The Composite Link Model  
and  
Models of Mortality

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Outline
- GLMs, Gompertz model
- CLMs, Gompertz, Makeham models
- Heligman-Pollard model
- Camarda-Eilers-Gampe model

The Pattern of Human Mortality

log(Mortality) for Swiss males in 1980
Gompertz model

\[ D_i \sim P(e_i \lambda_i), \quad i = 1, \ldots, n \]

\[ \log \mathbb{E}(D_i) = \log e_i + \log \lambda_i \]

\[ = \log e_i + \alpha_0 + \alpha_1 x_i \]

\[ \log \mathbb{E}(D) = \log e + X\alpha, \quad X = [1_n : x]. \]

Gompertz as a CLM

\[ \log \mathbb{E}(D) = \log e + X\alpha, \quad X = [1_n : x] \]

\[ \mathbb{E}(D) = \mu = C \exp(X\alpha), \quad C = \text{diag}(e) \]

\[ \mu = C \gamma, \quad \gamma = \exp(X\alpha) \]

Defn: \( C \) is the composition matrix

GLM

- Model matrix: \( X = [1_n : x] \)
- Link: log
- Error: Poisson
- Algorithm: Newton-Raphson: \( X'\tilde{W}X\tilde{\alpha} = X'\tilde{W}\tilde{z} \)
  - \( \mu = \mathbb{E}(D) \), mean
  - \( \tilde{W} = \text{diag}(\mu) \), weight matrix
  - \( \tilde{z} = X\tilde{\alpha} + \tilde{W}^{-1}(d - \tilde{\mu}) \), working variable
  - \( \tilde{\alpha}, \tilde{\mu}, \tilde{\Gamma} \) denotes current estimate, \( \hat{\alpha}, \hat{\mu}, \hat{\Gamma} \) updated estimate.

CLM Algorithm

- Model matrix, link and error as before
- Algorithm: Modified Newton-Raphson:
  \[ \tilde{X}'\tilde{W}\tilde{X}\tilde{\alpha} = \tilde{X}'\tilde{W}\tilde{z} \]
  \[ \tilde{X} = \tilde{W}^{-1}\tilde{\Gamma}\tilde{X}, \]
  where \( \tilde{W} = \text{diag}(\tilde{\mu}), \quad C = \text{diag}(e), \quad \tilde{\gamma} = \exp(X\tilde{\alpha}), \quad \tilde{\Gamma} = \text{diag}(\tilde{\gamma}), \quad \tilde{z} = \tilde{X}\tilde{\alpha} + \tilde{W}^{-1}(d - \tilde{\mu}). \]
GLM & CLM Paradigms

**GLM:** $\alpha \xrightarrow{X} \eta \xrightarrow{h(\cdot)} \mu$

**CLM:** $\alpha \xrightarrow{X} \eta \xrightarrow{h(\cdot)} \gamma \xrightarrow{C} \mu$

**Makeham as a CLM**

$D_i \sim \mathcal{P}[e_i (A + \exp(\theta_0 + \theta_1 x))], \; i = 1, \ldots, n$

$= \mathcal{P}[e_i (\exp(\delta) + \exp(\theta_0 + \theta_1 x))], \; A > 0$

$\Rightarrow \mathbb{E}(D) = \mu = C\gamma, \; \gamma = \exp(X\alpha)$

where

$C = [C_1 : C_2], \; C_1 = C_2 = \text{diag}(e)$

$C\gamma = C_1\gamma_1 + C_2\gamma_2, \; \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$

$= C_1 \exp(X_1\delta) + C_2 \exp(X_2\theta)$

$\exists$

$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \; X_1 = 1_n, \; X_2 = [1_n : x], \; \alpha = \begin{bmatrix} \delta \\ \theta_0 \\ \theta_1 \end{bmatrix}$

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The Pattern of Human Mortality

Gompertz & Makeham models

Makeham $\delta = 0.00013$

Infant/Youth
Accident
Senescence

Log(mortality) vs. Age

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Heligman-Pollard Model

\[ \frac{q_x}{1 - q_x} = f_I(x) + f_S(x) + f_A(x) \]

