Adaptive FE–BE Coupling for Strongly Nonlinear Transmission Problems with Friction II

Heiko Gimperlein* Ernst P. Stephan

Abstract

This article discusses the well-posedness and error analysis of the coupling of finite and boundary elements for transmission or contact problems in nonlinear elasticity. It concerns “pseudoplastic”, $p$–Laplacian-type Hencky materials with an unbounded stress–strain relation, as they arise in the modelling of ice sheets, non-Newtonian fluids or porous media. For $1 < p < 2$ the bilinear form of the boundary element method fails to be continuous in natural function spaces associated to the nonlinear operator. We propose a functional analytic framework for the numerical analysis and obtain a priori and a posteriori error estimates for Galerkin approximations to the resulting boundary/domain variational inequality. The a posteriori estimate complements recent estimates obtained for mixed finite element formulations of friction problems in linear elasticity.

1 Introduction

Let $n = 2$ or 3 and $Ω ⊂ \mathbb{R}^n$ be a bounded Lipschitz domain. We consider transmission and frictional contact problems between a nonlinear, uniformly $W^{1,p}(Ω)$–monotone operator in $Ω$ and the homogeneous Lamé equation in the exterior domain. Adaptive finite element / boundary element procedures provide an efficient and extensively investigated tool for the numerical solution when the nonlinear operator is uniformly elliptic [12]. Their analysis, however, does not apply to the above “pseudoplastic” material laws arising in the modelling of ice sheets, non-Newtonian fluids or porous media [1, 7], because for $p < 2$ the bilinear form of the boundary element method fails to be continuous on natural function spaces related to the nonlinear operator. This article provides a functional analytic framework to study the wellposedness and an error analysis of FE / BE coupling procedures in this situation.

Formulation of Problem: We consider the following contact problem for $(u, u_c) \in (W^{1,p}(Ω))^n \times (W^{1,2}_{loc}(Ω^c))^n$, where $p \in (1, \infty)$ and $\partial Ω = \Gamma_s \sqcup \Gamma_t$ is

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decomposed into two open subsets:

\[-\text{div} A'(\varepsilon(u)) = f \quad \text{in} \; \Omega, \]
\[-\mu \Delta u_c - (\lambda + \mu) \text{grad div} \; u_c = 0 \quad \text{in} \; \Omega^c, \]
\[A'(\varepsilon(u))\nu - T^* u_c = t_0 \quad \text{on} \; \partial \Omega, \]
\[u - u_c = u_0 \quad \text{on} \; \Gamma_t, \]

and a radiation condition \( u(x) = o(1), \) grad \( u(x) = \mathcal{O}(|x|^{-1}) \) resp. \( u(x) = \mathcal{O}(|x|^{-2}) \), grad \( u(x) = \mathcal{O}(|x|^{-2}) \) is satisfied for \( n = 2, 3 \) as \( |x| \to \infty \). On \( \Gamma_s \) contact conditions corresponding to Tresca friction are imposed. If \( v \) denotes the unit outer normal to \( \partial \Omega \), the conditions are given in terms of the normal and tangential components of \( u, u_n = \nu \cdot u \) and \( u_t = u - u_n \nu, \) and of the stress, \( \sigma_n(u) = -\nu A'(\varepsilon(u))\nu \) and \( \sigma_t(u) = -A'(\varepsilon(u))\nu - \sigma_n(u)\nu: \)

\[
\begin{align*}
\sigma_n(u) & \leq 0, \; u_{0,n} + u_{c,n} - u_n \leq 0, \; \sigma_n(u)(u_{0,n} + u_{c,n} - u_n) = 0, \\
|\sigma_t(u)| & \leq \mathcal{F}, \; \sigma_t(u)(u_{0,t} + u_{c,t} - u_t) + \mathcal{F}|u_{0,t} + u_{c,t} - u_t| = 0.
\end{align*}
\]

We have denoted the strains by \( \varepsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \) and the natural conormal derivative \( 2\mu \partial_\nu + \lambda \nu \text{div} + \mu \nu \times \text{curl} \) at the boundary by \( T^* \). The exterior problem is strongly elliptic provided \( \mu > 0, \lambda > -\mu \). The function \( A' : L^p(\Omega) \otimes \mathbb{R}^{n \times n} \to L^{p'}(\Omega) \otimes \mathbb{R}^{n \times n} \) is assumed to be a bounded, continuous and uniformly monotone operator, so that in particular for \( p \in (1, 2) \):

\[
\langle A'(x) - A'(y), x - y \rangle \lesssim (\|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)})^{p-2}\|x - y\|_{L^p(\Omega)}^2, \\
\langle A'(x) - A'(y), z \rangle \lesssim \|x - y\|_{L^p(\Omega)}^{p-1}\|z\|_{L^p(\Omega)}.
\]

When \( p \in [2, \infty), \) we require

\[
\langle A'(x) - A'(y), x - y \rangle \gtrsim \|x - y\|_{L^p(\Omega)}^p, \\
\langle A'(x) - A'(y), z \rangle \lesssim \|x - y\|_{L^p(\Omega)}^{p-2}\|x - y\|_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.
\]

We assume \( \Gamma_t \neq \emptyset, \) the compatibility condition \( \int_\Omega f + \langle t_0, 1 \rangle = 0 \) for \( n = 2 \) and that the data belong to the following spaces:

\( f \in (L^p(\Omega))^n, \) \( u_0 \in (W^{\frac{1}{2}, 2}(\partial \Omega))^n, \) \( t_0 \in (W^{-\frac{1}{2}, 2}(\partial \Omega))^n, \) \( 0 \leq \mathcal{F} \in L^\infty(\Gamma_s). \)

In Theorem 3.2 we will show that Problem (1) admits a unique weak solution \((u_1, u_2) \in W^{1,p}(\Omega)^n \times W^{1,2}_{loc}(\Omega^c)^n.\)

Examples include, in particular, \( p \)-Laplacian materials with \( A'(x) = |x|^{p-2}x \) as well as Carreau-type laws \( A'(x) = (|x|^{1-\delta}(1 + |x|^2)^\delta)^{\frac{p-2}{2} + \frac{\delta}{2}}x \) with \( \delta \in [0, 1]. \)

For the symmetric coupling of finite and boundary elements, the Poincaré–Steklov operator \( S \) of the Lamé equation on \( \Omega^c \) is used to reduce Problem (1) to a variational inequality in the Banach space

\[
X^p = \{(u, v) \in (W^{1,p}(\Omega))^n \times (W^{1,2-\frac{1}{2}p}(\Gamma_s))^n : u|_{\partial \Omega} + v \in W^{\frac{1}{2}, 2}(\partial \Omega)^n\},
\]
where \( r = \min\{p, 2\} \).

**Main Results:** This article complements the analysis of [6], which concerned a scalar \( p \)-Laplacian-type problem with frictional contact in the simpler case of “dilatant” material laws with \( 2 \leq p < \infty \). In [6] numerical approximations of the variational inequality could be studied in \( \tilde{X}^p = (W^{1,p}(\Omega))^n \times (W^{1/2}(\Gamma))_n \), as \( \tilde{X}^p = X^p \) for \( p \geq 2 \), with an emphasis on the transmission problem. Numerical examples confirmed the theoretical estimates.

Here we show that the space \( X^p \) provides the proper setting for the numerical analysis for all \( p \in (1, \infty) \), and we focus on the more intricate wellposedness and a sharp error analysis of the friction problem when \( p \in (1, 2) \): While the a posteriori estimate in [6] was aimed at the pure transmission problem, Theorem 6.1 gives a sharp a posteriori estimate for the error of Galerkin approximations to the variational inequality. It complements recent results for mixed finite element formulations of friction problems [9, 10, 11] and is new even in the elliptic case.

The existence of a unique \( X^p \)-solution is shown in Theorem 3.2, and Theorem 4.1 gives an a priori estimate for Galerkin approximations. Finally, in Section 6 we sketch the analysis when the discretization of the Poincaré–Steklov operator is included. As an example of the added difficulty when \( p \in (1, 2) \), the variational inequality no longer splits into an equality on \( \Omega \) and an inequality on \( \partial \Omega \), unless the artificial regularity assumption \( u|_{\partial \Omega} \in W^{1/2,2}(\partial \Omega)^2 \) is imposed.

The results in this article are stated for \( p \in (1, \infty) \), but we refer to [6] for most of the arguments when \( p \geq 2 \). Conversely, an appendix adapts the new a posteriori estimate for the frictional term to the setting considered there.

The mathematical differences between \( p < 2 \) and \( p \geq 2 \) are not artificial. They reflect the different physical behavior: While pseudoplastic materials like ice or molasses \((p < 2)\) get stiffer and stiffer under a smaller stress, possibly infinitely so, the opposite happens in the dilatant case like a thick emulsion of sand and water \((p > 2)\).

## 2 Preliminaries

Let \( \Omega \) be a bounded, open subset of \( \mathbb{R}^n \) with Lipschitz boundary \( \partial \Omega \). Set \( p' = \frac{p}{p - 1} \) whenever \( p \in (1, \infty) \). We will also denote \( r = \min\{p, 2\} \) and \( q = \max\{p, 2\} \).

