_x + \int_\Gamma (\partial_t(\tilde{\phi} - \phi)) \partial_t(\tilde{\omega} - \mathbf{u}_h) |_\Gamma \cdot n \, ds_x \right\} \, dt .
\end{align*}
\]

We estimate the separate terms. Using Young’s inequality we have for \( \epsilon > 0 \)
\[
\int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\omega} - \mathbf{u}) : \varepsilon(\partial_t(\tilde{\omega} - \mathbf{u}_h)) \, dx + \int_\Omega \partial_t^2(\tilde{\omega} - \mathbf{u}) \partial_t(\tilde{\omega} - \mathbf{u}_h) \, dx \right\} \, dt \\
\leq \sigma \frac{1}{(\Delta t)^2} \| \tilde{\omega} - \mathbf{u} \|_{0,1,\Omega}^2 + \epsilon \| \tilde{\omega} - \mathbf{u} \|_{0,1,\Omega}^2 .
\]
Next we estimate
\[
\int_0^\infty e^{-2\sigma t} \int_\Gamma (W(\tilde{\phi} - \phi) - (K^T - \frac{1}{2} I)(\tilde{\lambda} - \lambda))(\partial_t(\tilde{\phi} - \phi_h)) \, ds_x \, dt \\
\leq \left( \| W(\tilde{\phi} - \phi) \|_{0,-1/2,\Gamma} + \| (K^T - \frac{1}{2} I)(\tilde{\lambda} - \lambda) \|_{0,-1/2,\Gamma} \right) \| \tilde{\phi} - \phi_h \|_{1,1/2,\Gamma} .
\]
Using the inverse estimate as in (3.182) in [18]
\[
\| \tilde{\phi} \|_{1,1/2,\Gamma} \leq \frac{1}{\Delta t} \| \tilde{\phi} \|_{0,1/2,\Gamma} ,
\]
we further estimate with the mapping properties of the integral operators
\[
\leq \sigma \left( \| W(\tilde{\phi} - \phi) \|_{0,-1/2,\Gamma} + \| (K^T - \frac{1}{2} I)(\tilde{\lambda} - \lambda) \|_{0,-1/2,\Gamma} \right) \frac{1}{\Delta t} \| \tilde{\phi} - \phi_h \|_{0,1/2,\Gamma} \\
\leq \frac{1}{\sigma (\Delta t)^2} \| \tilde{\phi} - \phi \|_{0,1/2,\Gamma}^2 + \frac{1}{\sigma (\Delta t)^2} \| \tilde{\lambda} - \lambda \|_{1,-1/2,\Gamma}^2 + \epsilon \| \tilde{\phi} - \phi_h \|_{0,1/2,\Gamma}^2 .
\]
For the fourth term, we get:
\[
\int_0^\infty e^{-2\sigma t} \int_\Gamma \partial_t(\tilde{\omega} - \mathbf{u}) |_\Gamma \cdot n(\partial_t(\tilde{\phi} - \phi_h)) \, ds_x \, dt \\
\leq \sigma \| \partial_t(\tilde{\omega} - \mathbf{u}) |_\Gamma \cdot n \|_{0,-1/2,\Gamma} \| \partial_t(\tilde{\phi} - \phi_h) \|_{0,1/2,\Gamma} \\
\leq \| \tilde{\omega} - \mathbf{u} \|_{1,1/2,\Gamma} \| \tilde{\phi} - \phi_h \|_{1,1/2,\Gamma} \leq \frac{1}{\sigma (\Delta t)^2} \| \tilde{\omega} - \mathbf{u} \|_{1,1,\Omega}^2 + \epsilon \| \tilde{\phi} - \phi_h \|_{0,1/2,\Gamma}^2 .
\]
For the last term, analogously the trace theorem and the inverse estimate show:
\[
\int_0^\infty e^{-2\sigma t} \int_\Gamma \partial_t(\tilde{\phi} - \phi) \partial_t(\tilde{\omega} - \mathbf{u}_h) |_\Gamma \cdot n \, ds_x \, dt \leq \sigma \| \partial_t(\tilde{\phi} - \phi) \|_{0,1/2,\Gamma} \| \partial_t(\tilde{\omega} - \mathbf{u}_h) |_\Gamma \cdot n \|_{0,-1/2,\Gamma} \\
\leq \frac{1}{\Delta t} \| \tilde{\phi} - \phi \|_{1,1/2,\Gamma} \| \tilde{\omega} - \mathbf{u}_h \|_{0,1,\Omega} \leq \frac{1}{\sigma (\Delta t)^2} \| \tilde{\phi} - \phi \|_{1,1/2,\Gamma}^2 + \epsilon \| \tilde{\omega} - \mathbf{u}_h \|_{0,1,\Omega}^2 .
\]
Moving the terms with positive powers of $\epsilon$ to the left hand side, we conclude
\[
\left\| \bar{w} - u_h \right\|_{0,1,\Omega}^2 + \left\| \bar{\phi} - \phi_h \right\|_{0,1/2,\Gamma}^2 + \left\| \bar{\lambda} - \lambda_h \right\|_{0,-1/2,\Gamma}^2
\leq_\sigma \frac{1}{(\Delta t)^2} \left\| \bar{w} - u \right\|_{0,1,\Omega}^2 + \frac{1}{(\Delta t)^2} \left\| \bar{\phi} - \phi \right\|_{1,1/2,\Gamma}^2 + \frac{1}{(\Delta t)^2} \left\| \bar{\lambda} - \lambda \right\|_{1,-1/2,\Gamma}^2
+ \frac{1}{(\Delta t)^2} \left\| \bar{\phi} - \phi \right\|_{1,1/2,\Gamma}^2 + \frac{1}{(\Delta t)^2} \left\| \bar{\lambda} - \lambda \right\|_{1,-1/2,\Gamma}^2 + \left\| \bar{w} - u \right\|_{1,1,\Omega}^2 + \frac{1}{(\Delta t)^2} \left\| \bar{\phi} - \phi \right\|_{1,1/2,\Gamma}^2,
\]
and therefore
\[
\left\| u - u_h \right\|_{0,1,\Omega}^2 + \left\| \phi - \phi_h \right\|_{0,1/2,\Gamma}^2 + \left\| \lambda - \lambda_h \right\|_{0,-1/2,\Gamma}^2
\leq_\sigma \left(1 + \frac{1}{(\Delta t)^2}\right) \left( \left\| \bar{w} - u \right\|_{1,1,\Omega}^2 + \left\| \bar{\phi} - \phi \right\|_{1,1/2,\Gamma}^2 + \left\| \bar{\lambda} - \lambda \right\|_{1,-1/2,\Gamma}^2 \right).
\]
This proves the assertion. \hfill \Box