- \( \log f_I(x) = A(x + B)^C \), infant/child mortality is a power of \( x \)
- \( \log f_S(x) = G + Hx \), underlying mortality is linear in \( x \) (Gompertz)
- \( \log f_A(x) = D - E(\log x - \log F)^2 \), accident hump is quadratic in \( \log x \)

Consider Poisson model for the force of mortality

\[ \lambda_x = \frac{\mu_x}{e_x} = f_I(x) + f_S(x) + f_A(x) \]

With \( C = 1 \) we have a 3-component CLM or

SPE: Sum of Parametric Exponentials

Heligman-Pollard Model as CLM

\[ \mu = C \exp(X\alpha) = C_I \exp(X_I\alpha_I) + C_S \exp(X_S\alpha_S) + C_A \exp(X_A\alpha_A) \]

where

\[ C = [C_I : C_S : C_A], \quad C_I = C_S = C_A = \text{diag}(e) \]

\[ X = \text{blockdiag}\{X_I : X_S : X_A\} \]

\[ X_I = X_S = [1_n : x] \]

\[ X_A = [1_n : \log x : (\log x)^2] \]
The Method of $P$-splines

- Regression basis of $B$-splines:
  \[ B = \{ B_1, \ldots, B_c \} \]

- Smoothing penalty:
  \[ \lambda \left\{ (\alpha_1 - 2\alpha_2 + \alpha_3)^2 + \ldots + (\alpha_{c-2} - 2\alpha_{c-1} + \alpha_c)^2 \right\} \]
  \[ = \alpha' P \alpha, \quad P = \lambda D'D \]

- Estimation by penalized likelihood:
  \[ \ell_p(\alpha; \lambda) = \ell(\alpha) - \frac{1}{2} \alpha' P \alpha \]

- Choice of $\lambda$: minimize AIC, BIC, GCV

Penalized GLM or PGLM

- Model matrix: $B = [B_1(x), \ldots, B_c(x)]$
- Link: log
- Error: Poisson
- Penalty: $P = \lambda D'D$
- Algorithm: Newton-Raphson:
  \[ \left( B'WB + P \right) \hat{\alpha} = B'W \tilde{z} \]
Penalized CLM or PCLM

- Model matrix: $B$
- Mean: $\mathbb{E}(D) = \mu = C\gamma$, $\gamma = \exp(B\alpha)$, $C = \text{diag}(e)$
- Penalty: $P$
- Algorithm: Newton-Raphson:

$$
(\tilde{B}'\tilde{W}\tilde{B} + P)\hat{\alpha} = \tilde{B}'\tilde{W}\tilde{z}
$$

where $\tilde{B} = \tilde{W}^{-1}CTB$.

Two problems

- Adaptive smoothing required.
- Components not identified.

Solution

- Additive model with three components, as HP.
- Components smoothed separately

A Smooth 3-component Model of Mortality

$F$ of $M = \lambda = f_I(x) + f_S(x) + f_A(x)$

$$
\mathbb{E}(D) = \mu = C_I\gamma_I + C_S\gamma_S + C_A\gamma_A
$$

- $f_I(x) = \gamma_I = \exp(B_I\alpha_I)$, smooth, monotonic decreasing
- $f_S(x) = \gamma_S = \exp(B_S\alpha_S)$, smooth, monotonic increasing
- $f_A(x) = \gamma_A = \exp(B_A\alpha_A)$, smooth, concave

CLM: $\mu = C\gamma$, $\gamma = \exp(B\alpha)$

SSE: Sum of Smooth Exponentials
\[ \gamma = \begin{bmatrix} \gamma_I \\ \gamma_S \\ \gamma_A \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_I \\ \alpha_S \\ \alpha_A \end{bmatrix}, \]

\[ B = \begin{bmatrix} B_I & : & 0 & : & 0 \\ 0 & : & B_S & : & 0 \\ 0 & : & 0 & : & B_A \end{bmatrix}, \]

\[ C = [C_I : C_S : C_A] \]

where

\[ C_I = \begin{bmatrix} \text{diag}(e_I) \\ 0 \end{bmatrix}, \quad C_S = \text{diag}(e), \quad C_A = \begin{bmatrix} \text{diag}(e_A) \\ 0 \end{bmatrix}. \]

