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Comparison theorems and variable speed waves for a scalar reaction–diffusion equation

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This paper concerns the reaction–diffusion equation $u_t = u_{xx} + u^2(1-u)$. Previous numerical solutions of this equation have demonstrated various different types of wave front solutions, generated by different initial conditions. In this paper, the authors use a phase-plane form of comparison theorems for partial differential equations (PDEs) to confirm analytically these numerical results. In particular, they show that initial conditions with an exponentially decaying tail evolve to the unique exponentially decaying travelling wave, while initial conditions with algebraically decaying tails evolve either to an algebraically decaying travelling wave, or to the exponentially decaying wave, or to a perpetually accelerating wave, dependent upon the exact form of the decay of the initial conditions. We then focus on the case of accelerating waves and investigate their form in more detail, by approximating the full equation in this case with a hyperbolic PDE, which we solve using the method of characteristics. We use this approximate solution to derive a leading-order approximation to the wave speed.

1. Introduction

1.1. Background

Wave front solutions to scalar reaction–diffusion equations have been studied in the context of mathematical biology for many years. Work in this area dates back to the papers of Kolmogorov *et al.* [9] and Fisher [6] in 1937, on what is now known as the Fisher equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{1.1}$$

In [6], Fisher used this equation to study the spread of an advantageous gene, with density $u(x,t)$ at time t and position x in a one-dimensional spatial domain. A wave front solution of this equation is a smooth, moving transition between its two steady-states, $u = 0$ and $u = 1$, which approaches one steady-state as $x \to +\infty$ and the other as $x \to -\infty$. The Fisher equation is still used today as a very simple model in various applications, including ecological invasion [8] and wound healing [20]. Many of the more complicated reaction–diffusion models—now in widespread use in the study of a variety of wave-like phenomena in numerous areas of biology—have strong mathematical similarities to the Fisher equation.

Fisher's equation has been studied very extensively and it is well known that the equation has a travelling wave solution of constant shape and speed for all speeds $c \geqslant 2$ [9]. Moreover, for initial data that satisfy $u \to 1$ as $x \to -\infty$ and $u \sim A\exp(-\xi x)$ as $x \to +\infty$, the solution approaches the wave of speed

$$c = \begin{cases} \xi + 1/\xi & \text{for } \xi \leqslant 1, \\ 2 & \text{for } \xi \geqslant 1 \end{cases}$$

as $t \to \infty$ [10, 12, 18], while initial data with compact support approach the wave of minimum speed 2 [9, 12, 14].

This paper is concerned with the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u), \tag{1.2}$$

which was studied numerically in [19], and using asymptotic methods in [17], particularly with respect to initial conditions that decay algebraically to $u = 0$, rather than the much more frequently studied exponential decay. A wide range of wave front solutions, generated by different initial conditions, were observed by Sherratt and Marchant [19], using numerical solutions of this equation. A summary of their results can be found in § 1.3; the key new qualitative phenomenon identified in [19] was accelerating wave fronts. In [17], Needham and Barnes use matched asymptotic expansions to obtain the structure of the solution at large times. Very recently, Malham and Oliver [13] have shown that some aspects of this behaviour can be extended to systems of reaction–diffusion equations.

Our aim in this paper is to prove results on the dependence of wave form on initial conditions that are suggested by numerical solutions, by using a comparison theorem method that was developed by Fife and McLeod [5] for use in a phase-plane. In this first section, we describe the methods in [5] and adapt them for use with our equation (1.2). In § 2, we use the method to show that exponentially decaying initial conditions must evolve to the unique exponentially decaying travelling wave. In §§ 3, 4 and 5, we look at the various types of algebraically decaying initial conditions, and their evolution to either exponentially or algebraically decaying travelling waves, or to variable speed waves. In § 6, we concentrate on the case of accelerating waves, and investigate their form in more detail. We do this by neglecting a term in the governing equation, which we argue is small (although we do not attempt to justify formally the applicability of this simplification), and then study the solutions of the resulting equation, in order to gain some insight into the behaviour of the accelerating waves.

1.2. Existence of travelling waves

Travelling wave solutions to reaction–diffusion equations are transition fronts that move with constant shape and speed, connecting two steady-states of the kinetic term. This means that they can be studied by converting the reaction–diffusion equations into travelling wave ordinary differential equations (ODEs), given by looking for solutions that are functions of the similarity variable $z = x + ct$, where c is the wave speed. Using this, equation (1.2) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}z} = V, \qquad \frac{\mathrm{d}V}{\mathrm{d}z} = cV - U^2(1 - U), \tag{1.3}$$

where $u(x,t) = U(z)$, and we have the boundary conditions $U(-\infty) = 0$ and $U(+\infty) = 1$ if $c > 0$ (or vice versa if $c < 0$). Without loss of generality, we will take $c > 0$, giving a monotonically increasing wave. Changing the sign of c is equivalent to changing the sign of x (so that the wave is just reflected in the u-axis).

The system of two coupled ODEs (1.3) can then be studied using phase-plane techniques. In [19], the authors take $c > 0$ and then use the following argument to show the existence of travelling waves. The system (1.3) has two steady-states, $(0,0)$ and $(1,0)$. Linearization shows that the second of these steady-states is a saddle point, and has just one trajectory approaching it from the first quadrant. This trajectory will correspond to a travelling wave if it starts at $(0,0)$. The steady-state $(0,0)$ has one unstable eigenvector, $(1,c)$, with eigenvalue c, and one zero eigenvalue, with an unstable centre manifold $V = U^2/c$.

A simple argument is given in [19], showing that there can be only one value of $c > 0$ for which the trajectory going to $(1,0)$ leaves $(0,0)$ along the unstable eigenvector $(1,c)$. For this case, this speed is shown to be $c_{\text{crit}} = 1/\sqrt{2}$. For values of c less than this critical value, the trajectory to $(1,0)$ starts at infinity, so no travelling waves exist. For values of c greater than the critical value, the trajectory to $(1,0)$ leaves $(0,0)$ along the centre manifold, so travelling waves do exist for all these speeds. In summary, we have

$$\begin{aligned} c < c_{\text{crit}} &\quad\Rightarrow\quad \text{no travelling waves exist,}\\ c = c_{\text{crit}} &\quad\Rightarrow\quad \text{exponentially decaying travelling wave,}\\ c > c_{\text{crit}} &\quad\Rightarrow\quad \text{algebraically decaying travelling waves.} \end{aligned}$$

1.3. Summary of numerical results

Below is a summary of the observations from [19], where the authors solved (1.2) numerically from various initial conditions.

(i) Initial conditions that decay to zero exponentially, with $u(x,0) \sim A\exp(-\xi x)$ as $x \to \infty$, evolve rapidly to the exponentially decaying minimum speed wave, where the speed $c = 1/\sqrt{2} = c_{\text{crit}}$ (see figure 1*a*).

(ii) Initial conditions that decay to zero algebraically, with $u(x,0) \sim A/x$ as $x \to \infty$, evolve rapidly to algebraically decaying travelling waves with speed A if $A > 1/\sqrt{2}$. Otherwise, they evolve to a wave moving with the minimum speed of c_{crit} (see figure 1*b*).

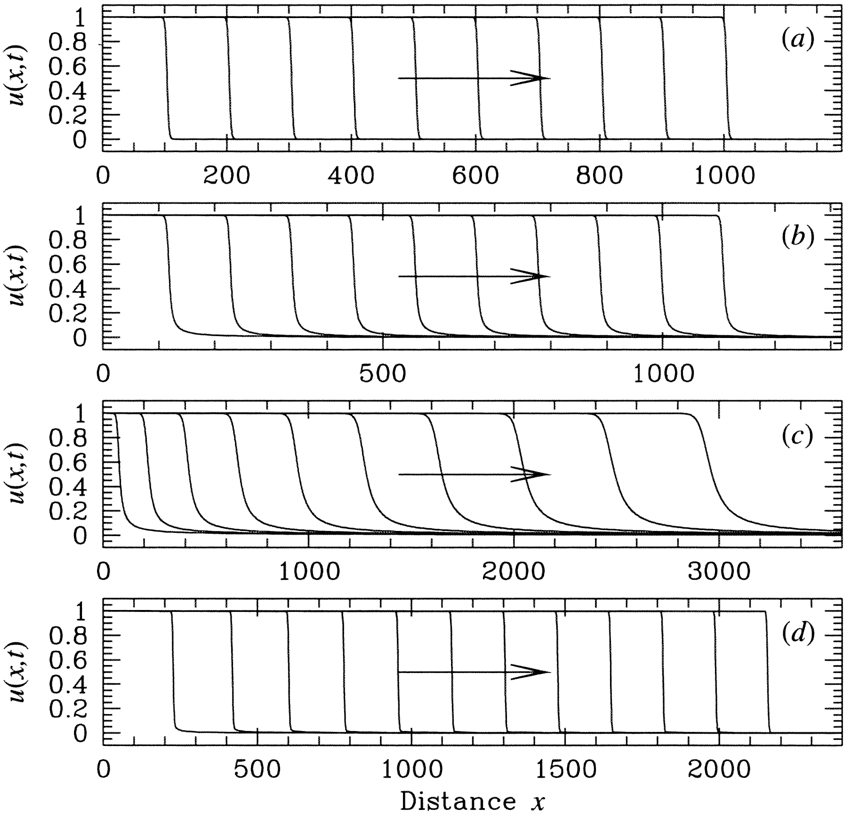

Figure 1. Graphs showing the results of numerical simulations of (1.2) from the various different initial conditions described in § 1.3: (*a*) exponentially decaying travelling wave; (*b*) algebraically decaying travelling wave; (*c*) accelerating wave; and (*d*) decelerating wave. The solutions were calculated using the method of lines and Gear's method.

(iii) Initial conditions that decay to zero algebraically, with $u(x,0) \sim A/x^{\alpha}$ as $x \to \infty$, $\alpha < 1$, evolve to an accelerating wave, with a slope that becomes shallower as time increases. That is, the speed depends on the height of the wave front, as well as time—higher parts of the front travel more slowly than lower parts (see figure 1*c*).

(iv) Initial conditions that decay to zero algebraically, with $u(x,0) \sim A/x^{\alpha}$ as $x \to \infty$, $\alpha > 1$, evolve to a decelerating wave, where the speed gradually approaches the minimum speed, and the slope becomes steeper as time increases. Thus in this case, the speed is an increasing function of u, but the speed at all heights approaches the value c_{crit} asymptotically (see figure 1*d*).

The object of this paper is to prove that the long-time behaviour of solutions from these various different initial conditions is as observed numerically in [19], by using comparison theorems for reaction–diffusion equations, and the method of sub- and super-solutions in the phase-plane. This method was developed in [5], where the authors prove convergence to the travelling wave (from initial conditions with a certain form) for the equation

$$u_t = u_{xx} + f_{\text{cubic}}(u), \tag{1.4}$$

where the kinetic term f_{cubic} has 'cubic form', the prototype kinetics being, for $\beta \in (0,1)$, $u(u-\beta)(1-u)$. Here, the subscripts x and t denote partial derivatives. For such kinetics, equation (1.4) has a unique travelling wave joining $u = 0$ and $u = 1$. This is because there is only one value of c for which a trajectory in the phase-plane joins $(0,0)$ to $(1,0)$, which are both saddle points (see [4, pp. 341–342] or [16] for proofs of this). Our task in studying (1.2) is somewhat more complicated than that for (1.4), since in our case there is a trajectory joining $(0,0)$ and $(1,0)$, corresponding to a travelling wave, for all c greater than or equal to the critical value c_{crit}.