4. Numerical results

In the numerical experiments for Problem (1), the variational formulation (5) is solved by choosing as ansatz function in the interior domain $\Omega$
\[
\begin{align*}
\mathbf{u}_{h,\Delta t}(x,t) &= \sum_{k=1}^{N_s} \sum_{i=1}^{N_z} \sum_{v=1}^{N_s} u_{k,i,v}(t) e_v \eta^i_k(x), 
\end{align*}
\]
where $\{\eta^i_k\}$ denotes the basis of piecewise linear hat functions for $V^1_h(\Omega)$ and
\[
\beta^m_{\Delta \tau}(t) = (\Delta t)^{-1}((t - t_{m-1})\gamma^m_{\Delta \tau}(t) - (t - t_{m+1})\gamma^{m+1}_{\Delta \tau}(t)) .
\]
Here $\gamma^m_{\Delta \tau}(t) = H(t - t_{m-1}) - H(t - t_m)$, with $H$ the Heaviside function. The test functions are given by
\[
\begin{align*}
\mathbf{w}_{h,\Delta t} &= \eta^i_k(x) \gamma^m_{\Delta \tau}(t) e_v,
\end{align*}
\]
for $l = 1, \ldots, N_s$, $n = 1, \ldots, N_t$ and $\mu = 1, 2, 3$.

The ansatz functions on $\Gamma$ are taken as
\[
\begin{align*}
\phi_{h,\Delta t}(x,t) &= \sum_{m=1}^{N_s} \sum_{i=1}^{N_s} \phi^m_i(t) \beta^m_{\Delta \tau}(t) \xi^i_h(x),
\end{align*}
\]
and
\[
\begin{align*}
\lambda_{h,\Delta t}(x,t) &= \sum_{m=1}^{N_s} \sum_{i=1}^{N_s} \lambda^m_i(t) \beta^m_{\Delta \tau}(t) \xi^i_h(x),
\end{align*}
\]
where $\{\xi^i_h\}$ denotes the basis of piecewise linear hat functions for $V^1_h$. The corresponding test functions are chosen as
\[
\begin{align*}
\tilde{w}_{h,\Delta t} &= \gamma^m_{\Delta \tau}(t) \xi^i_h(x), \\
m_{h,\Delta t} &= \gamma^m_{\Delta \tau}(t) \xi^i_h(x),
\end{align*}
\]
for $1 \leq n \leq N_t$ and $1 \leq j \leq N_s'$. 