Before analyzing a variational formulation of (16), we recall some properties of \( L^p \)-Sobolev spaces on \( \Omega \):

**Remark 2.1.** a) \( (W^{s,p}(\partial \Omega))^r = W^{-s,p'}(\partial \Omega) \) and \( W^{s,2}(\partial \Omega) = H^s(\partial \Omega) \).

b) \( W^{s,2}(\Omega) \hookrightarrow W^{s,p}(\Omega) \) and \( \|u\|_{W^{s,p}(\Omega)} \leq |\Omega|^{1/2} \|u\|_{W^{s,2}(\Omega)} \) for \( 1 < p \leq 2 \).

c) If \( \partial \Omega \) is smooth, pseudodifferential operators of order \( m \) with \( \mathbb{C}^{k\times k} \)-valued symbol in the Hörmander class \( S^m_{0,0}(\partial \Omega) \) map \( (W^{s,p}(\partial \Omega))^k \) continuously to \((W^{s-m, p}(\partial \Omega))^k \). For Lipschitz \( \partial \Omega \), at least the first–order Steklov–Poincaré
operator $S$ of the Lamé operator on $\Omega^c$ is continuous between $(W^{1,2}(\partial\Omega))^n$ and $(W^{-\frac{1}{2},2}(\partial\Omega))^n$.

\(\text{d) Points a) to c) imply that the quadratic form } \langle Su, u \rangle \text{ associated to } S \text{ is well-defined on } (W^{1-\frac{1}{p},p}(\partial\Omega))^n \text{ if } p \geq 2. S \text{ being elliptic, the form is unbounded for } p < 2 \text{ even if } \partial\Omega \text{ is smooth.}

The fundamental solution for the Lamé operator in $\mathbb{R}^2$,

$$G(x, y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \log(|x - y|^{-1}) \text{ Id} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^T}{|x - y|^2} \right\},$$

resp. $\mathbb{R}^3$

$$G(x, y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \frac{1}{|x - y|} \text{ Id} + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^T}{|x - y|^3} \right\},$$

allows to define layer potentials on $\partial\Omega$ associated to the exterior problem in the usual way:

- $\mathcal{V}\phi(x) = \int_{\partial\Omega} \phi(x') \ G(x, x') \ dx'$,
- $\mathcal{K}\phi(x) = \int_{\partial\Omega} \phi(x') \ \partial_{x'} G(x, x') \ dx'$,
- $\mathcal{K}'\phi(x) = \int_{\partial\Omega} \phi(x') \ \partial_{x'} G(x, x') \ dx'$,
- $\mathcal{W}\phi(x) = \partial_{x'} \int_{\partial\Omega} \phi(x') \ \partial_{x'} G(x, x') \ dx'$.

They extend from $C^\infty(\partial\Omega)^n$ to a bounded map $\begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}' \end{pmatrix}$ on the Sobolev space $W^{\frac{1}{2},2}(\partial\Omega)^n \times W^{-\frac{1}{2},2}(\partial\Omega)^n$. If (for $n = 2$) the capacity of $\partial\Omega$ is less than 1, which can always be achieved by scaling, $\mathcal{V}$ and $\mathcal{W}$ considered as operators on $W^{-\frac{1}{2},2}(\partial\Omega)^n$ are selfadjoint, $\mathcal{V}$ is positive and $\mathcal{W}$ non-negative. The Steklov-Poincaré operator for the exterior Lamé problem is given as

$$S = \mathcal{W} + (1 - \mathcal{K}') \mathcal{V}^{-1}(1 - \mathcal{K}) : W^{1,2}(\partial\Omega)^n \subset W^{-\frac{1}{2},2}(\partial\Omega)^n \rightarrow W^{-\frac{1}{2},2}(\partial\Omega)^n$$

and defines a positive and selfadjoint operator with the main property

$$T^* u_2|_{\partial\Omega} = -S(u_2|_{\partial\Omega})$$

for solutions $u_2$ of the Lamé equation on $\Omega^c$ satisfying the decay condition at $\infty$. $S$ therefore gives rise to a coercive and symmetric bilinear form $\langle Su, u \rangle$ on $W^{1,2}(\partial\Omega)^n$.

Existence of a unique solution to (1) will be shown using Korn’s inequality and coercivity:

**Proposition 2.2.** ([6], Proposition 2) Assume $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$ has positive $(n - 1)$-dimensional measure. Then there is a $C > 0$ such that

$$\|u\|_{1,p} \leq C(\|\varepsilon(u)\|_p + \|u|\Gamma\|_{L^1(\Gamma)}) \quad \text{for all } u \in (W^{1,p}(\Omega))^n.$$
3 Analysis of the boundary integral formulation

For \( r = \min\{p, 2\} \), we consider the space

\[ X^p = \{(u, v) \in (W^{1,p}(\Omega))^n \times (W^{1,2}(\Gamma_s))^n : u|_{\partial\Omega} + v \in W^{1,2}(\partial\Omega)^n \} \]

equipped with the norm

\[ \|u, v\|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,2}(\Gamma_s)} + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \cdot \]

Note that \( X^p = (W^{1,p}(\Omega))^n \times (W^{1,2}(\Gamma_s))^n \) when \( p \geq 2 \), so that we recover a vector-valued variant of the Banach spaces considered in [6].

**Lemma 3.1.** \( (X^p, \| \cdot \|_{X^p}) \) is a Banach space, and

\[ |u, v|_{X^p} = \|u\|_{W^{1,p}(\Omega)} + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \]

defines an equivalent norm on \( X^p \).

**Proof.** It is readily verified that \( \| \cdot \|_{X^p} \) defines a norm on \( X^p \). To show completeness, let \((u_j, v_j) \in X \) be a Cauchy sequence. Then \((u_j, v_j)\) converges to a limit \((u, v)\) in the Banach space \( W^{1,p}(\Omega)^n \times (W^{1,2}(\Gamma_s))^n \). Also \( u_j|\partial\Omega + v_j \) converges to a limit \( w \) in \( W^{1,2}(\partial\Omega)^n \). However, the continuity of the trace operator assures that \( u_j|\partial\Omega \to u|\partial\Omega \) in \( W^{1,\frac{1}{2}}(\partial\Omega)^n \). Therefore in \( W^{1,\frac{1}{2}}(\partial\Omega)^n \), hence also in \( W^{1,\frac{1}{2}}(\partial\Omega)^n \), \( u_j|\partial\Omega + v_j \) converges both to \( u|\partial\Omega + v \) and to \( w \). This means that \( u|\partial\Omega + v = w \in W^{1,2}(\partial\Omega)^n \), or \((u, v) \in X^p \).

To see the equivalence of norms, note that \( |u, v|_{X^p} \leq \|u, v\|_{X^p} \). On the other hand, the continuous inclusion of \( W^{1,2}(\partial\Omega) \) into \( W^{1,\frac{1}{2}}(\partial\Omega) \), of \( W^{1,\frac{1}{2}}(\partial\Omega) \) into \( W^{1,\frac{1}{2}}(\partial\Omega) \), and the continuity of the trace operator from \( W^{1,p}(\Omega) \) to \( W^{1,\frac{1}{2}}(\partial\Omega) \) imply

\[ \|u, v\|_{X^p} \leq \|u\|_{W^{1,p}(\Omega)} + \|u|\partial\Omega\|_{W^{1,\frac{1}{2}}(\partial\Omega)} + \|u|\partial\Omega + v\|_{W^{1,\frac{1}{2}}(\partial\Omega)} \]
\[ + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \]
\[ \leq \|u\|_{W^{1,p}(\Omega)} + \|u|\partial\Omega\|_{W^{1,\frac{1}{2}}(\partial\Omega)} + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \]
\[ + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \]
\[ \leq \|u\|_{W^{1,p}(\Omega)} + \|u|\partial\Omega + v\|_{W^{1,2}(\partial\Omega)} \]
\[ = |u, v|_{X^p} \cdot \]

The assertion follows. \( \Box \)

We consider a variational formulation of the contact problem in terms of the functional

\[ J(u, v) = \langle A(\varepsilon(u)), \varepsilon(u) \rangle + \frac{1}{2} \langle S(u|\partial\Omega + v), u|\partial\Omega + v \rangle - L(u, v) \]
on $X^p$. Here $A$ is derived from $A'$ by an explicit formula, $v = u_0 + u_c - u$,
\[ j(v) = \int_{\Gamma_s} F |v_2| , \]
and
\[ L(u, v) = \int_{\Omega} fu + \langle t_0 + Su_0, u|_{\partial \Omega} + v \rangle . \]
This paper investigates the numerical approximation of the following non-
mooth variational problem over the closed convex subset
\[ K = \{(u, v) : v_n \leq 0, \langle S1, u|_{\partial \Omega} + v - u_0 \rangle = 0\} \]
of $X^p$.
Find $(\hat{u}, \hat{v}) \in K$ such that
\[ J(\hat{u}, \hat{v}) + j(\hat{v}) = \min_{(u, v) \in K} J(u, v) + j(v) . \] (4)
Note that $j$ is Lipschitz, but not differentiable.

As in [6] one observes that Problem (4) is equivalent to the contact problem
(1). The existence of a unique solution to the latter is therefore a consequence
of the following theorem.