1.4. Comparison theorems in the phase-plane

In this subsection, we give a brief summary of the comparison theorem methods developed in [5]. As mentioned above, the proofs of convergence to travelling waves in [5] are not done in the conventional way using $u(x,t)$. Instead, the u–p phase-plane is used, where $p(u,t) = u_x(x,t)$. Thus the authors transform to u, t as independent variables, with p as the dependent variable. (This can be done because we are concerned with solutions that are monotonic in u; note that solutions with monotone initial conditions remain monotone [5].) The partial differential equation (PDE) satisfied by $p(u,t)$ is derived by first differentiating the equation $u_t = u_{xx} + f(u)$ with respect to x to give $u_{tx} = u_{xxx} + f'(u)u_x$. Then all of the partial derivatives of u with respect to x and t need to be determined in terms of the partial derivatives of p with respect to u and t, using $u_x = p$. This gives

$$\begin{aligned} u_{xx} &= pp_u, \\ u_{xxx} &= p^2 p_{uu} + pp_u{}^2, \\ u_{tx} &= p_u u_t + p_t \\ &= p_u(u_{xx} + f(u)) + p_t \\ &= p_u(pp_u + f(u)) + p_t \end{aligned}$$

(see [5] for further details). Substitution of these into $u_{tx} = u_{xxx} + f'(u)u_x$ then gives

$$p_t = p^2 \left(p_u + \frac{f(u)}{p} \right)_u. \tag{1.5}$$

If $u(x,0) = \phi(x)$ are the monotonic initial conditions for the u equation, then the corresponding initial conditions for the p equation are $p(u,0) = \Phi(u)$, where $\Phi(u) = \phi'(\phi^{-1}(u))$ (see [5, pp. 286–289]). The boundary conditions satisfied by a travelling wave solution are $p(0,t) = p(1,t) = 0$.

A travelling wave solution is then a steady-state of (1.5); that is, a solution $P(u)$ with

$$P_{uu} + \left(\frac{f(u)}{P} \right)_u = 0$$

or

$$P_u + \frac{f(u)}{P} = c,$$

where the constant c is easily identified as the wave speed (by comparison with the travelling wave equations (1.3)). Note that another travelling wave solution is then $-P(u)$, which has speed $-c$, reflecting the fact that wave fronts can travel in either direction. Without loss of generality, we consider only monotonically increasing waves, which move in the negative x-direction.

A sub-solution is a function $\underline{p}(u,t)$ such that

$$\frac{\underline{p}_t}{\underline{p}^2} \geqslant \underline{p}_{uu} + \left(\frac{f(u)}{\underline{p}}\right)_u \geqslant 0 \quad \text{and} \quad \underline{p}(u,0) = \underline{\Psi}(u) \leqslant \Phi(u).$$

Thus a sub-solution is non-decreasing with t, at fixed u. Similarly, a super-solution is a function $\overline{p}(u,t)$ such that

$$\frac{\overline{p}_t}{\overline{p}^2} \leqslant \overline{p}_{uu} + \left(\frac{f(u)}{\overline{p}}\right)_u \leqslant 0 \quad \text{and} \quad \overline{p}(u,0) = \overline{\Psi}(u) \geqslant \Phi(u),$$

and is non-increasing with t at fixed u. (Recall that $\Phi(u) = p(u,0)$.)

Comparison theorems are based on maximum principles for PDEs, and there is a long history of their application to reaction–diffusion equations. (The books by Fife [3] and Britton [2] describe the derivation of comparison theorems from maximum principles, and some of their applications.) For the equation $Nu \equiv u_t - u_{xx} - f(u) = 0$, the basic method uses the fact that $\underline{u} \leqslant \overline{u}$, where $\underline{u}$ is a sub-solution, which satisfies $N\underline{u} \leqslant 0$, and $\overline{u}$ is a super-solution, which satisfies $N\overline{u} \geqslant 0$. In [5], this form of comparison theorem is converted into those for the p PDE (1.5). Thus, by comparison theorems, we know that $\underline{p}(u,t) \leqslant p(u,t) \leqslant \overline{p}(u,t)$ for all times t. So, by constructing suitable sub- and super-solutions, one can determine the time evolution of $p(u,t)$ from initial conditions $\Phi(u)$.

1.5. The u–p phase-plane

In [5], the authors show that in order to prove $p(u,t) \to P(u)$ for (1.4), it is sufficient to find $\underline{\Psi}(u)$ and $\overline{\Psi}(u)$ satisfying the conditions for sub- and super-solutions, respectively. We give a statement of this result below, along with an outline of the proof.

THEOREM 1.1 (See [5]). *Let $f \in C^1[0,1]$ with $f(0) = f(1) = 0$. For some $\beta \in (0,1)$, suppose that $f < 0$ in $(0,\beta)$, and that in $(\beta,1)$ either $f > 0$ or $f \equiv 0$. Let P be the unique positive solution of $P_{uu} + (f/P)_u = 0$ on $(0,1)$ with $P(0) = P(1) = 0$, and let p be the positive solution of the initial value problem $p_t = p^2(p_{uu} + (f/p)_u)$ for $0 < u < 1$, $t > 0$, $p(0,t) = p(1,t) = 0$ and $p(u,0) = \Phi(u) > 0$ for $0 < u < 1$. Then*

$$p(u,t) \to P(u) \quad \text{as } t \to \infty,$$

uniformly for $u \in [0,1]$.

Outline of the proof. (The full proof can be found in [5].) To begin, choose any interval (A,B), $0 < A < \beta < B < 1$, and find a $\underline{\Psi}$ satisfying

$$\begin{aligned} \underline{\Psi}(A) = \underline{\Psi}(B) = 0, \quad \underline{\Psi} > 0 \quad &\text{on } (A,B), \\ \underline{\Psi}'' + (f/\underline{\Psi})' \geqslant 0 \quad &\text{on } (A,B) \end{aligned}$$

and $\underline{\Psi} \leqslant \Phi$. Now let $\underline{p}(u,t)$ be the solution of (1.5) from initial conditions $\underline{p}(u,0) = \underline{\Psi}(u)$. Then

$$\underline{p}(u,t) \to Q(u) \quad \text{as } t \to \infty,$$

where $Q(u)$ satisfies

$$Q(A') = Q(B') = 0, \quad Q > 0 \quad \text{on } (A', B'),$$
$$Q_{uu} + (f/Q)_u = 0,$$

where $0 \leqslant A' \leqslant A < B \leqslant B' \leqslant 1$. By comparison theorems, $p(u,t) \geqslant \underline{p}(u,t)$ for all times $t \geqslant 0$, so that

$$\lim_{t\to\infty} p(u,t) \geqslant Q(u).$$

However, the choice of A and B was arbitrary, so that we have

$$\lim_{t\to\infty} p(u,t) \geqslant P(u),$$

as one possible sub-solution limit $Q(u)$ is then $P(u)$. Next, find a $\overline{\Psi}$ satisfying

$$\overline{\Psi}(0) = \overline{\Psi}(1) = 0, \quad \overline{\Psi} > 0 \quad \text{on } (0,1),$$
$$\overline{\Psi}'' + (f/\overline{\Psi})' \leqslant 0 \quad \text{on } (0,1)$$

and $\overline{\Psi} \geqslant \Phi$. Let $\overline{p}(u,t)$ be the solution of (1.5) from the initial conditions $\overline{p}(u,0) = \overline{\Psi}(u)$. Then

$$\overline{p}(u,t) \to P(u) \quad \text{as } t \to \infty,$$

as $\overline{p}(u,t)$ is non-increasing with t at fixed u, with $\overline{p}(0,t) = \overline{p}(1,t) = 0$, so that it must evolve to a fixed point—the travelling wave P. By comparison theorems, $p(u,t) \leqslant \overline{p}(u,t)$ for all times $t \geqslant 0$, so that

$$\lim_{t\to\infty} p(u,t) \leqslant P(u).$$

Thus we have

$$P(u) \leqslant \lim_{t\to\infty} p(u,t) \leqslant P(u) \quad \Rightarrow \quad \lim_{t\to\infty} p(u,t) = P(u),$$

as required. □

When translated from $p(u,t)$ to $u(x,t)$, this result states that

$$u(x,t) - U(x - ct - \gamma(t)) \to 0 \quad \text{as } t \to \infty,$$

uniformly in x, where U is the travelling wave solution of (1.3), and $\gamma \in C^1[0,\infty)$ satisfies $\gamma'(t) \to 0$ as $t \to \infty$.

As stated previously, the theorem relies on there being a unique travelling wave solution joining $u = 0$ and $u = 1$ for the reaction–diffusion equation with a cubic form for the kinetic term. To study the time evolution from initial conditions in our case, where $f(u) = u^2(1-u)$, we need to construct, for each type of initial condition, sub- and super-solutions with only one travelling wave solution between them. This is possible because the various types of initial conditions can be distinguished in the phase-plane. Specifically, we have the following.

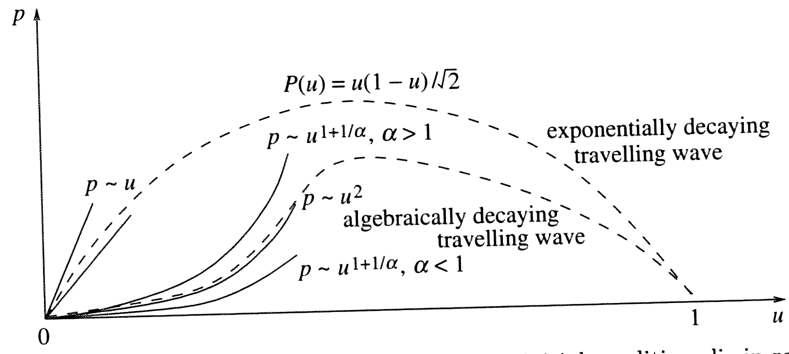

Figure 2. A schematic illustration of where the various initial conditions lie in relation to the two types of travelling wave, close to $u = 0$, in the u–p phase-plane.

(i) Initial conditions that decay as $u \sim Ae^{\xi x}$ as $x \to -\infty$ have $p \sim \xi u$ as $u \to 0$. For our case, with $f(u) = u^2(1-u)$, we know that the exponentially decaying travelling wave is $P(u) = u(1-u)/\sqrt{2}$.

(ii) Initial conditions that decay as $u \sim A/(-x)^{\alpha}$ as $x \to -\infty$ have $p \sim (\alpha/A^{1/\alpha})u^{1+1/\alpha}$ as $u \to 0$.

Figure 2 illustrates the relationship between these initial conditions in the u–p phase-plane.

The corresponding theorem in our case then follows.