11
As shown in Appendix B, this discretization of (5) leads to a time-stepping scheme, which solves a system of the following structure in each time step \( t \geq 3 \):

\[
\begin{pmatrix}
\frac{(\Delta t)}{2}A & + \frac{1}{(\Delta t)}M & [0, n_x RI]^T \\
[0, -RI h_x] & W^0 & K T^0 - \frac{1}{2} I \\
0 & -K^0 - \frac{1}{2} I & V^0
\end{pmatrix}
\begin{pmatrix}
\varphi^n \\
\lambda^n \\
u^n
\end{pmatrix} = \begin{pmatrix}
H^n + H^{n-1} - A \frac{(\Delta t)}{2} u^{n-1} + M \frac{1}{(\Delta t)} u^{n-2} + n_x RI \varphi^{n-1} \\
C^n + C^{n-1} + RI h_x u^{n-1} + \sum_{m=1}^{n-1} W^{n-m} \varphi^m - \sum_{m=1}^{n-1} K T^{n-m} \lambda^m + \frac{1}{2} \frac{(\Delta t)}{2} I \lambda^{n-1} \\
\sum_{m=1}^{n-1} K^{n-m} \varphi^m - \frac{1}{2} I \varphi^{n-1} - \sum_{m=1}^{n-1} V^{n-m} \lambda^m
\end{pmatrix}.
\]

The system in the first two time steps is similar, see Appendix B. We solve this system repeatedly until our desired time step \( N_t \) is reached.

**Example.** Let \( \Omega = [-1,1]^3 \). Using the discretization described above, we compute the solutions to the discrete system (5) for times \([0,4]\) and data \( v^{inc} \) corresponding to the exact solution

\[
u(x,t) = \begin{pmatrix}
(s(t - \frac{\pi}{2}))^{\frac{5}{2}} (H(-1 + t - \frac{\pi}{2}) - H(-3 + t - \frac{\pi}{2})) \\
0
\end{pmatrix}, \quad (18)
\]

\[
v(x,t) = \frac{|x-t|}{2\Delta t} (1 + \cos(\frac{\pi(x-t)}{\Delta t}))H(0.9 - ||x|-t|).
\]

We use uniform discretizations by tetrahedra as depicted in Figure 1 and a time step \( \Delta t \) such that \( \frac{\Delta t}{h} \approx 0.1414 \). We denote the number of grid points on an edge of the cube by \( n + 1 \). The finest mesh is then given by \( n = 24 \) and consists of 69120 tetrahedra, corresponding to \( \Delta t = 0.01667 \). The convergence of the numerical solution to the exact solution is studied as the mesh is refined, and we measure the error in terms of the \( L^2 \)-norm in space, resp. space-time.

Figure 2 shows the first component \( u_1 \) of the numerical and exact solutions at the corner point \( x_0 = (-1,-1,-1) \) as a function of time for \( n = 2, 4, 8 \). The behaviour of the solution in \( \Omega \), resp. \( \Omega^c \), is illustrated in Figure 3, which plots the \( L^2 \)-norms of \( u \), resp. of \( v \), as a function of time for \( n = 2, 8, 24 \). The \( L^2 \) error as a function of time is shown in Figure 4, corresponding to the numerical solutions depicted in Figure 3. All plots show excellent approximation of the simple behavior of the solution for short times and a monotonous convergence on the whole time interval. The absolute value of the error, however, turns out to be larger for the solution in \( \Omega^c \). Note that error does not seem to grow exponentially with time, as expected for a variational method. Figure 5 quantifies the convergence of the numerical solutions up to \( n = 24 \). It depicts the \( L^2 \)-norm of the error in space and time in terms of the space-time degrees of freedom, for the solution \( u \) in \( \Omega \), \( v \) in \( \Omega^c \), and the total error. While the error of the exterior solution \( v \) dominates, as noted above for Figure 4, the observed convergence rate of approximately 0.17 is naturally the same in the interior and the exterior. It corresponds to a convergence rate of 0.68 in \( h \).
Figure 1: Meshes for $[-1,1]^3$ with 27 ($n=2$), 125 ($n=4$) and 729 ($n=8$) nodes

Figure 2: Numerical and exact solutions $u_1(t, x_0)$, $x_0 = (-1, -1, -1)$

Figure 3: L2-norm of the numerical solution in $\Omega$, resp. $\Omega^c$
Figure 4: $L_2$-error of the numerical solution in $\Omega$, resp. $\Omega^c$

Figure 5: $L^2$-error in space and time in terms of the degrees of freedom
Appendix A. Integral operators and finite–boundary elements