**Theorem 3.2.** There exists a unique minimizer $(\hat{u}, \hat{v}) \in K$ of $J + j$ over $K$.

The crucial ingredient in the proof is a monotonicity estimate:

**Lemma 3.3.** The operator associated to $J$ is strongly monotone on $X^p$.

Let $r = \min\{p, 2\}$, $q = \max\{p, 2\}$ and $C > 0$. Then for every $(u_1, v_1), (u_2, v_2) \in X^p$ with $\|u_1, v_1\|_{X^p(\Omega)}, \|u_2, v_2\|_{X^p(\Omega)} < C$, there holds
\[ \|u_1 - u_2, v_2 - v_1\|_{X^p} \]
\[ \leq C \left\langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1) \right\rangle \]
\[ + \langle S((u_2 - u_1)|_{\partial \Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial \Omega} + v_2 - v_1 \rangle \]
\[ \leq C \|u_2 - u_1, v_2 - v_1\|_{X^p} . \]

**Proof.** The upper bound is a consequence of the estimates (2), (3) for
the nonlinear operator and the boundedness of $S$ from $W^{1,2}(\partial \Omega)^n$ to $W^{-1,2}(\partial \Omega)^n$.
For $p \geq 2$, we refer to [6], Lemma 3, for the proof of an analogous lower estimate.
When $p < 2$ the monotony of $A'$ resp. coercivity of $S$ imply for any $\delta \in (0, 1)$
\[ \langle A'(\varepsilon(u_2)) - A'(\varepsilon(u_1)), \varepsilon(u_2) - \varepsilon(u_1) \rangle \]
\[ + \langle S((u_2 - u_1)|_{\partial \Omega} + v_2 - v_1), (u_2 - u_1)|_{\partial \Omega} + v_2 - v_1 \rangle \]
\[ \geq \|\varepsilon(u_2 - u_1)\|_{L^{r}(\Omega)} + \|(u_2 - u_1)|_{\partial \Omega} + v_2 - v_1\|_{W^{1/2}(\partial \Omega)}^2 \]
\[ \geq \|\varepsilon(u_2 - u_1)\|_{L^{r}(\Omega)} + \|(u_2 - u_1)|_{\partial \Omega} + v_2 - v_1\|_{W^{1/2}(\Gamma_s)}^2 + \|u_2 - u_1\|_{W^{1/2}(\Gamma_s)}^2 \]
\[ + \|(u_2 - u_1)|_{\partial \Omega} + v_2 - v_1\|_{W^{1/2}(\partial \Omega)}^2 \]
\[ \geq \|\varepsilon(u_2 - u_1)\|_{L^{r}(\Omega)}^2 + \delta \|(u_2 - u_1)|_{\partial \Omega} + v_2 - v_1\|_{W^{1,2}(\Gamma_s)}^2 + \|u_2 - u_1\|_{W^{1,2}(\Gamma_s)}^2 \]
\[ + \|(u_2 - u_1)|_{\partial \Omega} + v_2 - v_1\|_{W^{1/2}(\partial \Omega)}^2 . \] (5)
In the last inequality we use the continuous inclusion \( W^{1,2}(\Gamma_s) \subset W^{1-\frac{1}{p}}(\Gamma_s) \).

Korn’s inequality, Proposition 2.2, implies
\[
\| \varepsilon(u_2 - u_1) \|_{L^p(\Omega)}^2 + \| u_2 - u_1 \|_{W^{1,2}(\Gamma_s)}^2 \gtrsim \| u_2 - u_1 \|_{W^{1,p}(\Omega)}^2 .
\]

Further note from the triangle inequality, the convexity of \( x \mapsto x^2 \) as well as the continuity of the trace map from \( W^{1,p}(\Omega) \) to \( W^{1-\frac{1}{p}}(\Gamma_s) \):
\[
\| u_2 - v_1 \|_{W^{1-\frac{1}{p}}(\Gamma_s)}^2 \leq \left( \| (u_2 - u_1) \|_{\Gamma_s} + \| v_2 - v_1 \|_{W^{1-\frac{1}{p}}(\Gamma_s)} + \| u_2 - u_1 \|_{\Gamma_s} \|_{W^{1-\frac{1}{p}}(\Gamma_s)} \right)^2 \\
\leq 2 \| (u_2 - u_1) \|_{\Gamma_s} + \| v_2 - v_1 \|_{W^{1-\frac{1}{p}}(\Gamma_s)}^2 + 2 \| u_2 - u_1 \|_{\Gamma_s}^2 \\
\leq 2 \| (u_2 - u_1) \|_{\Gamma_s} + \| v_2 - v_1 \|_{W^{1-\frac{1}{p}}(\Gamma_s)} + 2C' \| u_2 - u_1 \|_{W^{1,p}(\Omega)}^2 .
\]

The asserted estimate follows from (5), (6) and (7), after choosing \( \delta > 0 \) sufficiently small.

Strong monotony on all of \( X^p \) is shown similarly, but for large \( \| \varepsilon(u_2 - u_1) \|_{L^p(\Omega)} \), the exponent 2 in the lower bound has to be replaced by \( p \).

\( \square \)

**Proof (of Theorem 3.2).** By Lemma 3.3 the operator associated to \( J \) is bounded and strongly monotone. Existence and uniqueness for the perturbation \( J + j \) of \( J \) follow e.g. by applying the perturbation result [13], Proposition 32.36. \( \square \)

## 4 Discretization and a priori error analysis

Let \( \{ T_h \}_{h \in H} \) a regular triangulation of \( \Omega \) into disjoint open regular triangles \( (n = 2) \) resp. tetrahedra \( (n = 3) \) \( T \), so that \( \overline{\Omega} = \bigcup_{T \in T_h} \overline{T} \). Each element has at most one edge resp. face on \( \partial \Omega \), and the closures of any two of them share at most a single vertex, edge or face. Let \( h_T \) denote the diameter of \( T \in T_h \) and \( \rho_T \) the diameter of the largest inscribed ball. We assume that \( 1 \leq \max_{T \in T_h} \frac{h_T}{\rho_T} \leq R \) independent of \( h \) and that \( h = \max_{T \in T_h} h_T \). \( \mathcal{E}_h \) is going to be the set of all edges of the triangles / faces of the tetrahedra in \( T_h \). Associated to \( T_h \) is the space \( W^{1,p}_h(\Omega) \subset W^{1,p}(\Omega) \) of functions whose restrictions to any \( T \in T_h \) are linear.

The boundary \( \partial \Omega \) is triangulated by \( \{ l \in \mathcal{E}_h : l \subset \partial \Omega \} \). For \( r = \min\{ p, 2 \} \), \( W^{1-\frac{1}{r}}(\partial \Omega) \) denotes the corresponding space of continuous, piecewise linear functions, and \( \widetilde{W}^{1-\frac{1}{r}}_h(\Gamma_s) \) the subspace of those supported on \( \Gamma_s \). Finally, \( W^{1-\frac{1}{2}}(\partial \Omega) \subset W^{1-\frac{1}{2}}(\partial \Omega) \) is the space of piecewise constant functions, and \( X^p_h = W^{1,p}_h(\Omega) \times \widetilde{W}^{1-\frac{1}{2}}_h(\Gamma_s) \subset X^p \).

We denote by \( i_h : W^{1,p}_h(\Omega) \hookrightarrow W^{1,p}(\Omega) \), \( j_h : \widetilde{W}^{1-\frac{1}{2}}_h(\Gamma_s) \hookrightarrow \widetilde{W}^{1-\frac{1}{2}}(\Gamma_s) \) and \( k_h : W^{1-\frac{1}{2}}(\partial \Omega) \hookrightarrow W^{1-\frac{1}{2}}(\partial \Omega) \) the canonical inclusion maps.

The discrete problem involves the discretized functional
\[
J_h(u_h, v_h) = \langle A(\varepsilon(u_h)), \varepsilon(u_h) \rangle + \frac{1}{2} \langle S(u_h|_{\partial \Omega} + v_h), u_h|_{\partial \Omega} + v_h \rangle - L_h(u_h, v_h)
\]
on $X_h^p$. Here
\[ S_h = \frac{1}{2}(W + (I - K'))k_h(k_hVk_h)^{-1}k_h(I - K) \]
and
\[ L_h(u_h, v_h) = \int f u_h + \langle t_0 + S_h u_0, u_h + \partial v_h \rangle. \]

There exists $h_0 > 0$ such that the approximate Steklov–Poincaré operator $S_h$ is coercive uniformly in $h < h_0$, i.e. $\langle S_h u_h, u_h \rangle \geq \alpha_S \| u_h \|_{W^{1,2}(\partial \Omega)}^2$ with $\alpha_S$ independent of $h$. Therefore, as in the previous section the discrete minimization problem
\[ J(\hat{u}_h, \hat{v}_h) + j(\hat{v}_h) = \min_{(u_h, v_h) \in K \cap X_h^p} J(u_h, v_h) + j(v_h). \]
is associated to a perturbation of a strongly monotone operator on $X_h^p$ and admits a unique minimizer.

Our Galerkin method for the numerical approximation relies on an equivalent reformulation of the continuous and discretized minimization problems (4), (8) as variational inequalities:

Find $(\hat{u}, \hat{v}) \in K$ such that
\[ \langle A'(\hat{u}), \varepsilon(u - \hat{u}) \rangle + \langle S(\hat{u} + \hat{v}), (u - \hat{u}) + v - \hat{v} \rangle \\
+ j(v) - j(\hat{v}) \geq L(u - \hat{u}, v - \hat{v}) \]
for all $(u, v) \in K$.