**Shape penalties**

- Infant, monotonic decreasing: \( a_i < a_{i-1} \)
  \[ \kappa_I D' W_I D, \quad w_{ii} = 1, \text{if } a_i \geq a_{i-1}, \quad 0 \text{ otherwise.} \]

- Senescent, monotonic increasing: \( a_i > a_{i-1} \)
  \[ \kappa_S D' W_S D, \quad w_{ii} = 1, \text{if } a_i \leq a_{i-1}, \quad 0 \text{ otherwise.} \]

- Accident, concave: \( a_i - 2a_{i-1} + a_{i-2} < 0 \)
  \[ \kappa_A D' W_A D, \quad w_{ii} = 1, \text{if } a_i - 2a_{i-1} + a_{i-2} \geq 0, \quad 0 \text{ otherwise.} \]

- Strong penalties: \( \kappa_I = \kappa_S = \kappa_A = 10^5. \)

**Penalty matrix**

\[ P_I = \lambda_I D_I'D_I + \kappa_I D_I'D_{I2} \]
\[ P_S = \lambda_S D_S'D_S + \kappa_S D_S'D_{S2} \]
\[ P_A = \lambda_A D_A'D_A + \kappa_A D_A'D_{A2} \]

\[ P = \begin{bmatrix} P_I & : & 0 & : & 0 \\ 0 & : & P_S & : & 0 \\ 0 & : & 0 & : & P_A \end{bmatrix}. \]

Optimize over \( \lambda_I, \lambda_S \) and \( \lambda_A \).

**Initial values**

Fit to subsets of data.

**Questions?**

- How do we force smoothness?
- How do we force monotonicity and concavity?
- How do we overcome identifiability problems?

**Answers!**

- Smoothness penalties on \( \alpha_I, \alpha_S \) and \( \alpha_A \).
- Shape penalties on \( \alpha_I, \alpha_S \) and \( \alpha_A \).
- Good initial estimates of \( \alpha_I, \alpha_S \) and \( \alpha_A \).
The Pattern of Human Mortality

Smooth log(mortality) for Swiss males in 1980

Smooth force of mortality for Swiss males in 1980

Accident component on original scale
A general CLM for Poisson data

Model
\[ y \sim \mathcal{P}(\mu), \quad \mathbb{E}(y) = \mu, \quad \mu = C\gamma, \quad \gamma = h(\eta), \quad \eta = X\alpha. \]

Algorithm
\[
\tilde{X}'\tilde{W}\tilde{X}\tilde{\alpha} = \tilde{X}'\tilde{W}\tilde{z} \\
\tilde{X} = \tilde{W}^{-1}C\Delta X,
\]
where \( \tilde{W} = \text{diag}(\tilde{\mu}), \quad \tilde{\Delta} = \text{diag}(h'(\tilde{\eta}_i)), \)
\[
\tilde{z} = \tilde{X}\tilde{\alpha} + \tilde{W}^{-1}(y - \tilde{\mu}).
\]

Other Applications of CLM

1. Digit preference: heaping of age, height, weight, year of menopause, etc.
2. Smoothing variable width histograms
3. Mixtures of distributions
4. Additive structures on a transformed scale
5. Grouped data, eg, normal
CLM for Grouped Normal Data

\[ \alpha = \begin{bmatrix} -\mu / \sigma \\ 1 / \sigma \end{bmatrix}, \quad \eta = X \alpha = \begin{bmatrix} \frac{x_1 - \mu}{\sigma} \\ \frac{x_2 - \mu}{\sigma} \\ \vdots \end{bmatrix}, \]

\[ \gamma = h(\eta) = \Phi(\eta), \quad h'(\eta) = \phi(\eta), \quad \Delta = \text{diag}\{\phi(\eta)\}, \]

\[ \mu = C \gamma, \quad C = N \begin{bmatrix} 1 & 0 & 0 & \ldots \\ -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, N = \text{sample size}. \]

Results

Full MLE with multinomial: \( \hat{\mu} = 119.146, \hat{\sigma} = 22.924 \)

CLM with Poisson: \( \hat{\mu} = 119.148, \hat{\sigma} = 22.927 \)
References


