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ON THE EVOLUTION OF PERIODIC PLANE WAVES IN REACTION-DIFFUSION SYSTEMS OF λ–ω TYPE*

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Abstract. λ–ω systems are a class of simple reaction-diffusion equations with a limit cycle in the reaction kinetics. The author considers the solution of the system given by $\lambda(r) = \lambda_0 - r^p$, $\omega(r) = \omega_0 - r^p$ on a semi-infinite spatial domain with initial data decaying exponentially across the domain. Numerical evidence is presented, showing that this initial condition induces a wave front moving across the domain, with periodic plane waves behind the front. These periodic waves can move in either direction, depending on the parameter values. The author uses intuitive criteria to derive an expression for the speed of the advancing front, and by reducing the system to ordinary differential equations for similarity solutions, the amplitude and speed of the periodic plane waves are determined. The speed of these periodic waves varies continuously with the initial data. Perturbation theory is then used to obtain an analytical approximation to the solutions. These analytical predictions all compare very well with numerical solutions of the partial differential equations. Finally, the behavior that results when the periodic plane waves are of an amplitude such that they are linearly unstable as reaction-diffusion solutions is discussed.

Key words. reaction-diffusion, travelling waves, λ-ω systems

AMS subject classifications. 35, 92

1. Introduction. This paper concerns reaction-diffusion systems of the form

$$\partial u/\partial t = \partial^2 u/\partial x^2 + \lambda(r)u - \omega(r)v, \tag{1a}$$

$$\partial v/\partial t = \partial^2 v/\partial x^2 + \omega(r)u + \lambda(r)v, \tag{1b}$$

where $r = (u^2 + v^2)^{1/2}$, u and v are real-valued functions of space x and time t, and $\lambda(0)$ and $\omega(0)$ are both strictly positive. This type of equation, known as a λ–ω system, has the important property that any isolated zero of $\lambda(.)$ corresponds to a limit cycle in the reaction kinetics. As such, λ–ω systems have been widely used in prototype studies of reaction-diffusion equations in which the kinetics have a limit cycle. In particular, λ–ω systems have proved invaluable in the study of spiral waves (e.g., [5] and [7]) and periodic plane waves (e.g., [8] and [3]). Here we consider the latter phenomenon in the specific case of

$$\lambda(r) = \lambda_0 - r^p, \tag{2a}$$

$$\omega(r) = \omega_0 - r^p, \tag{2b}$$

where λ_0, ω_0 and p are real, strictly positive parameters.

Periodic plane waves are waves of constant shape and speed, that oscillate in both space and time. They correspond to limit cycle solutions of the system of ordinary differential equations satisfied by functions of the similarity variable $z = x + at$; a is the wave speed. Many of the key results concerning solutions of this form for λ–ω systems were derived in the seminal paper of Kopell and Howard [8]. In particular, these

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authors showed that (1) has an essentially unique one-parameter family of periodic plane wave solutions, given by

$$u = \hat{r}\cos\left[\omega(\hat{r})t \pm \lambda(\hat{r})^{1/2}x\right], \tag{3a}$$

$$v = \hat{r}\sin\left[\omega(\hat{r})t \pm \lambda(\hat{r})^{1/2}x\right] \tag{3b}$$

for any constant $\hat{r}$ such that $\lambda(\hat{r}) > 0$. For the system given by (2), this means that periodic plane waves exist for all amplitudes $\hat{r} < \lambda_0^{1/p}$. Moreover, Kopell and Howard [8] showed that the wave of amplitude $\hat{r}$ is linearly stable as a solution of the partial differential equations (1) if and only if

$$4\lambda(\hat{r})\left[1 + \left(\frac{\omega'(\hat{r})}{\lambda'(\hat{r})}\right)^2\right] + \hat{r}\lambda'(\hat{r}) \leq 0\,; \tag{4}$$

nonlinear stability has been investigated by Kapitula [6].

Here we consider an aspect of the dynamics of λ–ω systems that has previously been rather neglected, namely, the evolution of solutions of (1) into periodic plane waves from given initial data. Specifically, we will consider initial data of the form

$$u(x,0) = v(x,0) = A\exp(-\xi x) \tag{5}$$

on the semi-infinite domain $[0,\infty)$, with boundary conditions

$$\partial u/\partial x = \partial v/\partial x = 0 \text{ at } x = 0 \quad \text{and} \quad u,\, v \to 0 \quad \text{as } x \to \infty\,. \tag{6}$$

We do not attempt a direct analytical determination of the limiting form of the solution. Rather, we observe from numerical solutions that the system evolves to a wave front moving in the positive x direction, with periodic plane waves behind it. These periodic plane waves can move in either the positive or negative x directions, depending on the values of λ_0, ω_0, p and ξ. Given these numerical observations, we use simple intuitive criteria to derive an analytic expression for both the speed of the advancing front and the amplitude (and direction) of the periodic plane waves behind it. We go on to consider the possibility that the periodic plane waves could have an amplitude such that they are linearly unstable.