Let \( \Gamma \) be the boundary of a polyhedral domain \( \Omega \) in \( \mathbb{R}^3 \), consisting of curved, polygonal boundary faces. In \( \mathbb{R}^3 \setminus \Gamma \), a solution \( v \) to the homogeneous wave equation may be represented in terms of the jump of the Dirichlet and Neumann data across \( \Gamma \): 
\[
v = D\phi - S\lambda.
\]

Here
\[
S\lambda(x,t) = \int_0^\infty \int_\Gamma G(t-\tau,x,y) \lambda(y,\tau) \, dy \, d\tau,
\]
\[
D\phi(x,t) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_y}(t-\tau,x,y) \phi(y,\tau) \, dy \, d\tau,
\]
are the single, resp. double layer potential for the wave equation defined from the fundamental solution 
\[
G(\tau,x,y) = \frac{\delta(t-|x-y|)}{4\pi|t-x-y|}.
\]

The coupling method presented in this article relies on the resulting boundary integral operators:
\[
V\phi(x,t) = \int_0^\infty \int_\Gamma G(t-\tau,x,y) \phi(y,\tau) \, dy \, d\tau,
\]
\[
K\phi(x,t) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_y}(t-\tau,x,y) \phi(y,\tau) \, dy \, d\tau,
\]
\[
K^T\phi(x,t) = K\phi(x,t) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_x}(t-\tau,x,y) \phi(y,\tau) \, dy \, d\tau,
\]
\[
W\phi(x,t) = \int_0^\infty \int_\Gamma \frac{\partial^2 G}{\partial n_x\partial n_y}(t-\tau,x,y) \phi(y,\tau) \, dy \, d\tau.
\]

They are studied in space-time anisotropic Sobolev spaces \( H^r_{\sigma}(\mathbb{R}^+; \tilde{H}^s(\Gamma)) \) [20].

To define an explicit scale of Sobolev norms, fix a partition of unity subordinate to a covering of \( \tilde{\Gamma} \) by open sets \( B_i \) and diffeomorphisms \( \phi_i \) mapping each \( B_i \) into the unit cube \( c \subset \mathbb{R}^n \). They induce a family of norms from \( \mathbb{R}^d \):
\[
\|u\|_{s,\Gamma} = \left( \sum_{i=1}^n \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}\{((\alpha_i u) \circ \phi_i^{-1})(\xi)\}|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

\( \mathcal{F} \) here denotes the Fourier transform. The norms for different \( \omega \in \mathbb{C} \setminus \{0\} \) are equivalent.

Weighted Sobolev spaces in time for \( r \in \mathbb{R} \) and \( \sigma > 0 \): are defined as
\[
H^r_{\sigma}(\mathbb{R}^+) = \{ u \in \mathcal{D}'_+ : e^{-\sigma t}u \in \mathcal{S}' \text{ and } \|u\|_{H^r_{\sigma}(\mathbb{R}^+)} < \infty \}.
\]

Here, \( \mathcal{D}'_+ \) denotes the space of distributions on \( \mathbb{R} \) with support in \( [0, \infty) \), and \( \mathcal{S}' \) the subspace of tempered distributions. The Sobolev spaces are Hilbert spaces endowed with the norm
\[
\|u\|_{H^r_{\sigma}(\mathbb{R}^+)} = \left( \int_{-\infty}^{+\infty} \omega^{2r} |\hat{u}(\omega)|^2 \, d\omega \right)^{\frac{1}{2}}.
\]

The scale of space-time anisotropic Sobolev spaces combines the Sobolev norms in space and time:
Definition 5. For \( r, s \in \mathbb{R} \) and \( \sigma > 0 \) define

\[
H^r_{\sigma}(\mathbb{R}^+, H^s(\Gamma)) = \{ u \in D^r_{\sigma}(H^s(\Gamma)) : e^{-\sigma t}u \in S^r_{\sigma}(H^s(\Gamma)) \text{ and } \|u\|_{r, s, \Gamma} < \infty \}.
\]

\( D^r_{\sigma}(E) \) denotes the space of distributions on \( \mathbb{R} \) with support in \([0, \infty)\), taking values in \( E = H^s(\Gamma), \dot{H}^s(\Gamma) \), and \( S^r_{\sigma}(E) \) the subspace of tempered distributions. The Sobolev spaces are Hilbert spaces endowed with the norm

\[
\|u\|_{r, s, \Gamma} = \left( \int_{-\infty}^{+\infty + i\sigma} \|\omega\|^{2r} \|\hat{u}(\omega)\|^{2s}_{r, s, \Gamma} \, d\omega \right)^{\frac{1}{2}}.
\]

When \(|s| \leq 1\) one can show that the spaces are independent of the choice of \( \alpha_i \) and \( \phi_i \).