THEOREM 1.2 (Adapted). *Let* $f \in C^1[0,1]$, *with* $f(0) = f(1) = 0$, $f(u) > 0$ *for* $u \in (0,1)$. *Let* p *be the positive solution of the initial value problem* $p_t = p^2(p_{uu} + (f/p)_u)$ *for* $0 < u < 1, t > 0$, *with* $p(0,t) = p(1,t) = 0$ *and* $p(u,0) = \Phi(u) > 0$ *for* $0 < u < 1$. *Then*

$$p(u,t) \to P(u) \quad \textit{as } t \to \infty,$$

uniformly for $u \in [0,1]$, *if there exist* $\underline{\Psi}$, $\overline{\Psi}$ *such that* P *is the unique travelling wave solution satisfying*

$$\underline{\Psi}(u) \leqslant P(u) \quad \textit{for } u \in (0,B)$$

and

$$P(u) \leqslant \overline{\Psi}(u) \quad \textit{for } u \in (0,1),$$

where $0 < B < 1$ *is an arbitrary point such that* $\underline{\Psi}(B) = 0$, *as well as*

$$\underline{\Psi}'' + (f/\underline{\Psi})' \geqslant 0, \quad \underline{\Psi} \leqslant \Phi \quad \textit{for } u \in (0,B)$$

and

$$\overline{\Psi}'' + (f/\overline{\Psi})' \leqslant 0, \quad \overline{\Psi} \geqslant \Phi \quad \textit{for } u \in (0,1).$$

The proof of the above theorem is directly analogous to that in [5]. We use this theorem in later sections to prove that the various initial conditions evolve to the different travelling waves, as summarized in § 1.3.

2. Exponentially decaying initial conditions

To prove that initial conditions that decay exponentially to $u = 0$ evolve to the exponentially decaying travelling wave, we adapt the method used in [5] for a cubic kinetic function. Specifically, we construct the sub- and super-solutions in the case of kinetic $f(u) = u(u - \beta)(1 - u)$, and then consider the limit $\beta \to 0$; in this limit, $f(u) = u^2(1 - u)$, as required. We remark here that the actual definition of 'exponentially decaying' that we will use is that $u_x > \delta u$ close to $u = 0$ for some $\delta > 0$, as there are functions which violate this but which are still exponentially decaying in the usual sense.

2.1. Sub-solution

We now construct the sub-solution for exponentially decaying initial conditions, by following the method in [5, pp. 297–298], for construction of the initial conditions $\underline{\Psi}(u)$ for the sub-solution $\underline{p}(u, t)$.

First we let $f(u) = u(u-\beta)(1-u)$. We start by choosing a function $w \in C^2[\beta, B]$, positive in (β, B) and with $w(B) = 0$ $(\beta < B < 1)$, such that there exists a positive constant k with $(f(u)/w(u))' \geqslant k > 0$ in (β, B). We do this by setting

$$w(u) = \frac{f(u)}{g(u)} = \frac{u(u - \beta)(1 - u)}{g(u)}$$

and choosing g so that $g'(u) \geqslant k$ in (β, B), $g(B) = \infty$ and $g(\beta) > 0$. A simple $g(\cdot)$ that satisfies these conditions is $g(u) = 1/(B - u)$, but using this does not give us the required behaviour as $u \to 0$ in the limit $\beta \to 0$, as it gives $w \sim Bu^2$ as $u \to 0$, which is not above all of the algebraically decaying travelling waves near $u = 0$. We thus take $g(u) = u/(B - u)$ and $k = B/(B - \beta)^2$. Then

$$w(u) = (u - \beta)(1 - u)(B - u).$$

Now we set $\beta = 0$, giving $w(u) = u(1 - u)(B - u)$.

The function w may not work as $\underline{\Psi}$ straight away, as we need it to satisfy the conditions for a sub-solution; $\underline{\Psi}'' + (f/\underline{\Psi})' \geqslant 0$ and $\underline{\Psi} \leqslant \Phi$. We can do this by replacing w with εw, where ε is a small positive constant. Thus if ε is small enough, $\underline{\Psi}(u) = \varepsilon w(u)$ satisfies the conditions for a sub-solution, namely $\varepsilon w'' + (f/w)'/\varepsilon = \varepsilon w'' + g'/\varepsilon \geqslant 0$, since w'' is bounded as w is continuous on $(0, B)$, and $g'(u) \geqslant k > 0$ by construction. Then all we need to do is to ensure that ε is also sufficiently small that $\varepsilon w(u) < \Phi(u)$ for $u \in (0, B)$. Thus we have

$$\underline{\Psi}(u) = \varepsilon u(1 - u)(B - u),$$

which has the form needed near $u = 0$, namely $p \sim \varepsilon B u$, so that it lies above any algebraically decaying travelling waves near $u = 0$.

2.2. Super-solution

For the super-solution we use exactly the same initial condition $\overline{\Psi}(u)$ as in [5, p. 299], namely

$$\overline{\Psi}(u) = K(1 - (u - \tfrac{1}{2})^2),$$

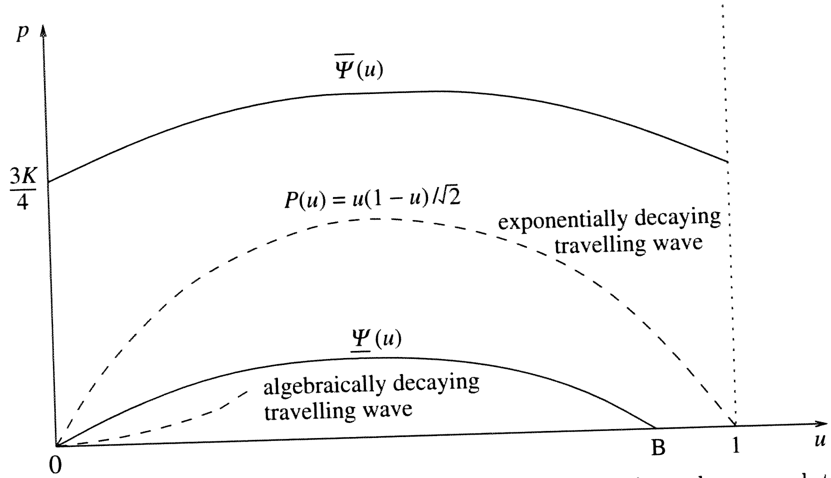

Figure 3. A schematic illustration of the initial conditions for sub- and super-solutions in the u–p phase-plane, for initial conditions which decay to zero exponentially. The existence of these sub- and super-solutions implies that such initial conditions must evolve to the exponentially decaying travelling wave, since all the algebraically decaying travelling waves lie below the sub-solution close to $u = 0$.

where K is a positive constant, along with the boundary conditions $\overline{p}(0,t) = \gamma_1(t)$ and $\overline{p}(1,t) = \gamma_2(t)$, where γ_1 and γ_2 are positive decreasing functions of t such that $\gamma_1(t), \gamma_2(t) \to 0$ as $t \to \infty$. It is easy to see that, if K is large enough, $\overline{\Psi}$ will satisfy the conditions for a super-solution, as $\overline{\Psi}''$ is large and negative whereas $(f/\overline{\Psi})'$ is small, and so $\overline{\Psi}'' + (f/\overline{\Psi})' < 0$. Of course, we also need to ensure that K is sufficiently large that $\Phi(u) < \overline{\Psi}(u)$ for $u \in (0,1)$.

2.3. Conclusion

Figure 3 illustrates the forms of $\overline{\Psi}$ and $\underline{\Psi}$. In effect, what we have done is bounded any exponentially decaying initial conditions away from any algebraically decaying travelling waves, which lie below them, by using a sufficiently small non-decreasing sub-solution.

We also have an upper non-increasing bound. Thus exponentially decaying initial conditions must evolve to the exponentially decaying travelling wave $P(u) = u(1-u)/\sqrt{2}$, which lies between the two bounds.

3. Algebraically decaying initial conditions, $\alpha = 1$

In this section we will attempt to show that any initial conditions with $\Phi \sim u^2/a$ near $u = 0$ evolve to the algebraically decaying travelling wave of speed a if $a > c_{\text{crit}}$, and to the exponentially decaying travelling wave otherwise. This means that we need to construct the sub- and super-solution initial conditions such that $\underline{\Psi}, \overline{\Psi} \sim u^2/a$ as $u \to 0$ if $a > c_{\text{crit}}$. For the case $a \leqslant c_{\text{crit}}$, the sub-solution will work in the same form, but we need not construct a super-solution of that form that works in this case—the super-solution for the exponentially decaying waves is then sufficient.

It should be remarked that while working in the u–p phase-plane yields effective sub-solutions, it does not seem to yield effective super-solutions in the case $\alpha = 1$, for reasons that we describe later. We therefore start by obtaining sub-solutions in the u–p plane, but work mainly in the x–u plane for super-solutions.

3.1. Sub-solution

We will consider the case for $a > c_{\text{crit}}$, and show that we can find a suitable sub-solution working in the u–p phase-plane. We write

$$\Phi(u) = \frac{u^2}{a}(1-u)(1+g(u)), \quad g(u) \to 0 \quad \text{as } u \to 0, \tag{3.1}$$

and we will show that this is itself a sub-solution, at least for u sufficiently small, provided that $g' \leqslant 0$, $g'' \geqslant 0$. We can then make a sub-solution $\underline{\Psi}$ for any Φ as in (3.1) by taking $\underline{\Psi}(u) = u^2(1-u)(1+g^*(u))/a$, where g^* is a convex function ($g^{*\prime\prime} \geqslant 0$) lying below g on $(0,1)$ and satisfying $g^*(0) = 0$ and $g^{*\prime} \leqslant 0$, and so $\underline{\Psi}$ satisfies all of the conditions required for it to be a sub-solution for Φ.

Thus what we have to do is prove, at least for u sufficiently small, that

$$\Phi'' + (f/\Phi)' \geqslant 0.$$

Substituting (3.1) into this gives the condition

$$\frac{2}{a}(1-3u)(1+g) + \frac{2g'}{a}(2u-3u^2) + \frac{u^2}{a}(1-u)g'' - \frac{ag'}{(1+g)^2} \geqslant 0,$$

and this is clearly so if $g' \leqslant 0$, $g'' \geqslant 0$ (u small).

This gives us a sub-solution over $(0,\nu)$, say. To obtain a sub-solution over $(0,1)$, we note that, by multiplying g if necessary by a large constant, C say, we may arrange that $1+g$ vanishes at ν. Then we set

$$\underline{\Psi}(u) = \begin{cases} \underline{\Psi}_\nu, & u \in (0,\nu_1), \\ \underline{\Psi}_B, & u \in (\nu_1, B), \end{cases} \tag{3.2}$$

where ν_1 ($0 < \nu_1 < \nu$) is determined below, $\underline{\Psi}_\nu$ is the sub-solution over $(0,\nu)$ obtained above and, for arbitrary B (near 1),

$$\underline{\Psi}_B = \varepsilon u(1-u)(B-u).$$

We then choose ν_1 sufficiently close to ν and ε sufficiently small that

$$\begin{aligned} &\underline{\Psi}_\nu(\nu_1) = \underline{\Psi}_B(\nu_1), \\ &\underline{\Psi}_\nu'(\nu_1) < 0, \quad \text{recalling that } \underline{\Psi}_\nu(\nu) = 0, \\ &\underline{\Psi}_B(u) < \Phi(u) \quad \text{for } u \in (\nu_1, B). \end{aligned}$$

Since $\underline{\Psi}_B$ satisfies $\underline{\Psi}_B'' + (f/\underline{\Psi}_B)' \geqslant 0$ on (ν_1, B) if ε is sufficiently small, and $\underline{\Psi}_\nu'(\nu_1) < \underline{\Psi}_B'(\nu_1)$, the function given in (3.2) is indeed a sub-solution, since we are taking the supremum at ν_1 of two sub-solutions $\underline{\Psi}_\nu$, $\underline{\Psi}_B$. Since B is arbitrary, we have the required sub-solution.