The discretized variant reads as follows:

Find $(\hat{u}_h, \hat{v}_h) \in K \cap X_h^p$ such that
\[ \langle A'(\hat{u}_h), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}_h + \hat{v}_h), (u_h - \hat{u}_h) + v_h - \hat{v}_h \rangle \\
+ j(v_h) - j(\hat{v}_h) \geq L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \]
for all $(u_h, v_h) \in K \cap X_h^p$.

**Theorem 4.1.** a) The following a priori estimate holds with $q = \max\{p, 2\}$:
\[ \| \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h \|_{X^p}^q \]
\[ \lesssim \inf_{(u_h, v_h) \in K \cap X_h^p} \left\{ \| \varepsilon(u - u_h) \|_{L^p(\Omega)} + \| (\hat{u} - u_h) + \hat{v} - v_h \|_{W^{1,2}(\partial \Omega)} \right. \\
\left. + \| \hat{v} - v_h \|_{L^1(\Gamma_s)} \right\} + \text{dist}_{W^{1,2}(\partial \Omega)}(V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W_h^{-1/2}(\partial \Omega))^2. \]
b) If $\hat{v} \in \tilde{W}^{1,2}(\Gamma_s)^n$, e.g. for $p \geq 2$ or $\Gamma_s = \emptyset$, the estimate can be improved to
\[ \| \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h \|_{X^p}^q \]
\[ \lesssim \inf_{(u_h, v_h) \in K \cap X_h^p} \left\{ \| \varepsilon(u - u_h) \|_{L^p(\Omega)} + \| (\hat{u} - u_h) + \hat{v} - v_h \|_{W^{1,2}(\partial \Omega)}^2 \right. \\
\left. + \| \hat{v} - v_h \|_{L^1(\Gamma_s)} \right\} + \text{dist}_{W^{1,2}(\partial \Omega)}(V^{-1}(1 - K)(\hat{u} + \hat{v} - u_0), W_h^{-1/2}(\partial \Omega))^2. \]

Here $\beta = \frac{2}{3-p}$ for $p < 2$ resp. $\beta = p'$ for $p \geq 2$. 

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Proof. Adding the continuous and discrete variational inequalities, we see that

\[
0 \leq \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \varepsilon(\hat{u}_h) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v} \rangle \\
+ j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v}) \\
+ \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) - \varepsilon(\hat{u}_h) \rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle \\
+ j(v_h) - j(\hat{v}_h) - L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) .
\]

Hence,

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}_h) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle \\
\leq \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}_h) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle \\
+ A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) - \varepsilon(\hat{u}) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v} \rangle \\
+ j(\hat{v}_h) - j(\hat{v}) - L(\hat{u}_h - \hat{u}, \hat{v}_h - \hat{v}) \\
+ \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) - \varepsilon(\hat{u}_h) \rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle \\
+ j(v_h) - j(\hat{v}_h) - L_h(u_h - \hat{u}_h, v_h - \hat{v}_h) \\
= \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h) - \varepsilon(\hat{u}) \rangle + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle \\
+ j(v_h) - j(\hat{v}) - L(u_h - \hat{u}_h, v_h - \hat{v}_h) - (L_h - L)(u_h - \hat{u}_h, v_h - \hat{v}_h) \\
+ \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle \\
+ \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h) - \varepsilon(\hat{u}) \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v} \rangle \\
+ \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle \\
+ j(v_h) - j(\hat{v}) - (L_h - L)(u_h - \hat{u}_h, v_h - \hat{v}_h) + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (u_h - \hat{u}_h)|_{\partial\Omega} + v_h - \hat{v}_h \rangle .
\]

Let \( p < 2 \). To bound \( \langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}_h) \rangle, \) we use the estimate (2) and Young’s inequality for any \( \delta > 0 \):

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}_h) \rangle \lesssim \| \varepsilon(\hat{u} - \hat{u}_h) \|^p_{L^p(\Omega)} \| \varepsilon(\hat{u} - u_h) \|^p_{L^p(\Omega)} \\
\lesssim \delta^{\frac{2-p}{2}} \| \varepsilon(\hat{u} - \hat{u}_h) \|^2_{L^p(\Omega)} + \delta^{-\frac{2}{2-p}} \| \varepsilon(\hat{u} - u_h) \|^2_{L^p(\Omega)} .
\]

On the other hand, for \( p \geq 2 \) the upper bound (3) yields

\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(\hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(\hat{u}_h) \rangle \lesssim \| \varepsilon(\hat{u} - \hat{u}_h) \|^p_{L^p(\Omega)} \| \varepsilon(\hat{u} - u_h) \|^p_{L^p(\Omega)} \\
\lesssim \delta^p \| \varepsilon(\hat{u} - \hat{u}_h) \|^p_{L^p(\Omega)} + \delta^{-q} \| \varepsilon(\hat{u} - u_h) \|^p_{L^p(\Omega)} .
\]

As for the second term, we use the boundedness of \( S \) from \( W^{\frac{1}{2}, 2}(\partial\Omega)^n \) to \( W^{-\frac{1}{2}, 2}(\partial\Omega)^n \) to estimate

\[
\langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)|_{\partial\Omega} + v_h \rangle \\
\lesssim \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v}_h \|_{W^{\frac{1}{2}, 2}(\partial\Omega)} \\
\lesssim \delta \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \|^2_{W^{\frac{1}{2}, 2}(\partial\Omega)} + \delta^{-1} \| (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \|^2_{W^{\frac{1}{2}, 2}(\partial\Omega)} .
\]

Without further assumptions on \( \hat{v} \), we estimate the second line using Cauchy–Schwarz by a multiple of

\[
\| \varepsilon(u_h - \hat{u}) \|_{L^p(\Omega)} + \| (u_h - \hat{u})|_{\partial\Omega} + v_h - \hat{v} \|_{W^{\frac{1}{2}, 2}(\partial\Omega)}
\]

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For part b), where \( \dot{v} \in \tilde{W}^{\frac{1}{2}, 2}(\Gamma_b) \), one may use the variational inequality for an improved estimate: Substituting \((u, v) = (u_h, \dot{v})\) and \((u, v) = (2u - u_h, \ddot{v})\) into the variational inequality on \(X^p\), we obtain

\[
(A'(\varepsilon(\ddot{v})), \varepsilon(u_h) - \varepsilon(\ddot{v})) + (S(\ddot{v}|_{\partial\Omega} + \dot{v}), (u_h - \ddot{v})|_{\partial\Omega}) = L(u_h - \ddot{v}, 0)
\]

With this, the second line reduces to \((S(\ddot{v}|_{\partial\Omega} + \dot{v}), v_h - \ddot{v}) + L(0, \dot{v} - v_h)\), i.e. to

\[-(t_0 - S(\ddot{v}|_{\partial\Omega} + \dot{v} - u_0), v_h - \ddot{v}) = -(A'(\varepsilon(\ddot{v})) \nu, v_h - \ddot{v}) \leq \|F||L^\infty(\Gamma_s)\|v_{n,h} - \ddot{v}_n\|L^1(\Gamma_s)\|
\]

For the third line,

\[j(v_h) - j(\ddot{v}) = \int_{\Gamma_s} F(|v_h| - |\ddot{v}|) \leq \int_{\Gamma_s} F(|v_h - \ddot{v}| - |\ddot{v}|) \leq \|F||L^\infty(\Gamma_s)\|v_h - \ddot{v}\|L^1(\Gamma_s)\|
\]

Finally, the last line simplifies as follows:

\[-(L_h - L)(u_h - \ddot{u}_h, v_h - \ddot{v}_h) + ((S_h - S)(\ddot{u}_h|_{\partial\Omega} + \dot{v}_h), (u_h - \ddot{u}_h)|_{\partial\Omega} + v_h - \ddot{v}_h)
\]

\[\leq \delta^{-1}||S_h - S)(\ddot{u}_h|_{\partial\Omega} + \dot{v}_h - u_0)|^2_{W^{-\frac{1}{2}, 2}(\partial\Omega)} + \delta\|(u_h - \ddot{u}_h)|_{\partial\Omega} + v_h - \ddot{v}_h||^2_{W^{-\frac{1}{2}, 2}(\partial\Omega)}
\]

The term involving \(S_h - S\) is known to be bounded by [3]

\[\text{dist}_{W^{-\frac{1}{2}, 2}(\partial\Omega)}(V^{-1}(1 - K)(\ddot{u} + \dot{v} - u_0), W^{-\frac{1}{2}, 2}(\partial\Omega))^2\]

To sum up, for general \(\ddot{v}\) we obtain for \(\alpha = \frac{p}{p-1}\), \(\beta = \frac{2}{p} (p < 2)\) resp. \(\alpha = p, \beta = p' (p \geq 2)\)

\[\langle A'((\ddot{u})), A'(\ddot{v})) + (S((\ddot{u} - \ddot{u}_h)|_{\partial\Omega} + \dot{v}_h), (\ddot{u} - \ddot{u}_h)|_{\partial\Omega} + v_h - \ddot{v}_h)\]

\[\leq \delta^{\alpha}||\ddot{u} - \ddot{u}_h||_{L^p(\Gamma)} + \delta||\ddot{u} - \ddot{u}_h||_{L^p(\Gamma)} + \|v_h - \ddot{v}_h||_{W^{-\frac{1}{2}, 2}(\partial\Omega)}
\]

The lowest exponents dominate.