Space-time anisotropic Sobolev spaces in a bounded domain \( \Omega \subset \mathbb{R}^d \) are defined analogously, starting from the standard Sobolev spaces \( H^s(\Omega) \) with norm \( \|u\|_{s, \Omega} = \inf_{v} \left( \int_{\Omega} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}v(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \) Here the infimum extends over all \( v \in H^s(\mathbb{R}^d) \) with \( v|_{\Omega} = u \).

We state the mapping properties of the boundary integral operators, see e.g. \cite{9, 20}:

Theorem 6. The following operators are continuous for \( r \in \mathbb{R}, \sigma > 0 \):

\[
V : H^{r+1}_{\sigma}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \to H^r_{\sigma}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)),
\]

\[
K^t : H^{r+1}_{\sigma}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \to H^r_{\sigma}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)),
\]

\[
K : H^{r+1}_{\sigma}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \to H^r_{\sigma}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)),
\]

\[
W : H^{r+1}_{\sigma}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \to H^r_{\sigma}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)).
\]

By a fundamental observation of Bamberger and Ha-Duong \cite{5}, \( V \partial_t \) satisfies a coercivity estimate in the norm of \( H^0_{\sigma}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \): \( \|\psi\|^2_{0, -\frac{1}{2}, \Gamma} \lesssim \langle V\psi, \partial_t \psi \rangle \).

From the mapping properties of Theorem 6 one also has the continuity of the bilinear form associated to \( V \partial_t \) in a bigger norm: \( \langle V\psi, \partial_t \psi \rangle \leq \|\psi\|^2_{1, -\frac{1}{2}, \Gamma} \). Similar estimates hold for \( W \partial_t : \|\phi\|^2_{0, \frac{1}{2}, \Gamma} \lesssim \langle W\phi, \partial_t \phi \rangle \leq \|\phi\|^2_{1, \frac{1}{2}, \Gamma} \). Proofs and further information may be found in \cite{20}.

We consider space-time discretizations based on tensor products of piecewise polynomials:

For simplicity, we assume that \( \Omega \) is a polygonal domain, with a quasi-uniform triangulation \( T_\Omega = \{T_1, \ldots, T_N\} \) by \( N \) tetrahedra. The induced quasi-uniform triangulation of the boundary \( \Gamma \), \( T_\Gamma = \{\Delta_1, \ldots, \Delta_N\} \), should consist of closed triangular faces \( \Delta_i \), such that each \( \Delta_i \) is a face of one \( T_j \) and at most one face of \( T_j \) is contained in \( \Gamma \).

We consider the space \( V^0_h(\Omega) \) of piecewise polynomial functions on \( T_\Omega \) of degree \( q \geq 0 \) in space (continuous if \( q \geq 1 \)). \( V^0_h \) consists of traces on \( \Gamma \) of functions in \( V^0_h(\Omega) \). The parameter \( h \) denotes the maximal diameter of an element in \( T_\Omega \).
We choose an equidistant temporal mesh on the positive half-line \( T_T = \{ [0, t_1), [t_1, t_2), \ldots \} \), where \( t_n = n(\Delta t) \). \( V^q_{\Delta t} \) is the space of piecewise polynomial functions of degree \( p \) on \( T_T \) (continuous and vanishing at \( t = 0 \) if \( p \geq 1 \), \( C^1 \) if \( p \geq 2 \)).

The space-time approximation spaces are given by tensor products of the approximation spaces in space and time, \( V^q_h \) and \( V^p_{\Delta t} \), associated to the space-time meshes \( T_{\Omega,T} = T_{\Omega} \times T_T \), respectively \( T_{S,T} = T_{S} \times T_T \). We write
\[
V^{p,q}_{\Delta t,h}(\Omega) := V^p_{\Delta t}(\Omega), \; V^{p,q}_{\Delta t,h} := V^p_{\Delta t} \otimes V^q_h.
\]

We write
\[
\sum_{m=1}^{N_t} \sum_{i=1}^{N_j} \varphi^m_i = \sum_{m=1}^{N_t} \sum_{i=1}^{N_j} \varphi^m_i,
\]

where
\[
Y^{n-m}(x,y) = (2(\Delta t))^{-1}(|x - y|^2 - 2|x - y|(n - m + 1)(\Delta t) + ((n - m + 1)(\Delta t))^2)\chi_{E_{n-m}} + (2(\Delta t))^{-1}(|x - y|^2 - 2|x - y|(n - m - 2)(\Delta t) + ((n - m - 2)(\Delta t))^2)\chi_{E_{n-m-2}} + (2(\Delta t))^{-1}(-2|x - y|^2 + 2|x - y|(n - m - 1)(\Delta t) + (n - m)(\Delta t)) - ((n - m - 1)(\Delta t))^2 + ((n - m)(\Delta t))^2)\chi_{E_{n-m-1}}.
\]