3.2. Super-solution

Section 3.1 has shown that we can find a sub-solution when all that we assume is that $\Phi \sim u^2/a$ as $u \to 0$, but it is easy to see that we cannot find super-solutions so easily. For a super-solution $\overline{\Psi}$, we must have

$$\overline{\Psi}' + (f/\overline{\Psi}) \quad \text{decreasing},$$

and at $u = 0$ we presumably need $\overline{\Psi} \sim u^2/a$, and so $\overline{\Psi}' + (f/\overline{\Psi}) = a$. Hence, for a super-solution, we need

$$\overline{\Psi}' + (f/\overline{\Psi}) \leqslant a.$$

Since the second term is non-negative, we have $\overline{\Psi}' \leqslant a$, $\overline{\Psi} \leqslant au$, which prevents arbitrarily large super-solutions. We therefore have to work in the x–u plane, and prove the following result.

PROPOSITION 3.1. *Suppose that the initial data ϕ satisfies the following conditions.*

(a) *ϕ lies between two translates of the travelling wave of speed a (greater than c_{crit}) as $x \to -\infty$.*

(b) *$\phi_{xx} - a\phi_x + f(\phi)$ has only a finite number of zeros.*

(c) *Either $\phi_x \sim \phi^2/a$ as $\phi \to 0$ or $\lim_{x\to\infty} \phi(x) = 1$.*

Then the solution converges to a translate of the travelling wave.

Before proving this, we make several remarks.

(1) The assumption that the initial data lie between two travelling waves is only for x sufficiently large and negative. Of course, by moving the translates sufficiently far apart, one can then ensure that it is also true for any bounded x. That it is not necessary to assume it as $x \to \infty$ is part of the proof.

(2) The assumption (a) is achieved if we assume that, as $u \to 0$,

$$\Phi(u) = p_a(u)[1 + \mathcal{O}(u)]. \tag{3.3}$$

For then, since $\Phi(u) = u_x(x, 0)$ and $p_a(u)$ is the derivative of the travelling wave of speed a, we have on integration that, for some constant x_0,

$$u(x, 0) = U(x - x_0) + \mathcal{O}(U^2). \tag{3.4}$$

But the difference between two translates of U is

$$\begin{aligned} U(x - x_1) - U(x - x_2) &= (x_2 - x_1)U'(\xi), \quad \xi \text{ between } x - x_1 \text{ and } x - x_2 \\ &\sim \frac{1}{a}(x_2 - x_1)U^2(\xi), \end{aligned} \tag{3.5}$$

and comparing (3.4) and (3.5) we see that the required assumption is satisfied.

(3) Assumption (a) is the only one required in order to establish a super-solution in the x–u plane, but, at least in our approach, we need (c) as well, in order to obtain a sub-solution. (Of course, the first alternative in (c) is sufficient on its own to give a sub-solution in the u–p plane.)

(4) Assumption (b) can certainly be relaxed, but it is convenient and not severely restrictive.

We first prove two lemmas.

LEMMA 3.2. *If the initial data lie below a travelling wave of speed a (greater than c_{crit}) as $x \to -\infty$ ($u \to 0$), we can find constants z_1, q_0 and μ, the last two positive, such that, for all z and t,*

$$v(z,t) \leqslant U(z - z_1) + q_0 \mathrm{e}^{-\mu t},$$

where U is the travelling wave of speed a and we are using travelling coordinates $z = x + at$, $v(z,t) = u(x,t)$.

Proof. The proof is an adaptation of the one in [4]. We look for a super-solution in the form

$$\overline{v}(z,t) = \begin{cases} U(z - z_0(t)) & \text{for } z < 0, \\ U(z - \xi(t)) + q(t) & \text{for } 0 < z < Z, \\ 1 & \text{for } z > Z, \end{cases}$$

where the various parameters have to be chosen so that $\overline{v}$ is continuous, so that

$$U(0 - z_0(t)) = U(0 - \xi(t)) + q(t), \tag{3.6}$$

$$U(Z - \xi(t)) + q(t) = 1. \tag{3.7}$$

We also require, if $\overline{v}$ is to be a super-solution, that, at $z = 0, Z$, the z-derivative on the left exceeds that on the right. This is automatic at $z = Z$, but at $z = 0$ we need

$$U'(0 - z_0(t)) \geqslant U'(0 - \xi(t)). \tag{3.8}$$

The differential inequalities that $\overline{v}$ must satisfy are trivially satisfied where $\overline{v} = 1$, but we need, for $z < 0$,

$$-z_0' U'(z - z_0(t)) \geqslant 0, \quad \text{i.e. } z_0' \leqslant 0, \tag{3.9}$$

and, for $0 < z < Z$,

$$-\xi' U'(z - \xi) + q' - U'' + aU' - f(U + q) \geqslant 0,$$

or, since U is a travelling wave,

$$-\xi' U' + q' + f(U) - f(U + q) \geqslant 0. \tag{3.10}$$

Finally, we require that

$$\overline{v}(z,0) \geqslant v(z,0). \tag{3.11}$$

To satisfy all of these conditions, we look first at (3.10). Given any fixed small number $\eta > 0$, there exists $\mu(\eta) > 0$ such that $f'(U) < -\mu$ for $U \in [1 - \eta, 1]$. Thus

$$f(U) - f(U + q) = -qf'(U + \theta q) > q\mu,$$

for some θ with $0 < \theta < 1$, and so (3.10) is satisfied for $U \in [1 - \eta, 1]$ if we take $q = q_0 \mathrm{e}^{-\mu t}$ and $\xi' \leqslant 0$. For $\eta \leqslant U \leqslant 1 - \eta$, equation (3.10) is satisfied, since $|f'(U)| \leqslant \kappa$, say, by choosing

$$-\xi' \geqslant q_0(\mu + \kappa)\mathrm{e}^{-\mu t}/\beta,$$

since $U'(s) \geqslant \beta$, say, if $\eta \leqslant U(s) \leqslant 1-\eta$. Thus, specifically, we could take

$$\xi' = -q_0(\mu+\kappa)e^{-\mu t}/\beta. \tag{3.12}$$

(We will find that we do not need to satisfy (3.10) for $U < \eta$.)

This determines q and ξ, except for q_0 and a constant of integration in ξ. We will, in fact, choose q_0 to be small, and then, from (3.12), we see that the range of ξ, $\xi(0)$–$\xi(\infty)$, is small. Once q and ξ are determined, $z_0(t)$ is determined from (3.6), and Z from (3.7). Equation (3.6) certainly implies that $\xi \geqslant z_0$, and indeed, for small q, we have $(\xi - z_0)U'(-\xi) \sim q$.

We still have a constant of integration in ξ, say $\xi(0)$, and we will choose it sufficiently large and positive that, for all ξ in its (small) range, we have $U(-\xi) > \eta$, $U''(-\xi) > 0$ and $f'(U) > 0$. Then (3.8) is satisfied, since, with $\xi \geqslant z_0$,

$$\begin{aligned} U'(-z_0) - U'(-\xi) &= (\xi - z_0)U''(-\theta), \quad z_0 < \theta < \xi \\ &\geqslant 0. \end{aligned}$$

To satisfy (3.9), we differentiate (3.6) to obtain

$$-z_0'U'(-z_0) = -\xi'U'(-\xi) + q' \geqslant f(U+q) - f(U) \geqslant 0,$$

so that $z_0' \leqslant 0$, as required.

Finally, equation (3.11) is satisfied, since the initial data lie below a travelling wave as $z \to -\infty$. (If necessary, we may translate the super-solution suitably.)

Once the super-solution $\overline{v}$ is established, the truth of the lemma follows immediately. □

LEMMA 3.3. *If the initial data lie above a travelling wave of speed a (greater than c_{crit}) as $x \to -\infty$ ($u \to 0$) and if the initial data satisfy $\phi(\infty) = 1$, then we can find constants z_2, q_0 and μ, the last two positive, such that, for all z and t,*

$$v(z,t) \geqslant U(z - z_2) - q_0 e^{-\mu t}.$$

Proof. The proof follows the lines of the previous proof with only minor modifications. We must find a sub-solution of the form

$$\underline{v}(z,t) = \begin{cases} U(z - z_0(t)) & \text{for } z < 0, \\ U(z - \xi(t)) - q(t) & \text{for } z > 0. \end{cases}$$

Since the initial data lie above a travelling wave and have $\phi(\infty) = 1$, we can satisfy the requirement $\underline{v}(z,0) \leqslant v(z,0)$, if necessary using a suitable translation, with $q(0)$ small. The argument then proceeds as before, by merely reversing most of the inequalities.

The argument will still work if $\phi(\infty)$ is sufficiently close to 1, but whether one can then dispense with any restriction on $\phi(\infty)$ (other than $\phi(\infty) > 0$) is unclear. Of course, if we assume $\Phi(u) \sim u^2/a$ as $u \to 0$, which implies an assumption on derivatives but not on $\phi(\infty)$, then we have a sub-solution from § 3.1. □

To prove the result stated earlier, on convergence to the travelling wave, we see that the solution $v(z,t)$ now lies, for all z and t, between two travelling waves, ignoring an exponentially small term. We also note that the number of zeros of

v_t (initially assumed finite) does not increase as t increases. To see this, note that $w = v_t$ satisfies the equation

$$w_t = w_{zz} - aw_z + f'(u)w, \tag{3.13}$$

and it is a well-known result that the number of zeros of w cannot increase. Briefly, if there are two zeros, $z_1(t)$ and $z_2(t)$, with $w(z, 0) > 0$ for $z_1(0) < z < z_2(0)$, then by applying the maximum principle to (3.13) in the region $z_1(t) < z < z_2(t)$, we see that, for all t, $w(z, t) > 0$, showing that no further zeros can develop. The number of zeros may of course decrease, either by the merger of two zeros or by a zero moving off to infinity, but it cannot increase.

We denote the smallest of the zeros by $z_1(t)$, with $Z_1 = \inf_t z_1(t)$. Then, for $z < Z_1$, we must have $w(z, t) = v_t(z, t) \geqslant 0$, say, so that v is monotonic in t. Since, however, we know that v is essentially bounded by travelling waves, we have that $v(z, t) \to V(z)$, say, for $z < Z_1$ and $t \to \infty$, and it is easy to see, using standard estimates that are given, for example, in [4], that $V(z)$ is of the form $U(z - z_0)$ for some z_0, and that $v_t \to 0$.