When \(\ddot{v} \in \tilde{W}^{\frac{1}{2}, 2}(\Gamma_b)\), the estimates yield:

\[\langle A'(\ddot{u}), A'(\ddot{v})) + (S((\ddot{u} - \ddot{u}_h)|_{\partial\Omega} + \dot{v}_h), (\ddot{u} - \ddot{u}_h)|_{\partial\Omega} + v_h - \ddot{v}_h)\]

\[\leq \delta^{\alpha}||\ddot{u} - \ddot{u}_h||_{L^p(\Gamma)} + \delta||\ddot{u} - \ddot{u}_h||_{L^p(\Gamma)} + \|v_h - \ddot{v}_h||_{W^{-\frac{1}{2}, 2}(\partial\Omega)}
\]

The lowest exponents dominate.
Note that as in Lemma 3.3, the monotony of \( A' \) and coercivity of \( S \) allow
to bound the left hand side from below by
\[
\|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^2(\Omega)} + \|((\hat{u} - \hat{u}_h))_{\partial \Omega} + \hat{v} - \hat{v}_h\|^2_{W^{1,2}(\partial \Omega)}.
\]
Choosing \( \delta > 0 \) sufficiently small, the claimed estimates follow. \( \square \)

\textit{Remark 4.2.} a) Theorem 4.1 proves convergence of the proposed FE–BE coupling
procedure for quasi–uniform grid refinements. However, generic weak
solutions to the contact problem (1) only belong to \( X^p \) and not to any higher-
order Sobolev space. Therefore the convergence can be arbitrarily slow as the
grid size \( h \) tends to 0.
b) Like for the \( p \)-Laplacian operators in [6], under additional assumptions on \( A' \)
slightly sharper estimates can be obtained with respect to certain quasinorms
on \( X^p \).

5 \textit{An a posteriori estimate}

If we consider the variational inequality (10) for \( v_h = \hat{v}_h \) and with \( u_h \mapsto u_h \)
resp. \( u_h \mapsto 2\hat{u}_h - u_h \), Problem (10) splits into an interior equation and an
inequality on the boundary: For all \((u_h, v_h) \in K \cap X^p_h\):
\[
\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h) \rangle + \langle S_h(\hat{u}_h)_{\partial \Omega} + \hat{v}_h), u_h \rangle_{\partial \Omega} = \int_\Omega f u_h + \langle t_0 + S_h u_0, u_h \rangle = L_h(u_h, 0),
\]
\[
\langle S_h(\hat{u}_h)_{\partial \Omega} + \hat{v}_h), v_h - \hat{v}_h \rangle + j(v_h) - j(\hat{v}_h) \geq \langle t_0 + S u_0, v_h - \hat{v}_h \rangle = L_h(0, v_h - \hat{v}_h) .
\]

(11)

For the continuous inequality, we only get a weaker assertion because \( u|_{\partial \Omega} + v \)
needs to be in \( W^{1,2}(\partial \Omega) \). Choosing \( u = \hat{u} + \hat{u}_h - u_h \), \( v = \hat{v} + \hat{v}_h - v_h \) for any
\((u_h, v_h) \in X^p_h \) with \( v_h \leq \hat{v} + \hat{v}_h \) transforms (9) into the estimate
\[
\langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u} + \hat{v}_h), (u_h - \hat{u}_h)_{\partial \Omega} + v_h - \hat{v}_h \rangle
\]
\[
\leq j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h) .
\]

(12)

In combination with the coercivity estimates, we may start to derive an a
posteriori estimate:
\[
\|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^2(\Omega)}^2 + \|((\hat{u} - \hat{u}_h))_{\partial \Omega} + \hat{v} - \hat{v}_h\|^2_{W^{1,2}(\partial \Omega)}
\]
\[
\lesssim \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle
\]
\[
+ \langle S((\hat{u} - \hat{u}_h))_{\partial \Omega} + \hat{v} - \hat{v}_h), (\hat{u} - u_h)_{\partial \Omega} + \hat{v} - v_h \rangle
\]
\[
+ \langle S((\hat{u} - \hat{u}_h))_{\partial \Omega} + \hat{v} - \hat{v}_h), (u_h - \hat{u}_h)_{\partial \Omega} + v_h - \hat{v}_h \rangle
\]
\[
\langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle
\]
\[
+ \langle S(\hat{u} + \hat{v}_h), (u_h - \hat{u}_h)_{\partial \Omega} + v_h - \hat{v}_h \rangle - \langle S(\hat{u}_h + \hat{v}_h), (u_h - \hat{u}_h)_{\partial \Omega} + v_h - \hat{v}_h \rangle .
\]

We consider the second and fourth term on the right hand side,
Applying the equality in (11) to

\[ \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{\partial \Omega} + v_h - \hat{v}_h \],

and inequality (12) to

\[ \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (u_h - \hat{u}_h) \rangle_{\partial \Omega} + v_h - \hat{v}_h \]

we estimate their sum by

\[ - L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h) \]

\[ - \langle S_h(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), v_h - \hat{v}_h \rangle + \langle (S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h) \rangle_{\partial \Omega} + v_h - \hat{v}_h \]

For

\[ \langle A'(\varepsilon(\hat{u})), \varepsilon(u_h - \hat{u}_h) \rangle + \langle S(\hat{u}|_{\partial \Omega} + \hat{v}), (\hat{u} - u_h) \rangle_{\partial \Omega} + \hat{v} - v_h \]

we use the variational inequality (9) with \((u, v) = (u_h, v_h)\) to conclude

\[ \|\varepsilon(\hat{u} - \hat{u}_h)\|_{W^{1,p}(\Omega)}^p + \|\varepsilon(\hat{u} - \hat{u}_h)|_{\partial \Omega} + \hat{v} - v_h\|_{W^{1,2^*(\partial \Omega)}}^2 \]

\[ \lesssim L(\hat{u} - u_h, \hat{v} - v_h) + j(v_h) - j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle \]

\[ - \langle S(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (\hat{u} - u_h) \rangle_{\partial \Omega} + \hat{v} - v_h \]

\[ - L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h) \]

\[ + ((S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (u_h - \hat{u}_h))_{\partial \Omega} + v_h - \hat{v}_h \]

\[ = \int_{\Omega} f(\hat{u} - u_h) + (t_0 + S u_0, (\hat{u}_h - u_h)|_{\partial \Omega} + \hat{v} - v_h) + j(\hat{v} + v_h - v_h) + j(v_h) - 2j(\hat{v}) \]

\[ - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle - \langle S_h(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (\hat{u} - u_h) \rangle_{\partial \Omega} + \hat{v} - v_h \]

\[ - L_h(u_h - \hat{u}_h, 0) + j(\hat{v} + \hat{v}_h - v_h) - j(\hat{v}) + L(u_h - \hat{u}_h, v_h - \hat{v}_h) \]

\[ + ((S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h), (\hat{u} - u_h))_{\partial \Omega} + \hat{v} - v_h \]

\[ = \int_{\Omega} f(\hat{u} - u_h) + j(\hat{v} + v_h - v_h) + j(v_h) - 2j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle \]

\[ + (t_0 - S_h(\hat{u}_h|_{\partial \Omega} + \hat{v}_h - u_h), (\hat{u} - u_h))_{\partial \Omega} + \hat{v} - v_h \]

\[ + ((S_h - S)(\hat{u}_h|_{\partial \Omega} + \hat{v}_h - u_h), (\hat{u} - u_h))_{\partial \Omega} + \hat{v} - v_h \]

The first term is estimated as usual for \(u_h = \hat{u}_h + \Pi_h(\hat{u} - \hat{u}_h)\) using the Hölder inequality and the properties of a Clement interpolation operator \(\Pi_h\) (see e.g. [2]):

\[ \int_{\Omega} f(\hat{u} - u_h) \lesssim \|\hat{u} - u_h\|_{W^{1,p}(\Omega)} \left( \sum_{T \subset \Omega} h_T^{p'} \|f\|_{L^{p'}(T)} \right)^{1/p'} \]

\( (p' = \frac{p}{p-1}) \)

Similarly, integrating by parts we obtain

\[ \langle A'(\varepsilon(\hat{u}_h)), \varepsilon(u_h - \hat{u}_h) \rangle = \sum_{E \subset \Omega} \int_E [A'(\varepsilon(\hat{u}_h))]_{\partial \Omega} + \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h) \rangle + \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h) \rangle_{\partial \Omega} \]

with

\[ \sum_{E \subset \Omega} \int_E [A'(\varepsilon(\hat{u}_h))]_{\partial \Omega} + \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h) \rangle_{\partial \Omega} \]

\[ \lesssim \|\hat{u} - u_h\|_{W^{1,p}(\Omega)} \left( \sum_{E \subset \Omega} h_E \|A'(\varepsilon(\hat{u}_h))\nu\|_{L^{p'}(E)} \right)^{1/p'} \].

