Stefania: The Euler equations as differential inclusion (De Lellis, Szekelyhidi) 1

Scheffer '93

Shnirelman '97

(EE) $\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p - f = 0 \\ \operatorname{div}(v) = 0 \end{cases}$

(LWP) (classical, Majda-Bertozzi '02)

(GWP) $n=2$

Def: $v \in L^2_{loc}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ is a weak sol. of (EE) if

$$\iint \left(v\, \partial_t \phi + \langle v \otimes v, \nabla \phi \rangle + \phi f \right) dx\, dt = 0$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n),\ \operatorname{div}(\phi) = 0$$

Thm 4.1 Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ open bdd

$\exists\, (v,p) \in L^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n \times \mathbb{R})$ weak sol. of (EE) with $f=0$

s.t. • $|v| = 1$ a.e. in Ω

• $v = 0$, $p = 0$ a.e. in Ω^c

Moreover $\exists\, ((v_k, p_k, f_k))_{k \in \mathbb{N}} \subseteq C_c^\infty(\Omega; \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$

• (v_k, p_k, f_k) solves (EE)

• $\|v_k\|_{L^\infty} + \|p_k\|_{L^\infty}$ unif. bounded

• $f_k \to 0$ in H^{-1}

• $(v_k, p_k) \to (v, p)$ in L^q $\forall q < \infty$.

Consider

(1) $\sum_{i=1}^{m} A_i \partial_i z = 0$, $A_i \in \mathbb{R}^d$, where $z : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^d$

$z(x) \in K \subseteq \mathbb{R}^d$ a.e. in Ω

Plane wave solutions : $z^a(x) = a\,h(\xi \cdot x)$, $h : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^d$, $\xi \in \mathbb{R}^m$

Wave cone : $\Lambda := \{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} : \sum_{i=1}^{m} \xi_i A_i a = 0 \}$

$a \in \Lambda \iff z^a$ solves (1) $\forall h$.

<u>Lemma 2.1:</u> $(v, u, q) \in L^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R})$ is a sol. of $\partial_t v + \operatorname{div}(u) + \nabla q = 0$ s.t. $u = v \otimes v - \frac{1}{n}|v|^2 I_n$ a.e.

$\iff$ (v, p) is weak sol. of (EE) , $p = q - \frac{1}{n}|v|^2$

$K := \{ (v, u) \in \mathbb{R}^n \times \mathcal{S}_0^n : u = v \otimes v - \frac{1}{n} \mathbb{1}_n : |v| = 1 \}$

$\mathcal{U} = \operatorname{int}(\operatorname{co}(K) \times [-1, 1])$

$z = (v, u, q)$ solves (EDI) and $z(x,t) \in \overline{\mathcal{U}}$ a.e.

$$\Lambda = \{ \underset{(v,u,q)}{\overset{\shortparallel}{a}} \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} : \det \underbrace{\begin{pmatrix} u + q I_n & v \\ v^T & 0 \end{pmatrix}}_{U} = 0 \}$$

$\operatorname{div}_{(x,t)} U = 0$

<u>Prop 2.2:</u> Let $a_0 \in \Lambda$, $v_0 \neq 0$, $\sigma = \{ t a + (1-t) a : t \in [0,1] \}$.

Then $\forall \varepsilon > 0 \ \exists z$ classical sol. of (EDI) s.t.

- $\operatorname{supp} z \subseteq B(0,1)$
- $z(x,t) \in B(\sigma, \varepsilon)$
- $\exists \alpha > 0 : \int |v| \, dx\, dt \geq \alpha |v_0|$

$$\mathbb{X}_0 := \{ z \in C^\infty(\mathbb{R}^n \times \mathbb{R}; \dots) \quad \operatorname{supp} z \subset \Omega, \ z \text{ solves (Edif)} \quad z(x,t) \in U \text{ that} \}$$ (2)

with w^* L^∞ topo, $\mathbb{X} = \overline{\mathbb{X}_0}$

Proof of Thm 4.1: $z_1 \equiv 0 \in \mathbb{X}_0$, $\eta_1 \in (0, \frac{1}{2})$, ρ mollif. kernel

$$\| z_1 - z_1 * \rho_{\eta_1} \|_{L^2(\Omega)} < \frac{1}{2}$$

Lemma 4.6: $\exists \beta > 0 \ \forall z \in \mathbb{X}_0 \ \exists (z_k)_{k \in \mathbb{N}} \subseteq \mathbb{X}_0 : z_k \xrightarrow{*} z$ in L^∞,

$$\| z_k \|^2_{L^2(\Omega)} \geq \| v \|^2_{L^2(\Omega)} + \beta \left(|\Omega| - \| v \|^2_{L^2} \right)^2$$

$\exists (z_k^{(1)})_{k \in \mathbb{N}} \subseteq \mathbb{X}_0$ s.t. $\exists N_1$ s.t. $\| (z_{N_1}^{(1)} - z_1) * \rho_{\eta_1} \|_{L^2(\Omega)} < \frac{1}{2}$

Call $z_2 = z_{N_1}^{(1)}$, choose $\eta_2 \in (0, \frac{1}{4})$ s.t.

$$\| z_2 - z_2 * \eta_{\eta_2} \|_{L^2(\Omega)} < \frac{1}{4}$$

$\exists (z_k^{(2)})_{k \in \mathbb{N}} \quad \dots \quad$ etc.

Show that (z_k) is unif bounded in L^∞: $\| z_k \|_{L^\infty} \leq \sup_{z \in U} (\underbrace{|v|}_{=1} + |u| + \underbrace{|q|}_{\leq 1})$

$\leq 3 + \frac{1}{4}$

$\Rightarrow \exists z \in L^\infty : z_k \xrightarrow{*} z$ in L^∞ $\Rightarrow z \in \mathbb{X}$

z_k converges strongly in L^2.

$$\| z_k - z \|_{L^2} \leq \underbrace{\| z_k - z_k * \rho_{\eta_k} \|_{L^2}}_{\to 0} + \underbrace{\| z_k * \rho_{\eta_k} - z * \rho_{\eta_k} \|_{L^2}}_{\to 0} + \underbrace{\| z * \rho_{\eta_k} - z \|_{L^2}}_{\to 0}$$

$\Rightarrow \| v \|_{L^2} \geq \| v \|_{L^2} + \beta (|\Omega| - \| v \|^2)^2$

$\Rightarrow |v| = \mathbb{1}_\Omega \quad \square$