Nash 2.6. 3.11.17

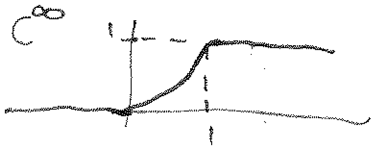
From last time: Reduced Problem to geometric ODE : $\dot{w}(t) = \mathcal{L}(w(t))\, \dot{h}(t)$
$w(t_0) = w_0$

Problem with the fact that for $w, \dot{h} \in C^k \Rightarrow \mathcal{L}(w)\, \dot{h} \in C^{k-2}$.
And $(1 - \mathcal{L}(w))^{-1} \dot{h}$ loses derivs.
Thus we need to extend the standard ODE theory.

Reg. problem: (R)

$$\dot{w}(t) = \mathcal{L}(S_{t^{-1}} w(t))\, \dot{h}(t)$$
$$W(t_0) = w_0$$
$$h(t) = S_{t^{-1}}\left[\psi(t-t_0)\dot{h} + \int_{t_0}^{t} 2d\left(S_{\tau^{-1}} w(\tau) - w(\tau)\right) \odot d\dot{w}(\tau)\, \psi(t-t_0)\, d\tau\right]$$

where $\psi \in C^\infty$



S_ε - lin. smoothing op., $\varepsilon \in (0,1]$

(5)
(a) $S_\varepsilon T(x)$ smooth in ε and x ; $\forall T$.
(b) $\|D^r S_\varepsilon T\|_0 \lesssim \varepsilon^{s-r} \|T\|_s$, $r \geq s$
(c) $\|D^r S_\varepsilon' T\|_0 \lesssim \varepsilon^{s-r-1} \|T\|_s$;
(d) $\|D^r (T - S_\varepsilon T)\|_0 \lesssim \varepsilon^{s-r} \|T\|_s$; $r \geq s$.

Details: deLellis.

- Strategy to global existence + conv. to a limit:
  (1) Local existence
  (2) Local $\rightsquigarrow$ Global.

Prop.: There exist t_0 and δ: If $\|h\|_3 \leq \delta$ then $\exists$ a solution of (R) on $[t_0, \infty)$ s.t. for suff. small $\varepsilon > 0$; $\|w(t) - w_0\|_3 + t^{-1}\|w(t) - w_0\|_4 \leq \varepsilon$,

(α) • $t^4 \|\dot{h}(t)\|_0 + \|\dot{h}(t)\|_4 \leq 1$, • $t^4 \|\dot{w}(t)\|_0 + \|\dot{w}(t)\|_4 \leq C$;
• $\exists \tilde{\delta} : [t_0, \infty) \to \mathbb{R}^+$ with $\tilde{\delta} \searrow 0$: $\|w(t) - w(s)\|_3 \leq \tilde{\delta}(s)$, $\forall t \geq s$

• (Loc. $\exists$) $\Rightarrow$ (Global $\exists$):

Prop: (a priori estimates) If bounds $\hat{(1)}, \hat{(2)}$ are satisfied w const. 2ε and 2 then $\hat{(1)}, \hat{(2)}$ hold.

This implies global existence provided local existence.

Continuation: Assume $J = [t_0, t_1]$ s.t. $\exists$ soln. w satisfying the bounds. Then $\exists$ extension of w to a soln. on $[t_0, t_2]$ which satisfies the first bounds with constants adjusted.

There exists a bootstrapping argument which recovers constants in the bounds.

This implies that t_2 can be iterated.

Local Existence: F.P. Iteration for (R).

$$\text{(w)} \qquad w(t) = w_0 + \int_{t_0}^{t} \mathcal{L}(S_{\tau^{-1}} w(\tau))\, \dot{h}(\tau)\, d\tau$$

$\lambda = \dot{h}$

$$= w_0 + \int_{t_0}^{t} \mathbf{W}(w(\tau), \lambda(\tau))\, d\tau$$

$$\mathcal{E}(t) = 2d(S_{t^{-1}} w(t) - w(t)) \odot d(\mathcal{L}(S_{t^{-1}} w(t))\lambda(t)) =: \mathcal{E}(w(t), \lambda(t))$$

$$(\lambda) \qquad \lambda(t) = \frac{d}{dt}\left(S_{t^{-1}}\left[\psi(t-t_0)h + \int_{t_0}^{t} \mathcal{E}(w(\tau), \lambda(\tau))\, \psi(t-\tau)\, d\tau\right]\right)$$

$$= \mu(t) - t^{-2} S'_{t^{-1}} \int_{t_0}^{t} \mathcal{E}(w(\tau), \lambda(\tau))\, \psi(t-\tau)\, d\tau + S_{t^{-1}} \int_{t_0}^{t} \mathcal{E}\psi'\, d\tau$$

One shows that $\mathcal{E}$ is loc. Lipschitz from C^4 to C^3, $\mathbf{W}$ smooth on C^4.