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It remains to consider the boundary contributions. To do so, recall the strong formulation of the contact conditions in terms of \( \sigma_n(u) \) and \( \sigma_t(u) \) on \( \Gamma_s \),

\[
\sigma_n(u) \leq 0 \, , \, v_n \leq 0 \, , \, \sigma_n(u)v_n = 0 \, , \\
|\sigma_t(u)| \leq \mathcal{F} \, , \, \sigma_t(u)v_t + \mathcal{F}|v_t| = 0 
\]

Then, substituting \( v_h = \hat{v}_h \), we obtain

\[
j(\hat{v} + \hat{v}_h - v_h) = j(\hat{v}) = \int_{\Gamma_s} \mathcal{F}|v_t| = -\langle \sigma_t(u), v_t \rangle = -\langle \sigma(u), v \rangle .
\]

Also,

\[
j(\hat{v}_h) - \langle A'(\varepsilon(\hat{u}_h))\nu, \hat{v}_h \rangle \leq \int_{\Gamma_s} \{ \mathcal{F}|v_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t} \} + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h})^+. 
\]

Together, the terms

\[
j(\hat{v} + \hat{v}_h - v_h) + j(v_h) - 2j(\hat{v}) - \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|_{\partial\Omega} \rangle_{\partial\Omega}
\]

are hence dominated by

\[
\int_{\Gamma_s} \{ \mathcal{F}|v_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t} \} + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h})^+ \\
- \langle A'(\varepsilon(\hat{u}_h))\nu, (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle_{\partial\Omega} + \langle \sigma(u) - \sigma(\hat{u}_h), \hat{v} \rangle .
\]

We split the \( \sigma \)-term into tangential and normal parts

\[
\langle \sigma(\hat{u}) - \sigma(\hat{u}_h), \hat{v} \rangle = \langle \sigma_n(\hat{u}) - \sigma_n(\hat{u}_h), \hat{v}_n \rangle + \langle \sigma_t(\hat{u}) - \sigma_t(\hat{u}_h), \hat{v}_t \rangle ,
\]

and estimate the normal part as follows (\( r' = \frac{r}{r+1} \)):

\[
\langle \sigma_n(\hat{u}) - \sigma_n(\hat{u}_h), \hat{v}_n \rangle \leq -\langle \sigma_n(\hat{u}_h)^+, \hat{v}_n \rangle \lesssim \| \sigma_n(\hat{u}_h)^+ \|_{W^{-1,\frac{r'}{r'}}(\Gamma_s)} .
\]

For the tangential contribution, involving the Tresca friction, we find it convenient to write \( \sigma_t(\hat{u}) = -\zeta \mathcal{F} \) with \( |\zeta| \leq 1 \) and \( |v_t| = \zeta |v_t| \). Then

\[
\langle \sigma_t(\hat{u}) - \sigma_t(\hat{u}_h), \hat{v}_t \rangle = -\langle \zeta \mathcal{F}, \hat{v}_t \rangle - \langle \sigma_t(\hat{u}_h), \hat{v}_t \rangle = -\langle \mathcal{F}, |\hat{v}_t| \rangle - \langle \sigma_t(\hat{u}_h), \hat{v}_t \rangle \\
\leq \langle \| \sigma_t(\hat{u}_h) \| - \mathcal{F} \rangle^+ + |\hat{v}_t| \lesssim \| \| \sigma_t(\hat{u}_h) \| - \mathcal{F} \|_{W^{-1,\frac{r'}{r'}}(\Gamma_s)} .
\]
We conclude
\[
\|\varepsilon(\hat{u} - \tilde{u}_h)\|_{L^p(\Omega)}^q + \|(\hat{u} - \tilde{u}_h)|\partial\Omega + \hat{v} - \tilde{v}_h\|_{W^{1/2,2}(\partial\Omega)}^q \\
\lesssim \|\varepsilon(\hat{u} - \tilde{u}_h)\|_{L^p(\Omega)}^q + \|(\hat{u} - \tilde{u}_h)|\partial\Omega + \hat{v} - \tilde{v}_h\|_{W^{1/2,2}(\partial\Omega)}^2 \\
\lesssim \|\hat{u} - \tilde{u}_h\|_{W^{1,p}(\Omega)} \left( \sum_{T \subset \Omega} h_T^{p'} \|f\|_{L^{p'}(T)}\right)^{1/p'} + \|\hat{u} - \tilde{u}_h\|_{W^{1,p}(\Omega)} \left( \sum_{E \subset \Omega} h_E \|A'(\varepsilon(\tilde{u}_h))\nu\|_{L^{p'}(E)}\right)^{1/p'} \\
+ \int_{\Gamma_s} \{F|v_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}\} + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h}) + \\
+ \|t_0 + S_h(u_0 - \hat{u}_h)|\partial\Omega + \hat{v}_h - A'(\varepsilon(\tilde{u}_h))\nu\|_{W^{1-1/p',p'}(\partial\Omega)}^q \\
+ \|\sigma_n(\hat{u}_h) + \|\hat{v}_{h,t}\|_{W^{1-1/p',p'}(\Omega)} + \|(|\sigma_t(\hat{u}_h)| - F) + \|\tilde{v}_{h,t}\|_{W^{1-1/p',p'}(\Omega)} \\
+ \|(S_h - S)(\tilde{u}_h)\|_{\partial\Omega + \hat{v}_h - u_0)\|_{W^{1-1/p',p'}(\Omega)}^2.
\]
Summing up:

**Theorem 5.1.** Let \( r = \min\{p, 2\} \) and \( q = \max\{p, 2\} \). The following a posteriori estimate holds:

\[
\|\hat{u} - \tilde{u}_h, \hat{v} - \tilde{v}_h\|_{X^p}^q \\
\lesssim \left( \sum_{T \subset \Omega} h_T^{p'} \|f\|_{L^{p'}(T)}\right)^{q'/p'} + \left( \sum_{E \subset \Omega} h_E \|A'(\varepsilon(\tilde{u}_h))\nu\|_{L^{p'}(E)}\right)^{q'/p'} \\
+ \|t_0 + S_h(u_0 - \hat{u}_h)|\partial\Omega + \hat{v}_h - A'(\varepsilon(\tilde{u}_h))\nu\|_{W^{1-1/p',p'}(\partial\Omega)}^q \\
+ \int_{\Gamma_s} \{F|v_{h,t}| + \sigma_t(\hat{u}_h)\hat{v}_{h,t}\} + \int_{\Gamma_s} (\sigma_n(\hat{u}_h)\hat{v}_{n,h}) + \\
+ \|\sigma_n(\hat{u}_h) + \|\hat{v}_{h,t}\|_{W^{1-1/p',p'}(\Omega)} + \|(|\sigma_t(\hat{u}_h)| - F) + \|\tilde{v}_{h,t}\|_{W^{1-1/p',p'}(\Omega)} \\
+ \|(S_h - S)(\tilde{u}_h)\|_{\partial\Omega + \hat{v}_h - u_0)\|_{W^{1-1/p',p'}(\Omega)}^2.
\]

**Remark 5.2.** Adapting the interpolation operator \( \Pi_h \) to include \( \hat{v} - \tilde{v}_h \) on \( \Gamma_s \), it might be possible to improve the term \( \|t_0 + S_h(u_0 - \hat{u}_h)|\partial\Omega + \hat{v}_h - A'(\varepsilon(\tilde{u}_h))\nu\|_{W^{1-1/p',p'}(\partial\Omega)}^q \) to \( \|t_0 + S_h(u_0 - \hat{u}_h)|\partial\Omega + \hat{v}_h - A'(\varepsilon(\tilde{u}_h))\nu\|_{W^{1-1/p',p'}(\partial\Omega)}^2 \).

### 6 Formulation in terms of layer potentials

In practice, one would like to estimate the numerical error without a priori information about \( S - S_h \). This is achieved by formulating the problem directly in terms of the layer potentials \( \mathcal{V}, \mathcal{W}, \mathcal{K}, \mathcal{K}' \) rather than \( S = \mathcal{V} + (1 - \mathcal{K}') \mathcal{V}^{-1}(1 - \mathcal{K}) \). The arguments are a notationally more involved variant of those in Section 5, and we only outline them.
We consider the space
\[ Y^p = X^p \times W^{-\frac{1}{2},2}(\partial \Omega)^n, \]
equipped with the norm
\[ \|u, v, \phi\|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}((\Gamma_2)}} + \|u|_{\partial \Omega} + v\|_{W^{\frac{1}{2},2}(\partial \Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial \Omega)}. \]

From Lemma 3.1 we conclude that \((Y^p, \| \cdot \|_{Y^p})\) is a Banach space and
\[ |u, v, \phi|_{Y^p} = \|u\|_{W^{1,p}(\Omega)} + \|v\|_{W^{\frac{1}{2},2}(\partial \Omega)} + \|\phi\|_{W^{-\frac{1}{2},2}(\partial \Omega)} \]
an equivalent norm on \(Y^p\). We consider the discretization in finite dimensional subspaces \(Y^p_h = X^p_h \times W^{-\frac{1}{2},2}(\partial \Omega)^n\) of \(Y^p\).