We now have to prove the same conclusion for $z > Z_1$, and to do this we use the procedure in [4]; we merely sketch the main argument here. We define the Lyapunov functional

$$F[v] = \int_{Z^*}^{\infty} \mathrm{e}^{-az} \left[\tfrac{1}{2} v_z^2 + \int_v^1 f(s)\, \mathrm{d}s \right] \mathrm{d}z,$$

where $Z^* < Z_1$ is fixed. Because $a > 0$, there is no difficulty over the convergence of the integral. Then

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= \int_{Z^*}^{\infty} \mathrm{e}^{-az} [v_z v_{zt} - f(v) v_t]\, \mathrm{d}z \\ &= -\mathrm{e}^{-az} v_z v_t |_{z=Z^*} - \int_{Z^*}^{\infty} \mathrm{e}^{-az} [v_{zz} - av_z + f(v)] v_t\, \mathrm{d}z. \end{aligned}$$

Defining

$$Q[v] = \int_{Z^*}^{\infty} \mathrm{e}^{-az} [v_{zz} - av_z + f(v)]^2\, \mathrm{d}z \geqslant 0,$$

then

$$\dot{F}[v] + Q[v] = \mathcal{O}(v_t(Z^*, t)).$$

Recalling that $v_t(Z^*, t) \geqslant 0$ and that F and v are bounded, we can integrate and see that

$$\int^{\infty} Q[v](t)\, \mathrm{d}t < \infty,$$

which, since $Q \geqslant 0$, implies that there exists a sequence $\{t_n\}$, $t_n \to \infty$, such that

$$Q[v](t_n) \to 0.$$

Indeed, since we can show that $\dot{Q}$ is bounded, we have, in fact,

$$Q[v](t) \to 0 \quad \text{as } t \to \infty.$$

(We have

$$\dot{Q}[v] = 2 \int_{Z^*}^{\infty} \mathrm{e}^{-az} v_t v_{tt}\, \mathrm{d}z,$$

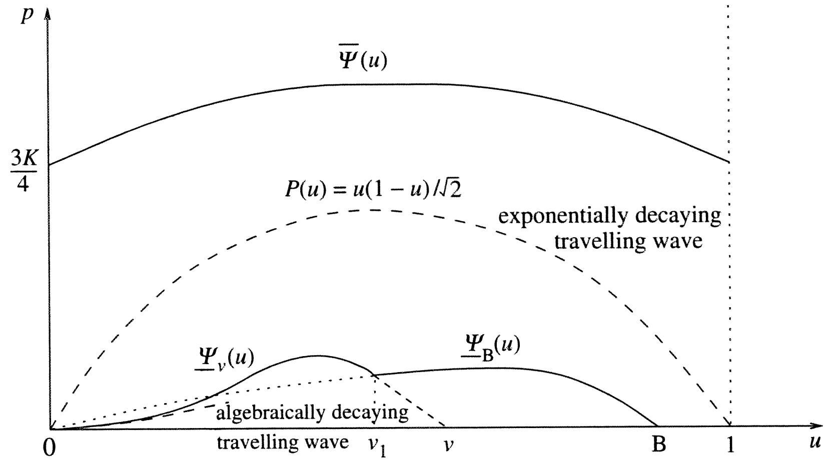

Figure 4. A schematic illustration of the initial conditions for sub- and super-solutions for initial conditions which decay to zero algebraically, as u^2/a with $a \leqslant c_{\text{crit}}$, in the phase-plane. These prove that such initial conditions must evolve to the exponentially decaying travelling wave, as all the algebraically decaying travelling waves lie below the sub-solution initial condition close to $u = 0$.

which is bounded because of standard estimates on v_t, v_{tt}.)

Further, bounds on v and its derivatives (which are easily derivable from the lemmas above) show that, for some sequence $\{t_n\}$, $v(z, t_n)$, together with the necessary derivatives, converge in any bounded z-interval I to a limit function $\tilde{v}(z)$, and that

$$\int_I \mathrm{e}^{-az}[v_{zz} - av_z + f(v)]^2 \, \mathrm{d}z \to \int_I \mathrm{e}^{-az}[\tilde{v}_{zz} - a\tilde{v}_z + f(\tilde{v})]^2 \, \mathrm{d}z = 0.$$

Therefore, $\tilde{v}_{zz} - a\tilde{v}_z + f(\tilde{v}) = 0$, so that $\tilde{v}(z)$ has the form $U(z - z_0)$. Since the compact interval I can be assumed to contain Z^*, we conclude that z_0 is the same as for the interval $z < Z^*$, and so convergence for all z is established.

As stated previously, for the case $a \leqslant c_{\text{crit}}$, we can use the same phase-plane super-solution as for exponentially decaying initial conditions.

3.3. Conclusion

For initial conditions $\Phi \sim u^2/a$ as $u \to 0$ with $a \leqslant c_{\text{crit}}$, we have a lower non-decreasing bound. This has bounded those initial conditions away from all the algebraically decaying travelling waves, which lie below it. We also have an upper non-increasing bound. Thus initial conditions must evolve to the only travelling wave solution that lies between the two bounds—the exponentially decaying travelling wave. Figure 4 illustrates the sub- and super-solution initial conditions in this case.

For initial conditions $\Phi \sim u^2/a$ as $u \to 0$ with $a > c_{\text{crit}}$, we have a lower non-decreasing bound. This has bounded those initial conditions away from any algebraically decaying travelling waves with speeds greater than a, which lie below it. However, we showed that it is impossible to find a phase-plane super-solution for initial conditions with ϕ too steep anywhere, and we thus have to revert to conventional comparison theorems to find a u–x super-solution in this case. In this way, we confine the solution from initial conditions ϕ between two translates of the travelling wave of speed a in the u–x plane (except for an exponentially small term and as $x \to \infty$), and then show evolution towards a translate of this same travelling wave, as required.

4. Algebraically decaying initial conditions, $\alpha > 1$

In this section we will consider algebraically decaying initial conditions with $\alpha > 1$ and construct sub- and super-solutions to show that these initial conditions evolve in the phase-plane to the exponentially decaying travelling wave, $P(u) = u(1-u)/\sqrt{2}$.

4.1. Sub-solution

If we have the algebraically decaying initial conditions $\Phi(u) \sim u^{1+1/\alpha}$ near $u = 0$, then our sub-solution initial conditions $\underline{\Psi}(u)$ need to lie below this for all $u \in (0,1)$ but above any algebraically decaying travelling waves near $u = 0$, and to satisfy $\underline{\Psi}'' + (f/\underline{\Psi})' \geqslant 0$. To do this, we take a form very similar to that for the exponentially decaying initial conditions in § 2, but with $\underline{\Psi} \sim u^{1+1/\alpha}$ as $u \to 0$. That is, we take

$$\underline{\Psi}(u) = \varepsilon u^{1+1/\alpha}(B-u)(1-u),$$

where ε is a small positive constant. Then if ε is sufficiently small, the conditions are satisfied, since $\underline{\Psi}''$ is bounded below by a small negative number, while $(f/\underline{\Psi})'$ is large and positive.

4.2. Super-solution

The super-solution initial conditions can be exactly the same as for the exponentially decaying case, that is, $\overline{\Psi}(u) = K(1-(u-\frac{1}{2})^2)$, with the boundary conditions $\overline{p}(0,t) = \gamma_1(t)$ and $\overline{p}(1,t) = \gamma_2(t)$, where γ_1, γ_2 are positive decreasing functions of t such that $\gamma_1(t), \gamma_2(t) \to 0$ as $t \to \infty$.

As an aside, we mention that it is not possible to find super-solution initial conditions with $\overline{\Psi} \sim u^{1+1/\alpha}$ as $u \to 0$. (This is expected, since there are no travelling waves with this behaviour near $u = 0$.) If $\overline{\Psi} \sim ku^{1+1/\alpha}$ near $u = 0$, then, also near $u = 0$,

$$\overline{\Psi}'' + \left(\frac{f}{\overline{\Psi}}\right)' \sim k\left(\frac{1+\alpha}{\alpha^2}\right)u^{-1+1/\alpha} + \frac{1}{k}\left(\frac{\alpha-1}{\alpha}\right)u^{-1/\alpha}.$$

Thus $\overline{\Psi}'' + (f/\overline{\Psi})' \to +\infty$ as $u \to 0$. This is a contradiction, since we need $\overline{\Psi}'' + (f/\overline{\Psi})' \leqslant 0$ for all $u \in (0,1)$.

4.3. Conclusion

We now have a lower non-decreasing bound for any initial conditions decaying to zero as $u^{1+1/\alpha}$ with $\alpha > 1$. This has bounded those initial conditions away from any algebraically decaying travelling waves, which lie below it near $u = 0$.

We also have an upper non-increasing bound. So the solution must evolve to the only steady-state solution lying between these two bounds—the exponentially decaying travelling wave, $P(u) = u(1-u)/\sqrt{2}$. Figure 5 illustrates $\overline{\Psi}$ and $\underline{\Psi}$ in this case.

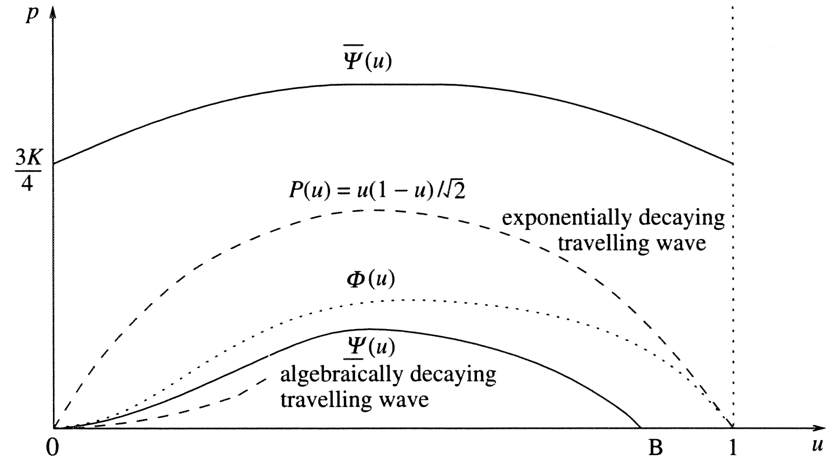

Figure 5. A schematic illustration of the initial conditions for sub- and super-solutions for initial conditions that decay to zero algebraically, as $u^{1+1/\alpha}$ with $\alpha > 1$, in the phase-plane. These prove that such initial conditions must evolve to the exponentially decaying travelling wave, as all the algebraically decaying travelling waves lie below the sub-solution close to $u = 0$.

Numerical solutions in this case indicate that there is a relatively long transient, during which the solution has the form of a decelerating wave (figure 1*d*). From the numerical solutions, one cannot determine whether the limiting wave form has speed $1/\sqrt{2}$ (exponentially decaying) or $1/\sqrt{2}^{+}$ (algebraically decaying); our analysis shows that the former case applies.

5. Algebraically decaying initial conditions, $\alpha < 1$

In this section we will look at algebraically decaying initial conditions with $\alpha < 1$ and construct sub- and super-solutions to show that these initial conditions evolve in the phase-plane to $p(u) \equiv 0$. Thus there is no limiting form to these accelerating waves, rather their form for $u(x,t)$ will continuously flatten as a function of x, as t increases. We have been unable to show this using purely sub- and super-solutions in the phase-plane; as in § 3, we have to combine this approach with more conventional sub- and super-solutions too; that is, $\overline{u}(x,t) \geqslant u(x,t) \geqslant \underline{u}(x,t)$. In § 6, we will study in more detail the long-time behaviour in this case.

5.1. Sub-solution

The sub-solution in this case is simply $\underline{p} \equiv 0$. As an aside, we mention that it is not possible in this case to find sub-solution initial conditions with the form $\underline{\Psi} \sim u^{1+1/\alpha}$ as $u \to 0$, with $\alpha < 1$. (This is as expected, since the solutions decrease to $p = 0$.) If $\underline{\Psi} \sim ku^{1+1/\alpha}$ near $u = 0$, then, also near $u = 0$,

$$\underline{\Psi}'' + \left(\frac{f}{\underline{\Psi}}\right)' \sim \frac{1}{\alpha}\left(\frac{\alpha+1}{\alpha}\right)u^{(1-\alpha)/\alpha} + \left(\frac{\alpha-1}{\alpha}\right)\frac{1}{u^{1/\alpha}} \sim -\left(\frac{1-\alpha}{\alpha}\right)\frac{1}{u^{1/\alpha}}.$$

Thus $\underline{\Psi}'' + (f/\underline{\Psi})' \to -\infty$ as $u \to 0$. This is a contradiction, since we need $\underline{\Psi}'' + (f/\underline{\Psi})' \geqslant 0$ for all $u \in (0, B)$ for some $B > 0$.