We can "see" that (w) and (λ) define a contraction on $(C^4$

$\rightsquigarrow$ can apply fixed point argument to (w, λ) in $X_\psi = C^0_t C$

It remains to show the a priori estimates ($\hat{\ }$) and to show that they are "self-improving".

3.

Prop. (3.6.1) delellis: Let t_0 be suff. large and $\varepsilon > 0$: if $\|u - w_0\|_2 \leq 4\varepsilon_1$ then u is a free embedding. Then $\exists \delta = \delta(t_0) > 0$ s.t. $\forall \|h\|_3 < \delta$ Any soln. of (R) on $[t_0, t_1]$ which satisfies; $\hat{(1)}, \hat{(2)}$ with const. $2\varepsilon_1 2$. Also satisfies $\hat{(1)}, \hat{(2)}$. Moreover, $\hat{(3)}$ holds. And if $t_2 = \infty$ then $\hat{(4)}$ holds.

Proof: $\hat{(1)}$ with 2ε and (d) of (S): $\|S_{t^{-1}} w - w_0\|_2 \leq 4\varepsilon \Rightarrow S_t w$ is free embedding

And $\mathcal{L}(S_{t^{-1}} w)$ smooth in W.

~~By assumption $\|w\|$~~

$$\|\mathcal{L}(S_{t^{-1}} w(t))\|_k \lesssim C(k)\left(1 + \|S_{t^{-1}} w\|_{k+2}\right) \overset{(b)\text{ of }(S)}{\lesssim} C(k)\left(1 + \|w\|_3^{t^{k-1}}\right)$$

$(\hat{3})$ is "easy" to show: $\|\mathcal{L}(S_{t^{-1}} w(t))\|_0 \leq C(k)\left(1 + \|S_{t^{-1}} w\|_2\right)$

$\lesssim \|w\|_2$.

Combining previous bounds gives:

$$\|\dot{w}\|_0 = \|\mathcal{L}(S_{t^{-1}} w)\dot{h}\|_0 \leq \|\mathcal{L}(S_{t^{-1}} w)\|_0 \|\dot{h}\|_0 \lesssim t^{-4}$$

$$\|\dot{w}\|_4 = \|\mathcal{L}(S_{t^{-1}} w)\|_4 \|\dot{h}\|_0 + \|\mathcal{L}(S_{t^{-1}} w)\|_0 \|\dot{h}\|_4$$

$$\lesssim C \quad \rightsquigarrow \quad (\hat{3}).$$

Pf. $(\hat{2})$: Recall $E(t)$; $L(t) = \int_{t_0}^t E(\tau)\varphi(t-\tau)\,d\tau$.

Then: $h(t) = S_{t^{-1}}\left(\varphi(t - t_0)h + L(t)\right)$.

Note: • $\|S_{t^{-1}} w - w_0\|_1 \overset{(d)}{\leq} C t^{-2} \|w\|_3 \underset{\hat{(1)}}{\lesssim} t^{-2}$.

• $\|\dot{w}\|_0 \lesssim t^{-3}$

These imply $\|E(t)\|_0 \lesssim t^{-5}$, from def. of $E(t)$.

Then $\|E(t)\|_3 \lesssim \|S_{t^{-1}} w - w\|_4 \|\dot{w}\|_1 + \|S_{t^{-1}} w - w\|_1 \|\dot{w}\|_4 \lesssim t^{-2}$. (**)

$$\Longrightarrow \quad \|L(t)\|_3 = \left\| \int_{t_0}^{t} E(\tau)\psi(t-\tau) \right\|_3 \lesssim \int_{t_0}^{t} \|E(\tau)\|_3 \, d\tau \lesssim t_0^{-1} \qquad (***)$$

Next, $\dot{h}(t) = \underbrace{\left(\frac{d}{dt} S_{t^{-1}}\right)\left(\psi(t-t_0)h + L(t)\right)}_{=:P(t)} + S_{t^{-1}}\left(\psi'(t-t_0)h + \dot{L}(t)\right)$

So, $\|\psi'(t-t_0) S_{t^{-1}} h\|_4 \lesssim \begin{cases} C t_0 \delta & t \in [t_0, t_0+1] \\ 0 & \text{o.w.} \end{cases}$

$$\|\dot{L}\|_0 \le \left\| \frac{d}{dt} \int_{t_0}^{t} E(\tau)\psi(t-\tau)\, d\tau \right\|_0 \le \int_{\max\{t_0, t-1\}}^{t} \|E(\tau)\|_0 \le t^{-5}$$

and $\|\dot{L}\|_3 \lesssim t^{-2}$

It is "easy" to see that $t^4 \|\dot{P}(t)\|_0 + \|P(t)\|_4 \lesssim C\|h\|_3 + \|L\|_3$
$$\lesssim C\delta + t_0^{-1}$$

Combining Above estimates:

$$t^4 \|\dot{h}\|_0 + \|\dot{h}\|_4 \le Ct^{-1} + C\delta(1 + t_0^5) + Ct_0^{-1}$$

for large t_0 need really small $\delta = \delta(t_0)$