Here, for \( l \in \mathbb{N}_0 \) we define the light cone \( E_l = \{ (x, y) \in \Gamma \times \Gamma : t_l \leq |x - y| \leq t_{l+1} \} \subset \Gamma \times \Gamma \), and \( \chi_{E_l}(x, y) = 1 \) if \( (x, y) \in E_l \), and \( = 0 \) otherwise.

The matrix \( W^{n-m} \) is therefore a sum of integrals over the four light cones \( E_{n-m+1}, E_{n-m}, E_{n-m-1} \) and \( E_{n-m-2} \). The accurate and efficient numerical evaluation of such integrals is reviewed in [14].
We next consider the discretization of the single layer potential. For the ansatz function we choose (15) and as test function we choose (17). After some calculations, we obtain

\[
\langle V \lambda_h, \Delta t, \bar{m}_{h, \Delta t} \rangle_{\Gamma \times \mathbb{R}^+} = \int_0^\infty \int_{\Gamma} V \lambda_h, \Delta t(x, t) \cdot \bar{m}_{h, \Delta t}(x, t) ds_x dt
\]

\[
= \sum_{n=1}^{N_t} \sum_{i=1}^{N_x} \lambda_i^n \int_{E_{n-m}} \left( -(n-m+1) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|} + \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)} \right) ds_y ds_x
\]

\[
+ \int_{E_{n-m-1}} \left( (2n-m-1) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|} - 2 \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)} \right) ds_y ds_x
\]

\[
+ \int_{E_{n-m-2}} \left( -(n-m-2) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|} + \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)} \right) ds_y ds_x
\]

\[
= \sum_{n=1}^{N_t} \sum_{i=1}^{N_x} (V - m_{n-m}) \lambda_i^n = \sum_{n=1}^{N_t} (V - m_{n-m}) \lambda_i^n.
\]

We next consider the retarded adjoint double layer potential:

\[
\langle K^T \lambda_h, \Delta t, \bar{w}_{h, \Delta t} \rangle_{\Gamma \times \mathbb{R}^+} = \int_0^\infty \int_{\Gamma} K^T \lambda_h, \Delta t \bar{w}_{h, \Delta t} ds_x dt
\]

\[
= \sum_{n=1}^{N_t} \sum_{i=1}^{N_x} \lambda_i^n \int_{E_{n-m}} n_x \cdot \left( (n-m+1) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|^2} - \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x
\]

\[
+ \int_{E_{n-m-1}} n_x \cdot \left( (2n-m-1) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|^2} - 2 \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x
\]

\[
+ \int_{E_{n-m-2}} n_x \cdot \left( (n-m-2) \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi|x-y|^2} - \frac{\xi_i^\Delta_h(y) \xi_i^\Delta_h(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x
\]

\[
= \sum_{n=1}^{N_t} \sum_{i=1}^{N_x} (K^T)^{n-m} \lambda_i^n = \sum_{n=1}^{N_t} (K^T)^{n-m} \lambda_i^n.
\]
The related term for the mass matrix is given by

\[ \langle \frac{1}{2} \lambda_{h, \Delta t}, u_{h, \Delta t} \rangle_{\Gamma \times \mathbb{R}_+} = \frac{1}{2} \int_0^\infty \int_{\Gamma} \sum_{m=1}^{N_i} \sum_{i=1}^{N_s} \lambda_i^m \beta_{\Delta t}^m(t) \xi_h^i(x) \gamma_h^m(t) \xi_h^i(x) ds_x dt \]

\[ = \frac{1}{2} \sum_{m=1}^{N_i} \sum_{i=1}^{N_s} \lambda_i^m \left( \int_{\Gamma} \xi_h^i(x) \xi_h^i(x) ds_x \right) \left( \int_0^\infty \beta_{\Delta t}^m(t) \gamma_h^m(t) dt \right) \]

\[ = \frac{1}{2} \sum_{i=1}^{N_s} \lambda_i^m \left( \int_{\Gamma} \xi_h^i(x) \xi_h^i(x) ds_x \right) \frac{(\Delta t)}{2} \begin{cases} \lambda_i^1, & n = 1 \\ \lambda_i^n + \lambda_i^{n-1}, & n \geq 2 \end{cases} \]

\[ = \frac{1}{2} \sum_{i=1}^{N_s} I_{i, i} \lambda_i = \frac{1}{2} I \lambda_I, \]

where

\[ \lambda_I = \frac{(\Delta t)}{2} \begin{cases} \lambda_i^1, & n = 1 \\ \lambda_i^n + \lambda_i^{n-1}, & n \geq 2 \end{cases} . \]

