Magnitude Homology: Opinions & Speculations

Richard Hepworth
University of Aberdeen

I will only consider metric spaces

$MH_{k,\ell}(X)$ - magnitude homology

$BMH_k(X)$ - blurred magnitude homology

$\mathbb{G}\mathbb{M}$ = graded modules

$\mathbb{P}\mathbb{M}$ = persistence modules

<u>Opinion</u>: Magnitude homology should categorify magnitude.

<u>Finite spaces</u>: Yes!

$$|tX| = \sum_{\ell \geq 0} \sum_{k \geq 0} \sum_{\substack{x_0 \neq x_1 \neq \cdots \neq x_k \\ d(x_0,x_1)+\cdots+d(x_{k-1},x_k)=\ell}} (-1)^k e^{-t\ell} = \sum_{\ell \geq 0} \sum_{k \geq 0} (-1)^k \operatorname{rk} MH_{k,\ell}(X) e^{-t\ell}$$

$$(t \gg 0)$$

<u>Infinite spaces</u>: No! Balls!

So we must upgrade magnitude homology without sacrificing geometry.

Opinion: It's nice that $MH_{x,x}(X)$ splits things up according to "total length" ℓ, with contribution $e^{-t\ell}$ for each piece of length ℓ.

Speculation: Magnitude is a Laplace transform

$$|tX| = \mathcal{L}\{f_X\}(t) = \int_{\ell=0^-}^{\infty} f_X(\ell)\, e^{-\ell t}\, d\ell$$

for $t \gg 0$

distributional magnitude ?

Finite spaces: Yes! f_X [redacted] $= \sum_{\ell \geq 0} \sum_{k \geq 0} \sum_{\substack{x_0 \neq \cdots \neq x_k \\ d(x_0,x_1)+\cdots = \ell}} (-1)^k \delta_\ell$

(No convergence issues!)

Infinite Spaces? $\quad |t[0,1]| = 1 + \frac{1}{2}t$

$$= \int_{0^-}^{\infty} \left[\delta_0(\ell) + \tfrac{1}{2}\delta_0'(\ell)\right] e^{-\ell t}\, d\ell$$

$$= \mathcal{L}\left\{\delta_0 + \tfrac{1}{2}\delta_0'\right\}(t)$$

Opinion: $\mathbb{G}\mathbb{M}$ and $\mathbb{P}\mathbb{M}$ need upgrading too.

For X finite, $\quad |tX| = \mathcal{L}\left\{\sum_{k\geq 0}(-1)^k \sum_{\ell\geq 0} \mathrm{rk}\, MH_{k,\ell}(X)\,\delta_{\ell}\right\}(t)$

$$= \mathcal{L}\left\{\sum_{k\geq 0}(-1)^k\ \mathrm{rk}\, BMH_k(X)'\right\}(t)$$

I don't see how such rank functions can ever produce something like $\delta_0 + \frac{1}{2}\delta_0'$.

<u>Speculation</u>: There is a category $\mathbb{LM}$ of "<u>length modules</u>" — some sort of linear object living over $[0,\infty)$ — with these properties:

- $L \otimes M$
- $L \longmapsto tL$
- "rank distributions" $\mathrm{rk}\, L : [0,\infty) \longrightarrow \mathbb{R}$
- $\mathrm{rk}(L \otimes M) = \mathrm{rk}\, L * \mathrm{rk}\, M$
- $\mathrm{rk}(tL)(\ell) = \frac{1}{t}\, \mathrm{rk}(L)(\ell/t)$

And <u>rich enough</u> that $\mathrm{rk}\, L$ can take values like $\delta_x^{(n)}$.

The <u>magnitude of L</u> is then

$$|tL| = \mathcal{L}\{\mathrm{rk}\, L\}(t) = \int_{\ell=0}^{\infty} \mathrm{rk}\, L(\ell)\, e^{-t\ell}\, d\ell$$

Speculation: There is $\mathcal{MH}_*(X)$ taking values in $\mathbb{LM}$ such that

$$|_{\mathbb{E}} X| = \sum_{k=0}^{\infty} (-1)^k \, |_{\mathbb{E}} \mathcal{MH}_k(X)|$$

and

$$\mathcal{MH}_*(X) = \operatorname*{colim}_{\substack{F \subseteq X \\ \text{finite}}} \mathcal{MH}_*(F)$$

<u>Speculation</u>: $X \subseteq \mathbb{R}^{n,\text{odd}}$ cpt cvx sm dom

$$|tX| \sim t^n \frac{\mathrm{vol}_n(X)}{n!\,\omega_n} + t^{n-1}\frac{\mathrm{vol}_{n-1}(\partial X)}{(n-1)!\,\omega_{n-1}} + \cdots$$

$$= \mathcal{L}\left\{ \frac{\mathrm{vol}_n(X)}{n!\,\omega_n}\delta_0^{(n)} + \frac{\mathrm{vol}_{n-1}(\partial X)}{(n-1)!\,\omega_{n-1}}\delta_0^{(n-1)} + \cdots \right\}(t)$$

Observe that $t \to \infty$ corresponds to germ of $\mathcal{M}\mathcal{H}_*(X)$ at 0.
So $\mathcal{M}\mathcal{H}_*(X)$, near 0, should decompose as contributions for $\mathrm{vol}_n(X)$, $\mathrm{vol}_{n-1}(\partial X), \ldots,$ and these parts will be categorifications of $\mathrm{vol}_n(X)$, $\mathrm{vol}_{n-1}(\partial X), \ldots$

Summary opinion:

Before finding the true magnitude homology, we must first categorify $\mathrm{vol}_n(X)$, $\mathrm{vol}_{n-1}(\partial X)$, ... Before doing that, we must find the categories they inhabit. These must have a notion of dimension taking all values in $[0,\infty)$.

values of volumes

And on the side:

Can we gain anything by regarding $|tX|$ as a Laplace transform?

Ideas:

- Von Neumann dimension
- $\mathrm{Rep}(S_t)$, $t \in \mathbb{R}$
- $\mathbb{GM} \longleftarrow \mathbb{PM} \longleftarrow \cdots$ continue the sequence?

$$\begin{array}{ccc} \mathbb{GM} & \longleftarrow & \mathbb{PM} \\ \mathrm{gr}\,L & \longmapsfrom & L \end{array}$$