5.2. Super-solution

If we have the algebraically decaying initial conditions $\Phi(u) \sim \eta u^{1+1/\alpha}$ near $u = 0$, where η is a positive constant, then our super-solution initial conditions need to lie above this, but still below the algebraic travelling waves near $u = 0$, which have $p \sim u^2$. To satisfy this, we will take the initial conditions for our super-solution in this case to have $\overline{\Psi}(u) \sim \varepsilon u^{2+b}$ near $u = 0$, where $0 < b < 1/\alpha - 1$, say $b = \frac{1}{2}(1/\alpha - 1)$.

As for the exponentially decaying initial conditions, we introduce the boundary condition $\overline{p}(1, t) = \gamma_2(t)$, where γ_2 is a positive decreasing function of t, and $\gamma_2(t) \to 0$ as $t \to \infty$.

We have not been able to construct a super-solution for all possible initial conditions Φ, only those which do not become too large anywhere (i.e. where the initial u profile is not too steep anywhere). This is because we need the super-solution initial conditions, $\overline{\Psi}$, to be concave near $u = 0$, so that they lie below the algebraically decaying travelling waves there, and this means that we need the coefficient of the leading order term of $\overline{\Psi}$ as $u \to 0$ to be sufficiently *small*, not large, as we would like. That is,

$$\overline{\Psi} = \varepsilon u^{2+b}(C - u),$$

with $C \geqslant 1$, $b = \frac{1}{2}(1/\alpha - 1)$, satisfies $\overline{\Psi}'' + (f/\overline{\Psi})' \leqslant 0$ only if ε is sufficiently small, and so only works for initial conditions Φ that lie below $\overline{\Psi}$ for all $u \in (0, 1)$. More specifically, given Φ, we can find a $\overline{\Psi}$ which will lie above Φ on some region close to zero, but we cannot necessarily make $\overline{\Psi}$ large enough to be above Φ outside of this region, for larger values of $u < 1$.

5.3. Initial conditions not covered by the super-solution from $\overline{\Psi}$

In § 5.2, we derived a super-solution that works for a limited class of initial conditions Φ. We now use this to show that the initial conditions not covered by this super-solution still evolve to an accelerating wave. We will do this by constructing sub- and super-solutions of a more conventional form, that is, $\underline{u}(x, t)$ and $\overline{u}(x, t)$ such that $N\overline{u} \geqslant Nu = 0 \geqslant N\underline{u}$ for all x and for all $t \geqslant 0$, and $\overline{u}(x, t = 0) \geqslant u(x, t = 0) \geqslant \underline{u}(x, t = 0)$. Then we will show that these functions imply that the $u(x, t)$ solution (from initial conditions ϕ) must evolve to an accelerating wave.

Given initial conditions Φ, we consider whether they lie completely below our super-solution initial conditions $\overline{\Psi}$. If they do, we are done, as the previously used form of sub- and super-solutions work—we have bounded the initial conditions away from any travelling waves, so that they must evolve to the only steady-state left between the two bounds, namely $p \equiv 0$. If, however, Φ lies above $\overline{\Psi}$ anywhere, then we consider

$$\Psi^*(u) = \min\{\overline{\Psi}(u), \Phi(u)\}$$

(figure 6*a*). For u sufficiently small, say $u \leqslant \delta_1$, $\Phi(u) \leqslant \overline{\Psi}(u)$, and so $\Psi^*(u) = \Phi(u)$. Also, by choosing $C > 1$, we have $\overline{\Psi}(u) > 0$ for u sufficiently close to 1, say $\overline{\Psi} \geqslant \eta > 0$ for $\delta_2 \leqslant u \leqslant 1$; for these values of u, we again have $\Psi^*(u) = \Phi(u)$. It therefore follows that

$$\int_{1/2}^{u} \frac{\mathrm{d}s}{\Psi^*(s)} - \int_{1/2}^{u} \frac{\mathrm{d}s}{\Phi(s)}$$

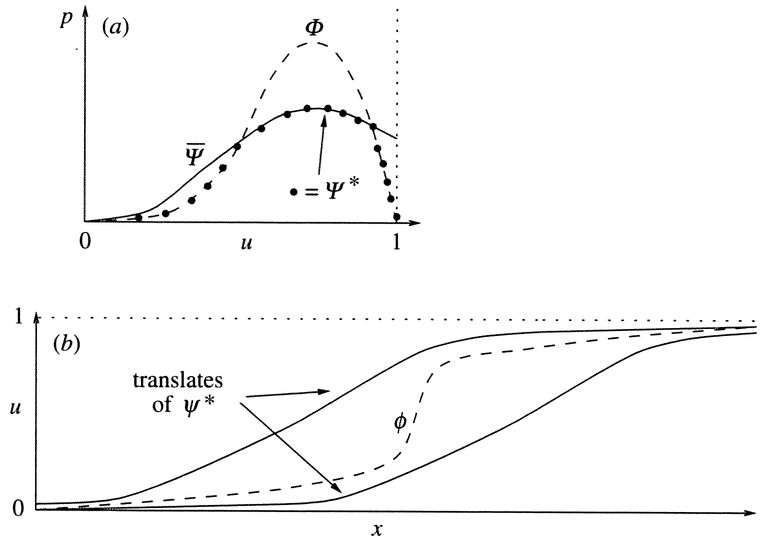

Figure 6. A schematic illustration of how we can use the phase-plane super-solution, which cannot be arbitrarily large, to help show that initial conditions that are not covered by it still must evolve to an accelerating wave. (a) Given initial conditions Φ, which are not less than the phase-plane super-solution initial conditions $\overline{\Psi}$ everywhere, we define a curve Ψ^* as the minimum of Φ and $\overline{\Psi}$. The argument of § 5.2 then applies to Ψ^*, which must evolve to an accelerating wave, with $p \to 0$ as $t \to \infty$. (b) We transform Φ and Ψ^* back to their u–x form, ϕ and ψ^*, and translate copies of the latter to the left and right, so that they lie, respectively, completely above and below ϕ. The time evolution of these curves, via (1.2), then satisfies the conditions for conventional sub- and super-solutions, and so $u(x,t)$ (from initial conditions ϕ) is trapped in between two wave fronts, which we know are accelerating waves.

is bounded independently of u in $[0,1]$, since the two integrands are identical whenever $0 \leqslant u \leqslant \delta_1$. Also, of course, $\Psi^* \leqslant \overline{\Psi}$, so that the argument of § 5.2 applies to Ψ^*, which thus evolves to an accelerating wave.

We now convert both Ψ^* and Φ back to forms with u written as a function of x, namely $u = \psi^*(x)$ and $u = \phi(x)$, defined by

$$x = \int_{1/2}^{u} \frac{\mathrm{d}s}{\Psi^*(s)} \quad \text{and} \quad x = \int_{1/2}^{u} \frac{\mathrm{d}s}{\Phi(s)}, \tag{5.1}$$

respectively. It follows that if $\phi(x_1) = \psi^*(x_2)$, then $|x_1 - x_2|$ is bounded. We can therefore translate one copy of ψ^* to the right by a finite distance and have it lie completely below ϕ, and another to the left a finite distance and have it lie completely above ϕ. This is illustrated in figure 6b; note that these translations correspond to different lower limits of integration in (5.1).

We then let $\underline{u} = \mathcal{F}(x,t)$ and $\overline{u} = \mathcal{F}(x+\kappa,t)$ be the solutions of (1.2), with initial conditions the left and right translates of ψ^*, respectively; the positive constant κ is the difference between the translations. These are then conventional super- and sub-solutions for u, as they satisfy $N\overline{u} \geqslant Nu = 0 \geqslant N\underline{u}$ for all x and for all $t > 0$ (in fact, $N\overline{u} = N\underline{u} = 0$) and $\overline{u}(x,t=0) \geqslant u(x,t=0) \geqslant \underline{u}(x,t=0)$. Thus $\overline{u} \geqslant u \geqslant \underline{u}$ for all x and for all $t \geqslant 0$, by the original form of comparison theorems for PDEs.

It remains to show that the fact that these sub- and super-solutions evolve to accelerating waves means that the solution $u(x,t)$ with $u(x,0) = \phi(x)$ must also evolve to an accelerating wave, i.e. that $u_x \to 0$ as $t \to 0$, uniformly in u. We have that $\mathcal{F}(x+\kappa,t) \geqslant u(x,t) \geqslant \mathcal{F}(x,t)$; also it is known that u_{xx} is bounded (proved, for example, in [4]), say $|u_{xx}| < \Gamma$. Thus, for $\epsilon > 0$,

$$\begin{aligned} u_x(x,t) &< [u(x+\epsilon,t) - u(x,t)]/\epsilon + \epsilon\Gamma \\ &\leqslant [\mathcal{F}(x+\epsilon+\kappa,t) - \mathcal{F}(x,t)]/\epsilon + \epsilon\Gamma \\ &= (1+\kappa/\epsilon)\mathcal{F}_x + \epsilon\Gamma, \end{aligned}$$

with $\mathcal{F}_x$ evaluated between x and $x+\kappa+\epsilon$. Since $\mathcal{F}_x \to 0$, uniformly in u, as $t \to \infty$, and ϵ can be arbitrarily small, we must also have $u_x \to 0$, uniformly in u.

5.4. Conclusion

We have an upper non-increasing bound for all initial conditions decaying to zero as $u^{1+1/\alpha}$, with $\alpha < 1$, which do not become too large (i.e. ϕ too steep) anywhere. This has bounded those initial conditions away from any algebraically decaying travelling waves, which lie above it near $u = 0$ (and the exponentially decaying travelling wave, which is above them). Thus there are no travelling wave solutions to which these initial conditions can evolve; hence they must evolve so that $p(u,t) \to 0$ as $t \to \infty$, as this is the lower non-decreasing bound. For initial conditions that become too large to be covered by our super-solution, we can use the phase-plane super-solution which we do have to construct more conventional sub- and super-solutions, of $u(x,t)$ form. We have then trapped our solution from initial conditions $\phi \sim A/(-x)^\alpha$ as $x \to -\infty$, with $\alpha < 1$, in between two accelerating waves, and can show that it too must evolve to an accelerating wave.

6. The behaviour of the accelerating waves

In the previous section, we looked at algebraically decaying initial conditions, with $u(x,0) \sim A/(-x)^\alpha$ as $x \to -\infty$, or, equivalently, $p(u,0) \sim (\alpha/A^{1/\alpha})u^{1+1/\alpha}$ as $u \to 0$, with $\alpha < 1$. We proved that the long-time behaviour of these waves was that $u_x \to 0$ uniformly in u. However, this does not give us any information about the nature of this approach. In this section, we will study this in more detail, by making an approximation to our p PDE. We do this by neglecting the term $p^2 p_{uu}$ and solving the remaining hyperbolic PDE using the method of characteristics. Intuitively, we expect $p^2 p_{uu}$ to be negligible at large times compared to the other terms in (1.5), which are linear in p, since $p \to 0$ as $t \to \infty$, but we make no attempt to justify formally this simplification. In this section, we will simply explore the solution of the equation given by neglecting the $p^2 p_{uu}$ term, in order to gain some insight into the behaviour of the accelerating waves.