In order to show coercivity, we use a theoretical stabilization as in [5]: Let \(r_1, \ldots, r_D\) a basis of the space of rigid body motions, and consider their orthogonal projections \(\xi_1, \ldots, \xi_D\) onto \(L^2(\partial \Omega)\). The arguments in [5], Lemma 4 and Proposition 5, show that \(|u, v, \phi|_{Y^p}\) is equivalent to the norm
\[ |u, v, \phi|_{Y^p, s} = \|\varepsilon(u)\|_{L^p(\Omega)}^2 + \langle \mathcal{W}(u|_{\partial \Omega} + v), u|_{\partial \Omega} + v \rangle + \langle \phi, \mathcal{V}\phi \rangle \]
\[ + \sum_{j=1}^D |\langle \xi_j, (1 - K)(u|_{\partial \Omega} + v) + \mathcal{V}\phi \rangle|^2. \]

On \(Y^p_h\), we have the following equivalent formulation of the contact problem (1): Find \((\hat{u}, \hat{v}, \hat{\phi}) \in K' = (K \cap Y^p) \times W^{-\frac{1}{2},2}(\partial \Omega)^n\) such that for all \((u, v, \phi) \in K'\):
\[ B(\hat{u}, \hat{v}, \hat{\phi}; u - \hat{u}, v - \hat{v}, \phi - \hat{\phi}) + \xi(v) - j(\hat{v}) \geq \Lambda(u - \hat{u}, v - \hat{v}, \phi - \hat{\phi}) \]
with
\[ B(u, v, \phi; \hat{u}, \hat{v}, \hat{\phi}) = \langle A'(\varepsilon(u)), \varepsilon(\hat{u}) \rangle + \langle \mathcal{W}(u|_{\partial \Omega} + v), u|_{\partial \Omega} + v \rangle + \langle \phi, \mathcal{V}\phi + (1 - K)(u|_{\partial \Omega} + v) \rangle, \]
\[ \Lambda(u, v, \phi) = \langle t_0 + \mathcal{W}u_0, u|_{\partial \Omega} + v \rangle + \int_{\Omega} f u + \langle \phi, (1 - K)u_0 \rangle. \]
The discretized problem is obtained by restricting to \(Y^p_h\), and we denote its solution by \((\hat{u}_h, \hat{v}_h, \hat{\phi}_h)\). We also consider a stabilized problem that for all \((u_h, v_h, \phi_h) \in K' \cap Y^p_h\)
\[ \tilde{B}(\hat{u}_{s,h}, \hat{v}_{s,h}, \hat{\phi}_{s,h}; u_h - \hat{u}_{s,h}, v_h - \hat{v}_{s,h}, \phi_h - \hat{\phi}_{s,h}) + \xi(v_h) - j(\hat{v}_{s,h}) \geq \tilde{\Lambda}(u_h - \hat{u}_{s,h}, v_h - \hat{v}_{s,h}, \phi_h - \hat{\phi}_{s,h}), \]
when
\[ \tilde{B}(u, v, \phi; \hat{u}, \hat{v}, \hat{\phi}) = B(u, v, \phi; \hat{u}, \hat{v}, \hat{\phi}) \]
\[ + \sum_{j=1}^D \langle \xi_j, \mathcal{V}\phi + (1 - K)(u|_{\partial \Omega} + v) \rangle \langle \xi_j, \mathcal{V}\phi + (1 - K)(u|_{\partial \Omega} + v) \rangle \]

15
respectively

\[ \tilde{\Lambda}(u, v, \phi) = \Lambda(u, v, \phi) + \sum_{j=1}^{D} (\xi_j, (1 - K)u_0) \langle \xi_j, (1 - K)(u|_{\partial\Omega} + v) \rangle. \]

Because the variational inequality (15) is an equality in \( \phi \), as in [5], Proposition 3, the solution to the stabilized and nonstabilized problems coincide, \((\hat{u}_h, \hat{v}_h, \hat{\phi}_h) = (\hat{u}_{s,h}, \hat{v}_{s,h}, \hat{\phi}_{s,h})\). However, the stabilized variational inequality is coercive in the stabilized norm (13):

\[
\|\varepsilon(\hat{u} - \hat{u}_h)\|_{L^p(\Omega)} + \langle \mathcal{V}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle
\]
\[
+ \langle \mathcal{V}(\hat{\phi} - \hat{\phi}_h), \hat{\phi} - \hat{\phi}_h \rangle + \sum_{j=1}^{D} |\langle \xi_j, (1 - K)((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h) \rangle|^2
\]
\[
\lesssim (A'(\varepsilon(\hat{u}))) - A'(\varepsilon(\hat{u}_h))
\]
\[
+ \langle \mathcal{V}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + (K' - 1)(\hat{\phi} - \hat{\phi}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle
\]
\[
+ ((1 - K)((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h), \hat{\phi} - \hat{\phi}_h)
\]
\[
+ \sum_{j=1}^{D} |\langle \xi_j, (1 - K)((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + \mathcal{V}(\hat{\phi} - \hat{\phi}_h) \rangle|^2
\]

Proceeding as in Section 5, we obtain:

**Theorem 6.1.** Let \( r = \min\{p, 2\} \) and \( q = \max\{p, 2\} \). The following a posteriori estimate holds:

\[
\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|_{q, p}^q
\]
\[
\lesssim \left( \sum_{E \subset \Omega} h_E^{q'} \|f\|_{L^p(T)}^{q'/p'} \right)^{q'/p'} + \left( \sum_{E \subset \Omega} h_E^{q'} \|A'(\varepsilon(\hat{u}_h))\nu\|_{L^{q'}(E)}^{q'/p'} \right)^{q'/p'}
\]
\[
+ \|\hat{t}_0 - \mathcal{V}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (K' - 1)\hat{\phi}_h - A'(\varepsilon(\hat{u}_h))\nu\|_{W^{1+\frac{1}{4}, r'}(\partial\Omega)}^{q'/p'}
\]
\[
+ \|\mathcal{V}\hat{\phi}_h + (1 - K)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\|_{W^{1+\frac{3}{4}, r'}(\partial\Omega)}^2
\]
\[
+ \int_{\Gamma_s} \|\mathcal{F}\hat{v}_{h,t} + \sigma_t(\hat{u}_h)\hat{\phi}_{h,t} \| + \int_{\Gamma_s} \|\hat{\sigma}_{n}(\hat{u}_h)\hat{\phi}_{n,h} \| + \int_{\Gamma_s} \|\hat{\sigma}_{n}(\hat{u}_h)\|_{W^{1+\frac{1}{4}, r'}(\Gamma_s)} + \|\mathcal{F}\hat{v}_{h,t} \|_{W^{1+\frac{3}{4}, r'}(\Gamma_s)}.
\]

7 Appendix – An improved error estimate for the scalar \( p \)-Laplacian

Consider the following scalar transmission problem for \( p \geq 2 \):

\[
-\text{div} A'(\nabla u) = f \quad \text{in } \Omega,
\]
\[
-\Delta u_c = 0 \quad \text{in } \Omega_c,
\]
\[
A'(u)\nu - \partial_{n}u_c = t_0 \quad \text{on } \partial\Omega,
\]
\[
u - u_c = u_0 \quad \text{on } \Gamma_t,
\]

(16)
On $\Gamma_s$, contact conditions corresponding to Tresca friction are imposed in terms of the stress $\sigma(u) = -A'(\nabla u)\nu,$
$$|\sigma(u)| \leq g, \quad \sigma(u)(u_0 + u_c - u) + g|u_0 + u_c - u| = 0.$$ A radiation condition holds for $|x| \rightarrow \infty$:
$$u(x) = a + o(1),$$
and for simplicity of notation we assume $a = 0$. Here $A' : L^p(\Omega)^2 \rightarrow L^{p'}(\Omega)^2$

is assumed to be a bounded, continuous and uniformly monotone operator, so that in particular

$$\langle A'(x) - A'(y), x - y \rangle \geq \|x - y\|_{L^p(\Omega)}^p,$$

$$\langle A'(x) - A'(y), z \rangle \lesssim \|x\|_{L^p(\Omega)} + \|y\|_{L^p(\Omega)}^{p-2}\|x - y\|_{L^p(\Omega)}\|z\|_{L^p(\Omega)}.$$ The data belong to the following spaces:

$$f \in L^{p'}(\Omega), \quad u_0 \in W^{1/2,2}(\partial\Omega), \quad t_0 \in W^{-1/2,2}(\partial\Omega), \quad 0 \leq g \in L^\infty(\Gamma_s), \quad a \in \mathbb{R}.$$ In addition, $\int_\Omega f + t_0 = 0$. We are looking for weak solutions $(u, u_c) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^\circ)$.