Furthermore

\[ \langle K \phi_{h, \Delta t}, \dot{m} \rangle_{\Gamma \times \mathbb{R}_+} = \int_0^\infty \int_{\Gamma} K \phi_{h, \Delta t} \dot{m}_{h, \Delta t} ds_x dt \]

\[ = \sum_{m=1}^{N_i} \sum_{i=1}^{N_s} \varphi_i^m \left[ \int_{E_{n-m}} n_y^i(x - y) \left( -(n - m + 1) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi |x - y|^3} + \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} \right) ds_y ds_x \right. \]

\[ + \int_{E_{n-m-1}} n_y^i(x - y) \left( (2(n - m) - 1) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi |x - y|^3} - 2 \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} \right) ds_y ds_x \]

\[ + \int_{E_{n-m-2}} n_y^i(x - y) \left( -(n - m - 2) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi |x - y|^3} + \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} \right) ds_y ds_x \]

\[ + \sum_{m=1}^{N_i} \sum_{i=1}^{N_s} \varphi_i^m \left[ \int_{E_{n-m}} -n_y^i(x - y) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} ds_y ds_x + \int_{E_{n-m-1}} 2 n_y^i(x - y) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} ds_y ds_x \right. \]

\[ \left. + \int_{E_{n-m-2}} -n_y^i(x - y) \frac{\xi_h^i(y) \xi_h^j(x)}{4 \pi (\Delta t) |x - y|^2} ds_y ds_x \right] = \sum_{m=1}^{N_i} \sum_{i=1}^{N_s} K_{i,j}^{n-m} \varphi_i^m = \sum_{m=1}^{N_i} K^{n-m} \varphi^m \]
For the second coupling term:

\[
\langle \frac{1}{2} \phi_{h, \Delta t}, \mathbf{w}_{h, \Delta t} \rangle_{G \times \mathbb{R}_+} = \frac{1}{2} \int_0^\infty \int_G \sum_{n=1}^{N_s} \sum_{m=1}^{N_s} \phi_i^m \beta_i^n(t) \xi_i^m(t) \eta_i^n(t) ds dt
\]

\[
= \frac{1}{2} \sum_{i=1}^{N_s} \sum_{m=1}^{N_s} \phi_i^m (\int_G \xi_i^m(x) \eta_i^n(x) ds) (\int_0^\infty \beta_i^n(t) dt)
\]

\[
= \frac{1}{2} \sum_{i=1}^{N_s} \sum_{m=1}^{N_s} \phi_i^m (\int_G \xi_i^m(x) \eta_i^n(x) ds) \begin{cases} -\varphi_i^n, & n = 1 \\ -\varphi_i^n + \varphi_i^{n-1}, & n \geq 2 \end{cases},
\]

with

\[
\varphi_i = \begin{cases} -\varphi_i, & n = 1 \\ -\varphi_i^n + \varphi_i^{n-1}, & n \geq 2 \end{cases}.
\]

It remains to consider the coupling contributions. For \( j = 1, \ldots, N_s \)

\[
\langle \mathbf{u}_{h, \Delta t}, \mathbf{w}_{h, \Delta t} \rangle_{G \times \mathbb{R}_+} = \int_0^\infty \int_G \mathbf{u}_{h, \Delta t} \cdot \mathbf{w}_{h, \Delta t} ds dt
\]

\[
= \sum_{\nu=1}^3 \sum_{i=1}^{N_s} (\int_G \eta_i^\nu(x) \xi_i^\nu(x) ds) \begin{cases} u_{i,\nu}^1, & n = 1 \\ u_{i,\nu}^n - u_{i,\nu}^{n-1}, & n \geq 2 \end{cases}
\]

\[
= \sum_{\nu=1}^3 \sum_{i=1}^{N_s} (R \eta_i^\nu)_{i,\nu} \varphi_T = R \eta_i^\nu \varphi_T,
\]

with

\[
\varphi_T = \begin{cases} u_{i,\nu}^1, & n = 1 \\ u_{i,\nu}^n - u_{i,\nu}^{n-1}, & n \geq 2 \end{cases}.
\]