6.1. Solving the hyperbolic PDE

Neglecting the term $p^2 p_{uu}$ in (1.5) gives

$$p_t + f(u)p_u = f'(u)p, \tag{6.1}$$

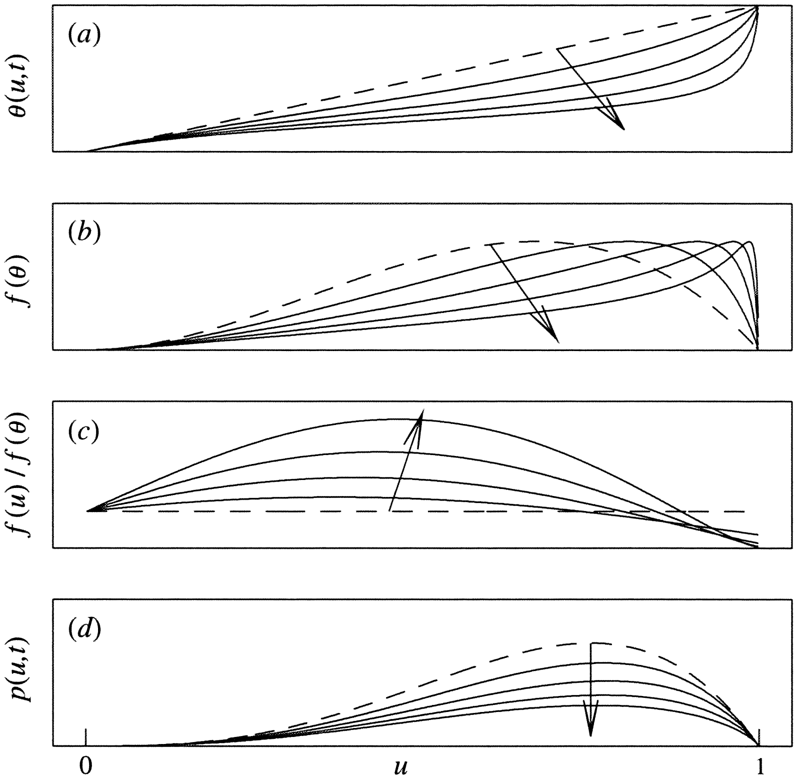

Figure 7. Sequence of graphs from solving (6.1) numerically using the method of characteristics. The first graph shows $\theta(u,t)$, calculated as $G^{-1}(G(u)-t)$ by using linear interpolation on the values of $G(u)-t$. Then $f(\theta)$ and $f(u)/f(\theta)$ are calculated, (where $f(u)=u^2(1-u)$). Finally, $p(u,t)=p_0(\theta)\cdot f(u)/f(\theta)$ is plotted, where we have taken $p_0(u)=u^3(1-u)$. The dashed lines are the solutions at $t=0$ and the solid lines show solutions at time increments of 1, up to $t=4$. The arrows indicate the direction in which the solutions evolve as t increases. The final graph clearly shows that $p(u,t)$ is decreasing to zero as t increases, but at a decreasing rate.

which is a first-order hyperbolic equation, and as such can be solved using the method of characteristics. The characteristics are given by

$$\mathrm{d}t=\frac{\mathrm{d}u}{f(u)}=\frac{\mathrm{d}p}{pf'(u)},$$

which gives the two characteristic functions

$$\mathcal{C}_1(t,u)=t-G(u)\quad\text{and}\quad\mathcal{C}_2(p,u)=\frac{p}{f(u)},$$

where

$$G(u)=\int\frac{\mathrm{d}u}{f(u)}.$$

The arbitrary constant of integration in G corresponds simply to a translation in t.

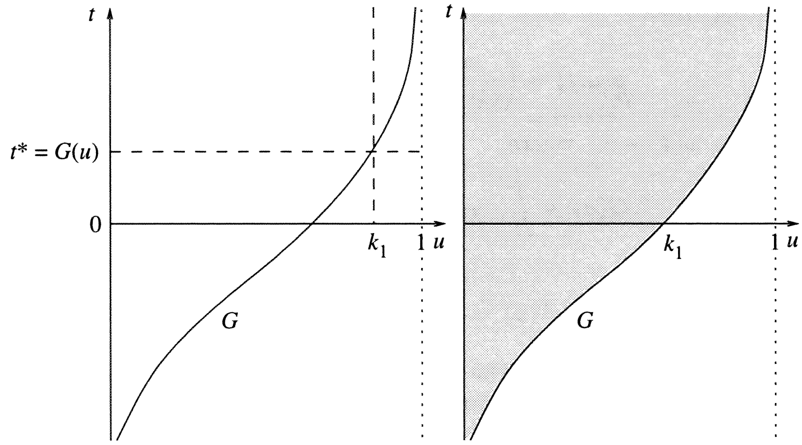

Figure 8. Graphs illustrating the function $G(u)$, and the region of validity of solution (6.2), at each $0 < u < 1$.

The solution for p is then given by eliminating the parameter θ, say, from

$$\frac{p}{f(u)} = \frac{p_0(\theta)}{f(\theta)} \quad \text{and} \quad G(u) - t = G(\theta),$$

where $p_0(u) = p(u, t = 0)$. Figure 7 shows a solution calculated numerically using the above method. The first graph shows $\theta(u,t)$, calculated as $G^{-1}(G(u) - t)$ by using linear interpolation on the values of $G(u) - t$. Then $f(\theta)$ and $f(u)/f(\theta)$ are calculated, where $f(u) = u^2(1-u)$. Finally, $p(u,t) = p_0(\theta) \cdot f(u)/f(\theta)$ is plotted, where we have taken $p_0(u) = u^3(1-u)$. It can be clearly seen (figure 7*d*) that $p(u,t)$ is decreasing to zero as t increases, and at a rate that decreases with time. However, the form of this decrease requires more detailed calculation.

6.2. General initial conditions

For a general initial condition $p(u, t = 0) = p_0(u)$, an exact solution is not possible; however, we can gain an approximate solution at large time. If we let $p_0(u) = u^{1+1/\alpha} h(u)$, where $h(u) = \mathcal{O}(1)$ as $u \to 0$, with $h(0) = \alpha/A^{1/\alpha}$, then

$$\mathcal{C}_2 = \frac{p_0(\theta)}{f(\theta)} = \frac{\theta^{1+1/\alpha} h(\theta)}{\theta^2(1-\theta)} = \theta^{(1-\alpha)/\alpha} H(\theta),$$

say, with $H(0) = \alpha/A^{1/\alpha}$. Also,

$$\mathcal{C}_1 = -\int_\theta \frac{\mathrm{d}s}{f(s)} = \frac{1}{\theta} + \ln\left(\frac{1-\theta}{\theta}\right),$$

and since $\theta|_{u=\text{const.}} \to 0$ as $t \to \infty$, we have that $\theta = \mathcal{C}_1^{-1}$ to leading order at large times. Thus

$$\mathcal{C}_2 = \left(\frac{\alpha}{A^{1/\alpha}}\right) \theta^{(1-\alpha)/\alpha} = \left(\frac{\alpha}{A^{1/\alpha}}\right) \frac{1}{\mathcal{C}_1^{(1-\alpha)/\alpha}}$$

to leading order. Hence, for fixed $u \in (0,1)$,

$$p(u,t) = \left(\frac{\alpha}{A^{1/\alpha}}\right)\frac{f(u)}{(t-G(u))^{(1-\alpha)/\alpha}} \tag{6.2}$$

to leading order as $t \to \infty$, where

$$G(u) = \int_{k_1}^{u} \frac{1}{f(s)}\,\mathrm{d}s = F(k_1) - F(u),$$

with

$$F(u) = \frac{1}{u} + \ln\left(\frac{1-u}{u}\right).$$

For use as an approximation to the solution at finite time, equation (6.2) has a problem with a singularity, since, for any u, there will be a time $t^*(u)$ such that $G(u) = t^*(u)$. The equation can thus only be used as an approximation at a given $0 < u < 1$, provided $t > t^*(u)$, so that $t - G(u) = t - t^*(u) > 0$, as illustrated in figure 8. In practice, this is not restrictive, as we can make $t^*(u) < 0$ for any given u by choosing the value of k_1 such that $u < k_1 < 1$, since k_1 is the arbitrary lower limit of integration of $G(\cdot)$. That is, for $0 < u < k_1$, the solution (6.2) is valid for all $t > 0$; it is only for $k_1 < u < 1$ that the approximation can only be used for $t > t^*(u) > 0$. Thus choosing k_1 closer to 1 means that we can make our solution valid from $t = 0$ onwards for a large range of u values. Note that (6.2) tells us that $p \to 0$ algebraically, as $1/t^{(1-\alpha)/\alpha}$. Furthermore, the decay of p is faster for smaller values of α. This means that the $u(x,t)$ wave fronts spread and flatten more rapidly for smaller values of α.

Since $p = u_x$, we can then integrate (6.2) with respect to x, to give

$$x = \left(\frac{A^{1/\alpha}}{\alpha}\right)\int_{u_0(t)}^{u(x,t)} \frac{(t-G(s))^{(1-\alpha)/\alpha}}{f(s)}\,\mathrm{d}s, \tag{6.3}$$

where $u_0(t) = u(0,t)$; this is a valid approximation at a given u, provided that $t > t^*(u) = G(u)$.

The function $u_0(t) = u(0,t)$ requires further investigation, however. We have

$$\frac{\partial u}{\partial x} = p(u,t),$$

and integrating with respect to x, followed by differentiation with respect to t, gives

$$u_t = \frac{p(u,t)}{p(u_0,t)}\frac{\mathrm{d}u_0}{\mathrm{d}t} + p(u,t)p_u(u,t) + f(u) - p(u,t)p_u(u_0,t) - p(u,t)\frac{f(u_0)}{p(u_0,t)}.$$

Substituting $u_t = u_{xx} + f = pp_u + f$ gives

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = p_u(u_0,t)p(u_0,t) + f(u_0). \tag{6.4}$$

This is a compatibility condition, required for (6.3) to hold. At large times, the pp_u in (6.4) is actually negligible, so that by integrating the remaining terms we get

$$F(u_0) = -t + k_2, \tag{6.5}$$

where k_2 is an arbitrary constant of integration.

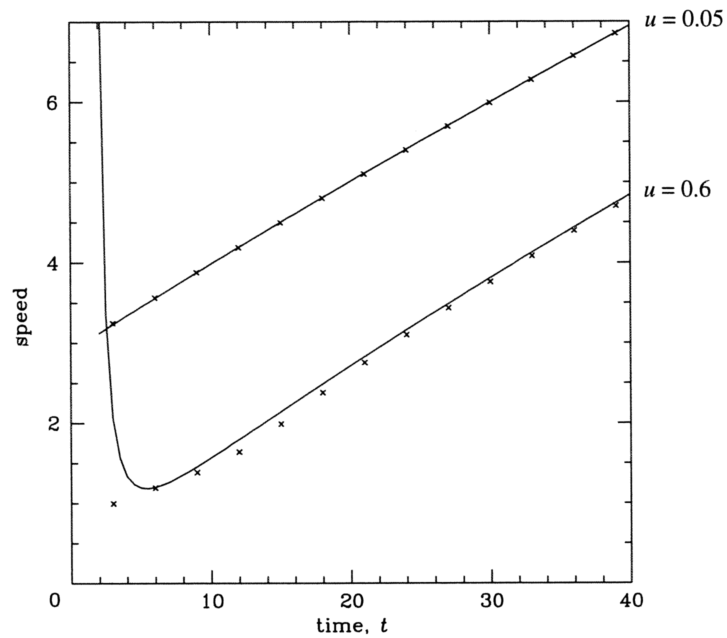

Figure 9. A comparison of our approximation to the speed of the accelerating waves, equation (6.7) (lines), with numerically calculated ones (points) for two values of u. The fit is extremely good in both cases. We have chosen k_1 such that it is the maximum value of $u < 1$ in the initial conditions for the numerical solution.