The above contact problem is equivalent to the following variational inequality in the space

$$X^p = W^{1,p}(\Omega) \times \tilde{W}^{1/2,2}(\Gamma_s), \quad \tilde{W}^{1/2,2}(\Gamma_s) = \{u \in W^{1/2,2}(\partial\Omega) : \text{supp } u \subset \tilde{\Gamma}_s\}.$$ Find $(\hat{u}, \hat{v}) \in X^p$ such that for all $(u, v) \in X^p$,

$$\langle A'(\nabla \hat{u}), \nabla u \rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), u|_{\partial\Omega} \rangle = \int_\Omega f u + \langle t_0 + Su_0, u|_{\partial\Omega} \rangle = L(u, 0),$$

$$\langle S(\hat{u}|_{\partial\Omega} + \hat{v}), v - \hat{v} \rangle + j(v) - j(\hat{v}) \geq \langle t_0 + Su_0, v - \hat{v} \rangle = L(0, v - \hat{v}).$$

We obtain a variant of Galerkin orthogonality in the interior:

$$\langle A'(\nabla \hat{u}) - A'(\nabla \hat{u}_h), \nabla u_h \rangle + \langle S(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h, u_h|_{\partial\Omega} \rangle + \langle (S - S_h)(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v}_h - u_0, u_h|_{\partial\Omega} \rangle = 0.$$ As in [6], Theorem 2, the monotony of $A'$ and coercivity of $S$ imply

$$\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h\|_{X^p}^p \lesssim \|\hat{u} - \hat{u}_h\|_{W^{1,p}(\Omega)}^p + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{1,2}(\partial\Omega)}^2 \lesssim \langle A'(\nabla \hat{u}) - A'(\nabla \hat{u}_h), \nabla (\hat{u} - \hat{u}_h) \rangle + \langle S(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h, (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle.$$ Using the variational equality in $\Omega$, the right hand side becomes

$$\langle A'(\nabla \hat{u}) - A'(\nabla \hat{u}_h), \nabla (\hat{u} - \hat{u}_h) \rangle + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h \rangle = L(\hat{u} - \hat{u}_h, 0) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), \hat{v} - \hat{v}_h \rangle - \langle A'(\nabla \hat{u}_h), \nabla (\hat{u} - \hat{u}_h) \rangle - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} \rangle - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), \hat{v} - \hat{v}_h \rangle.$$
Let \( u_h \in W_h^{1,p}(\Omega) \) arbitrary and \((e, \tilde{e}) = (\dot{u} - \dot{u}_h, \dot{v} - \dot{v}_h)\), \( e_h = \dot{u} - u_h \), whence \( e - e_h = \dot{u} - u_h \). With the help of Galerkin orthogonality in \( \Omega \), the right hand side turns into

\[
L(e - e_h, 0) - \langle S(\dot{u}_0 + \hat{\nu}), \epsilon \rangle - \langle A'(\nabla \dot{u}_h), \nabla (e - e_h) \rangle - \langle S(\dot{u}_h|_{\partial \Omega} + \hat{\nu}_h), e - e_h \rangle
+ \langle (S_h - S)(\dot{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0), e_h \rangle.
\]

Recall that \( L(e - e_h, 0) = \int_{\Omega} f(e - e_h) + \langle t_0 + S u_0, (e - e_h)|_{\partial \Omega} \rangle \). In [6] it was shown for a suitable interpolant \( e_h = \pi e \) and any \( \varepsilon > 0 \),

\[
\int_{\Omega} f(e - e_h) \lesssim \varepsilon |e|_{(1, \tilde{\alpha}, p)}^2 + C(\varepsilon) \eta_f^2 + \varepsilon \eta_{gr}^2
\]

where

\[
\eta_{gr}^2 = \sum_{K \in T_h} \int_K G_{p, \delta}(\nabla \dot{u}_h, \nabla \dot{u}_h - G_h \dot{u}_h),
\]

\[
\eta_f^2 = \sum_{K \in T_h} \int_K G_{p', 1}(|\nabla \dot{u}_h|^{p-1}, h_K(f - f_K)),
\]

involve the gradient recovery resp. the approximation error of \( f \). Integrating by parts in the term \( -\langle A'(\nabla \dot{u}_h), \nabla (e - e_h) \rangle \) yields two terms,

\[-\sum_{I \subset \partial \Omega} \int_I \nu \cdot A'(\nabla \dot{u}_h) \; (e - e_h)\]

and

\[-\sum_{I \subset \partial \Omega} \int_I A_1(e - e_h) \lesssim \eta_{gr}^2 + \varepsilon (|e|_{(1, \tilde{\alpha}, p)}^2 + \eta_{gr}^2).
\]

Altogether we conclude

\[
\| \dot{u} - \dot{u}_h, \dot{v} - \dot{v}_h \|_{X_\pi} \lesssim \| \dot{u} - \dot{u}_h \|_{(1, \tilde{\alpha}, p)} + \| (\dot{u} - \dot{u}_h)|_{\partial \Omega} + \dot{v} - \dot{v}_h \|_{W^{1,2}(\partial \Omega)}
\lesssim 2\varepsilon |e|_{(1, \tilde{\alpha}, p)}^2 + C(\varepsilon) \eta_f^2 + (1 + 2\varepsilon) \eta_{gr}^2
+ \langle \nu \cdot A'(\nabla \dot{u}_h) + S(\dot{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0) - t_0, \pi e - e \rangle
+ \langle S((\dot{u} - \dot{u}_h)|_{\partial \Omega} + \hat{\nu} - \dot{\nu}_h), \dot{v} - \dot{\nu}_h \rangle
+ \langle (S_h - S)(\dot{u}_h|_{\partial \Omega} + \hat{\nu}_h - u_0), \pi e \rangle.
\]

We write the second–to–last term as \( \langle \sigma(\dot{u}) - \sigma(\dot{u}_h), \dot{v} - \dot{\nu}_h \rangle \) and the friction
conditions as $\sigma(\dot{u}) = -\zeta g$, $|\dot{v}| = \zeta \dot{v}$ for some $|\zeta| \leq 1$. Then

$$
\langle \sigma(\dot{u}) - \sigma(\dot{u}_h), \dot{v} - \dot{v}_h \rangle
$$

$$
= -\langle \zeta g, \dot{v} \rangle - \langle \zeta(\dot{u}_h), \dot{v} \rangle + \langle \zeta g, \dot{v}_h \rangle
$$

$$
= -g(\langle \sigma(\dot{u}_h), \dot{v} \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle)
$$

$$
\leq \langle \|\sigma(\dot{u}_h) - g\|, |\dot{v}| \rangle + \langle \zeta g, \dot{v}_h \rangle - \langle \|\sigma(\dot{u}_h)|, |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle
$$

$$
\leq \langle \|\sigma(\dot{u}_h) - g\| + |\dot{v}_h| + \langle g, |\dot{v}_h| \rangle - \langle \|\sigma(\dot{u}_h)|, |\dot{v}_h| \rangle + \langle \sigma(\dot{u}_h), \dot{v}_h \rangle
$$

$$
\leq \|\|\sigma(\dot{u}_h) - g\|\|_{W^{-\frac{1}{2},2}(\Gamma_s)}\|\dot{v} - \dot{v}_h\|_{W^{\frac{1}{2},2}(\Gamma_s)}
$$

$$
+ \langle \|\sigma(\dot{u}_h)|, |\dot{v}_h| \rangle - \langle \sigma(\dot{u}_h), \dot{v}_h \rangle
$$

$$
= \|\|\sigma(\dot{u}_h) - g\|\|_{W^{-\frac{1}{2},2}(\Gamma_s)}\|\dot{v} - \dot{v}_h\|_{W^{\frac{1}{2},2}(\Gamma_s)}
$$

$$
+ \int_{\Gamma_s} \|\sigma(\dot{u}_h)| - g\|_{W^{-\frac{1}{2},2}(\Gamma_s)}\|\dot{v} - \dot{v}_h\|_{W^{\frac{1}{2},2}(\Gamma_s)} + 2 \int_{\Gamma_s} (\sigma(\dot{u}_h)\dot{v}_h)_+ .
$$

This proves the following a posteriori estimate:

**Theorem 7.1.** Let $f \in L^p(\Omega)$ and denote by $(\epsilon, \tilde{\epsilon})$ the error between the Galerkin solution $(\dot{u}_h, \dot{v}_h) \in X_h^p$ and the true solution $(\dot{u}, \dot{v}) \in X^p$. Then

$$
\|\dot{u} - \dot{u}_h, \dot{v} - \dot{v}_h\|_{X^p} \leq \eta_{gr}^2 + \eta_f^2 + \eta_S^2 + \eta_\theta^2 + \eta_g^2,
$$

where

$$
\eta_{gr}^2 = \sum_{K \in T_h} \int_K G_{p,\delta}(\nabla \dot{u}_h, \nabla \dot{u}_h - G_h \dot{u}_h),
$$

$$
\eta_f^2 = \sum_{K \in T_h} \int_K G_{p',1}(\|\nabla \dot{u}_h\|^{p-1}, h_K(f - f_K)),
$$

$$
\eta_S^2 = \text{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)} \left( V^{-1}(1 - K)(\dot{u} - u_0), W^{-\frac{1}{2},2}(\partial\Omega) \right)^2
$$

$$
\eta_\theta^2 = \|\nu \cdot A'(\nabla \dot{u}_h) + S(\dot{u}_h|\partial\Omega + \dot{v}_h - u_0) - t_0\|_{W^{-1} + \frac{1}{2},p'}
$$

$$
\eta_g^2 = \|\|\sigma(\dot{u}_h)| - g\|_{W^{-\frac{1}{2},2}(\Gamma_s)} + \int_{\Gamma_s} \|\sigma(\dot{u}_h)| - g\|_{W^{-\frac{1}{2},2}(\Gamma_s)} + 2 \int_{\Gamma_s} (\sigma(\dot{u}_h)\dot{v}_h)_+ .
$$

**Remark 7.2.** As $p \geq 2$, we are here able to split both the discretized and the continuous variational inequality into an equation in $\Omega$ and an inequality on $\partial\Omega$. This explains the slightly different form of the frictional terms compared to Theorems 5.1 and 6.1.

**References**