For the second coupling term:

\[
\langle \phi_{h, \Delta t}, \mathbf{w}_{h, \Delta t} \rangle_{G \times \mathbb{R}_+} = \int_0^\infty \int_G \phi_{h, \Delta t} \cdot n \mathbf{w}_{h, \Delta t} ds dt
\]

\[
= \sum_{i=1}^{N_s} (\int_G \xi_i^m(x) n \cdot \eta_i^m(x) ds) \begin{cases} \varphi_i^1, & n = 1 \\ \varphi_i^n - \varphi_i^{n-1}, & n \geq 2 \end{cases}
\]

\[
= \sum_{i=1}^{N_s} (n_R \eta_i^m)_{i,\nu} \varphi_T = n_R \varphi_T.
\]

with

\[
\varphi_T = \begin{cases} \varphi_i^1, & n = 1 \\ \varphi_i^n - \varphi_i^{n-1}, & n \geq 2 \end{cases}.
\]

For completeness we mention the right hand side: Set \( h = \hat{\nu}^{inc} n \) and \( g = \frac{\partial \hat{\nu}^{inc}}{\partial n} \).
We approximate the time integral by the trapezoidal rule, so that:

\[ -\langle \dot\psi_n \phi, \Phi \rangle \approx -\langle h, \Phi \rangle = -\frac{(\Delta t)}{2} \int (h^n + h^{n-1}) \eta_h^j(x) \epsilon_{\mu} ds_x =: H^n + H^{n-1} \]

\( \langle \frac{\partial \psi_n}{\partial t}, \dot{\Phi} \rangle = \langle g, \dot{\Phi} \rangle = \frac{(\Delta t)}{2} \int (g^n + g^{n-1}) \xi_h^j(x) ds_x =: G^n + G^{n-1} \),

where \( h^n = h(x, t_n) \) and \( g^n = g(x, t_n) \).

The resulting system of equations therefore becomes:

\[
Au_A + Mu_M - RIn_x u_T + n_x RI \varphi - \frac{1}{2} I \lambda I = \sum_{m=1}^{N_t} K^{m-1} \lambda^m - \frac{1}{2} I \lambda I
\]

\[
+ \frac{1}{2} I \varphi - \sum_{m=1}^{N_t} K^{m-1} \varphi^m + \sum_{m=1}^{N_t} V^{m-1} \lambda^m = H^n + H^{n-1} + G^n + G^{n-1}.
\]

The matrices \( W^k, R^k, T^k, V^k \) vanish if the index \( k \) is negative. Therefore we get for (B.1) in the first time step \((n = 1)\):

\[
\left( \frac{(\Delta t)}{2} A + \frac{1}{(\Delta t)} M \right) \begin{bmatrix} [0, n_x RI]^T \\ [0, -RIn_x] \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -W^0 \\ K^0 - \frac{1}{2} I \end{bmatrix} = \begin{bmatrix} H^1 + H^0 \\ 0 \end{bmatrix}.
\]

For the second time step \((n = 2)\) we obtain

\[
\left( \frac{(\Delta t)}{2} A + \frac{1}{(\Delta t)} M \right) \begin{bmatrix} [0, n_x RI]^T \\ [0, -RIn_x] \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -W^0 \\ K^0 - \frac{1}{2} I \end{bmatrix} = \begin{bmatrix} H^2 + H^1 - A(\frac{(\Delta t)}{2}) u^1 + M(\frac{2}{\Delta t}) u^1 + n_x RI \varphi^1 \\ G^2 + G^1 + RIn_x u_T + W^1 + K^1 \lambda^1 + \frac{(\Delta t)}{2} I \lambda^1 \end{bmatrix},
\]

using \( u^1, \varphi^1 \) and \( \lambda^1 \) from above. For later time steps \( n \geq 3 \) we conclude:

\[
\left( \frac{(\Delta t)}{2} A + \frac{1}{(\Delta t)} M \right) \begin{bmatrix} [0, n_x RI]^T \\ [0, -RIn_x] \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -W^0 \\ K^0 - \frac{1}{2} I \end{bmatrix} = \begin{bmatrix} H^n + H^{n-1} - A(\frac{(\Delta t)}{2}) u^{n-1} + M(\frac{2}{\Delta t}) u^{n-1} + n_x RI \varphi^{n-1} \\ G^n + G^{n-1} + RIn_x u_T^{n-1} + \sum_{m=1}^{n-1} W^{m-1} \varphi^{m} - \sum_{m=1}^{n-1} K^{m-1} \lambda^m + \frac{1}{2} I \lambda^{n-1} \end{bmatrix},
\]

This system is solved repeatedly until reaching time step \( N_t \geq 3 \).

References