6.3. The wave speed

The term 'wave speed' denotes the quantity $\partial x/\partial t|_{u=\text{const.}}$. This is a useful property of the wave to consider, as it is a simple measure of the behaviour of the wave front solutions implied by (6.2). In this subsection, we will show that a formula for the wave speed can be arrived at using the previous method of characteristics solution for p, and that this formula fits well with an approximation derived in [19].

In terms of $u(x,t)$ as the dependent variable, the speed is given by

$$c(u,t) = -\frac{u_t}{u_x}.$$

We require a formula in terms of $p(u,t)$ rather than $u(x,t)$, and this is given by

$$c(u,t) = -\frac{u_t}{u_x} = -\frac{u_{xx} + f(u)}{u_x} = -\frac{pp_u + f(u)}{p} = -\left(\left.\frac{\partial p}{\partial u}\right|_{t=\text{const.}} + \frac{f(u)}{p(u,t)}\right). \quad (6.6)$$

Substituting our leading-order solution for $p(u,t)$, (6.2), into (6.6) gives

$$c(u,t) = -\left(\frac{\alpha(1-\alpha) + \alpha^2 u(2-3u)(t-G(u)) + A^{2/\alpha}(t-G(u))^{2/\alpha-1}}{\alpha A^{1/\alpha}(t-G(u))^{1/\alpha}}\right). \quad (6.7)$$

At large times, equation (6.7) is dominated by its final term, so that

$$c(u,t) = -\frac{A^{1/\alpha}}{\alpha}\left(t - F(k_1) + \frac{1}{u} + \ln\left(\frac{1-u}{u}\right)\right)^{1/\alpha-1} \quad (6.8)$$

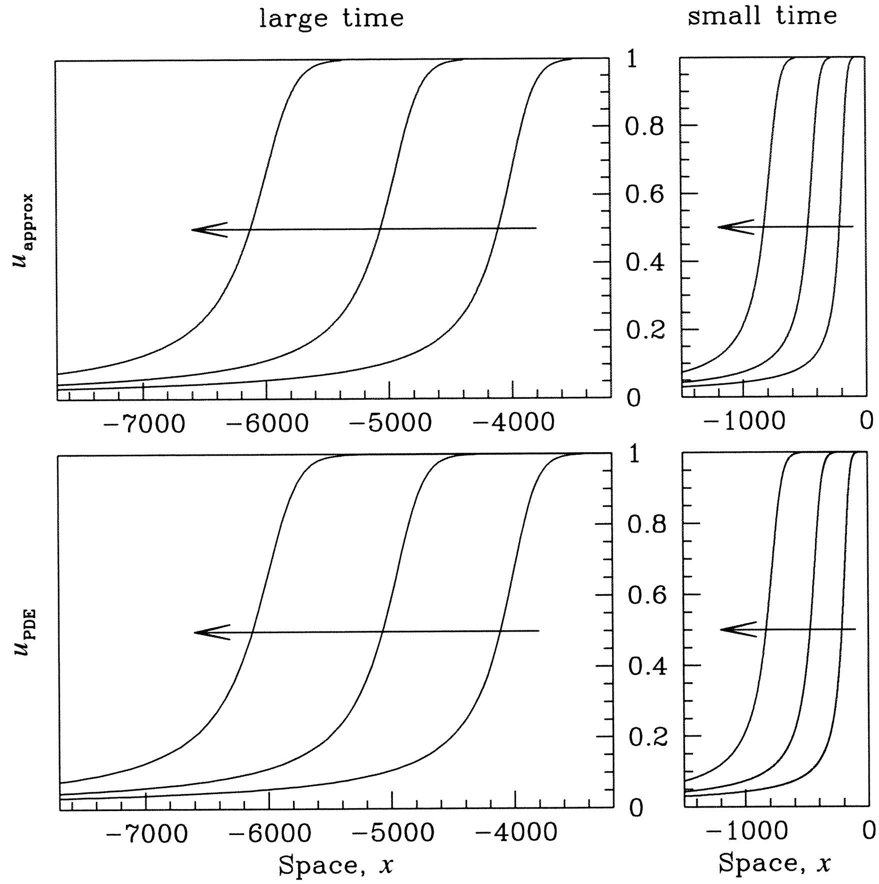

Figure 10. A comparison of a numerical solution of the PDE (1.2), u_{PDE} (lower two graphs), with our approximate analytic solution of it, u_{approx}, from (6.11) (upper two graphs), at small and large times ($t = 10, 20, 30$ and $t = 80, 90, 100$, respectively). The upper graphs are for $F(k_1) = k_2 = -8.4$. The arrows indicate the direction in which the waves evolve as t increases.

to leading order as $t \to \infty$, where we have used

$$-G(u) = F(u) - F(k_1) = 1/u + \ln((1-u)/u) - F(k_1).$$

Again, this equation is valid at a given $u \in (0, 1)$, provided $t > t^*(u) = G(u)$. A cruder version of this approximation was derived in [19], using heuristic methods. Needham and Barnes [17] derived an asymptotic formula for the wave speed at $u = \frac{1}{2}$ (eqn (4.17) in [17]), which agrees exactly with our approximation (6.8). Figure 9 shows that our approximation compares well with numerically calculated speeds for two different values of u.

6.4. Analytic solution from initial conditions $p_0(u) = u^3(1-u)$

To get a better picture of how the decrease of $p(u, t)$ to zero is occurring, we now attempt to derive an exact solution of (6.1), using the method of characteristics, for one particular initial function $p_0(u) = u^3(1-u)$ (i.e. $\alpha = \frac{1}{2}$, $A = 1/\sqrt{2}$). Solution is more straightforward in this case, since

$$C_2 = \frac{p_0(\theta)}{f(\theta)} = \theta,$$

so that

$$C_1 = -\int_\theta \frac{ds}{f(s)} = \frac{1}{\theta} + \ln\left(\frac{1-\theta}{\theta}\right) = \frac{1}{C_2} + \ln\left(\frac{1-C_2}{C_2}\right).$$

The above equation is transcendental, so we cannot rearrange to get a formula for $p(u,t)$ as we would like. However, the leading-order approximation (6.3) derived above is still valid, and the integral can in this case be calculated explicitly. This gives

$$x = \int_{u_0(t)}^{u(x,t)} \frac{t - G(s)}{f(s)}\,ds = -\tfrac{1}{2}F(u)^2 + (F(k_1)-t)F(u) + F(u_0)(t - F(k_1) + \tfrac{1}{2}F(u_0)), \tag{6.9}$$

with $u_0(0) = k_3$, where $u_0(\cdot)$ is determined by (6.4). The arbitrary constants $F(k_1)$ and k_3 then correspond to translations in t and x, respectively. To simplify (6.9) further, we use our leading-order compatibility condition (6.5), so that

$$x = -\tfrac{1}{2}F(u)^2 + (F(k_1) - t)F(u) + (k_2 - t)(\tfrac{1}{2}t - F(k_1) + \tfrac{1}{2}k_2), \tag{6.10}$$

where $F(u_0(0)) = F(k_3) = k_2$ (by (6.5)), and we require $k_3 \leqslant k_1 < 1$, so that $t^*(u = k_3) \leqslant 0$. The formula (6.10) simplifies considerably if we set $k_3 = k_1$, giving

$$x = -\tfrac{1}{2}(F(u) - F(k_1) + t)^2. \tag{6.11}$$

A comparison of this solution with a numerical solution of the reaction–diffusion equation shows good correspondence. Figure 10 compares these two sets of fronts.

7. Discussion

We have studied large-time solutions of the scalar reaction–diffusion equation $u_t = u_{xx} + u^2(1-u)$ from initial conditions $u(x,0) = \phi(x)$, and shown that they have a variety of different forms, dependent upon the exact form of decay of ϕ.

(i) If $\phi \sim A\exp(\xi x)$ as $x \to -\infty$, then u evolves to the unique exponentially decaying travelling wave.

(ii) If $\phi \sim A/(-x)$ as $x \to -\infty$, then u evolves to the algebraically decaying travelling wave of speed A if $A > 1/\sqrt{2}$. If $A \leqslant 1/\sqrt{2}$, then u evolves to the exponentially decaying travelling wave.

(iii) If $\phi \sim A/(-x)^\alpha$ as $x \to -\infty$ with $\alpha > 1$, then u evolves to a decelerating wave, where the speed gradually approaches the minimum speed $1/\sqrt{2}$, and higher parts of the wave front move faster than lower parts.

(iv) If $\phi \sim A/(-x)^\alpha$ as $x \to -\infty$ with $\alpha < 1$, then u evolves to an accelerating wave, where higher parts of the wave front move slower than lower parts, and so the slope of the front decreases with time.

In the main, we have used a phase-plane method of comparison theorems, based on Fife and McLeod's in [5], although for the case of algebraic initial conditions with $\alpha < 1$ or $\alpha = 1$, this method alone proved insufficient for initial conditions ϕ which were too steep anywhere; in these cases, we had to couple the phase-plane method with more conventional comparison theorems [4] to complete the proofs.

An obvious extension to this work would be to generalize it to other kinetic functions f with the same form as $u \to 0$, that is, $f(\cdot)$ such that $f(0) = f(1) = 0$, $f(u) > 0$ for $0 < u < 1$, $f'(0) = 0$, $f'(1) < 0$ and $f''(0) = 2$. In most cases, this is a very simple procedure; the only problem comes for the case $\phi \sim A/(-x)$ $(\alpha = 1)$. In this case, the sub-solution had to be made up from two parts, and the first of these does not easily extend to such general f. An even more general kinetic function f, with $f(0) = f(1) = 0$ and $f(u) > 0$ for all $0 < u < 1$, was considered in [21]. There, the author studies evolution from initial conditions to wave fronts, but purely using u comparison theorems, and with no specific mention of the case $f'(0) = 0$ or its associated algebraically decaying waves.

This paper forms part of a growing body of work on wave fronts in reaction–diffusion equations that are nonlinear at leading order ahead of the front. This phenomenon has been studied in most detail for models of auto-catalytic chemical reactions, in which travelling waves correspond to moving transition fronts. The system of equations modelling this reaction exhibits a wide range of travelling wave behaviour, which is still not fully understood (see [7, 15] for review), and it was in this system that waves with algebraically decaying tails were first identified [1]. The use of a scalar reaction–diffusion equation with kinetics $u^m(1-u)$ as a caricature model for this behaviour was introduced in [19], and the equation has recently been studied by Needham and Barnes [17], in the case $m > 1$, using matched asymptotic expansions. Very different behaviour occurs for $m < 1$, and this has been studied recently by McCabe *et al.* [11], who derive conditions for travelling wave solutions for the extended kinetics $u^a(1-u) - ku^b$ $(0 < a, b < 1)$. Here, we have presented a detailed analysis for kinetics $u^2(1-u)$, giving the first proofs that the various travelling wave solutions are the long-time limits of solutions evolving from simple initial data. Taken together with previous results, this shows how much richer the travelling wave behaviour is in reaction–diffusion equations with entirely nonlinear kinetics.

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