2. Numerical results. We have investigated numerically the solutions of the system (1) and (2), subject to (5) and (6), for a wide range of values of the parameters λ_0, ω_0, p, A, and ξ. In almost every case, the behavior is of the same basic form (Fig. 1). The exponentially decaying initial solution evolves to a wave front moving in the positive x direction. Behind this front are regular spatiotemporal oscillations, which appear to move with constant shape and speed in either the positive or negative x directions. The parameter A affects the time course of the evolution, but has no effect on the ultimate behavior. Similarly, the zero-flux boundary condition at $x = 0$ is irrelevant, and essentially the same behavior results if one fixes the values of $u(0,t)$ and $v(0,t)$ (at any prescribed values). The only exceptions that we have found to this type of behavior are discussed and explained in § 5.

An obvious possibility is that the observed regular spatiotemporal oscillations are periodic plane waves; strictly, we mean by this that the oscillations approach periodic plane waves as $t \to \infty$, away from the left-hand boundary. We confirmed this possibility numerically in the following way. From numerical solutions of $u(x,t)$

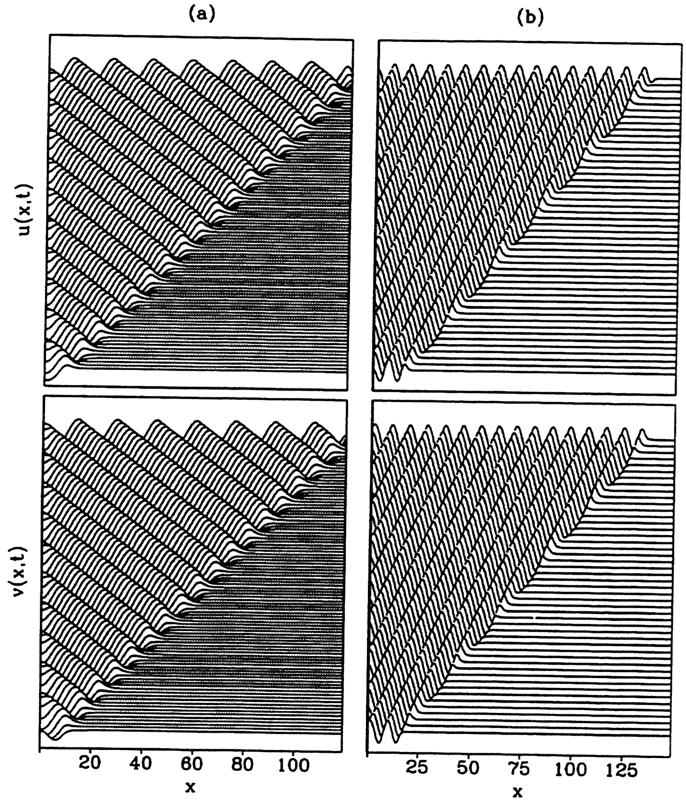

FIG. 1. *Typical numerical solutions of* (1) *with* (2) *subject to* (5) *and* (6). *In both cases a front moves across the domain at constant speed, with regular spatiotemporal oscillations behind this front. In* (a) *these oscillations appear to move in the negative x direction with constant shape and speed, while in* (b) *they move in the positive x direction. The solutions for u and v are plotted as functions of space x at successive times t, with the vertical separation of solutions proportional to the time interval. The parameter values are:* (a) $\xi = 0.8$, $\lambda_0 = 1$, $\omega_0 = 2$, $p = 1.8$, $0 < t < 60$; (b) $\xi = 4$, $\lambda_0 = 3$, $\omega_0 = 1$, $p = 3$, $0 < t < 40$. *Here and in all other figures, a value of $A = 0.1$ is used, but the solutions are essentially independent of the value of A. The equations were solved using the method of lines and Gear's method, with a uniform space mesh of* 500 *points; the results are not sensitive to the number of space mesh points. This numerical scheme was also used in the solutions of* (1) *presented in other figures. In numerical solutions, the right hand boundary is necessarily finite and we took the boundary conditions to be $u = v = 0$, with the domain length sufficiently large that further increase had a negligible effect on the solution. Note that it is not feasible to solve the equivalent system* (7) *numerically except for very short times, since θ increases rapidly and without bound: recall that $\theta_t \longrightarrow \omega_0$ as $x \longrightarrow \infty$.*

and $v(x,t)$, it is straightforward to calculate $r = (u^2+v^2)^{1/2}$. A typical form of r as a function of space at a given time is illustrated in Fig. 2a: in the region in which regular spatiotemporal oscillations are observed, r appears to be constant. Having determined this constant value, r_0 say, we then superimposed the periodic plane wave solution of that amplitude on the numerical solution of the partial differential equations. The form of this periodic plane wave is given in (3): as a function of time t at a fixed position x it is $u = r_0 \cos[\omega(r_0)t + C_1]$, $v = r_0 \sin[\omega(r_0)t + C_1]$; and as a function of x at fixed t it is $u = r_0 \cos[\lambda(r_0)^{1/2}x + C_2]$, $v = \pm r_0 \sin\left[\sqrt{\lambda(r_0)}x + C_2\right]$. Here C_1 and C_2 are constants that must be chosen so that these solutions are in phase with the partial differential equation waves at some arbitrary reference point; the choice of $\pm$

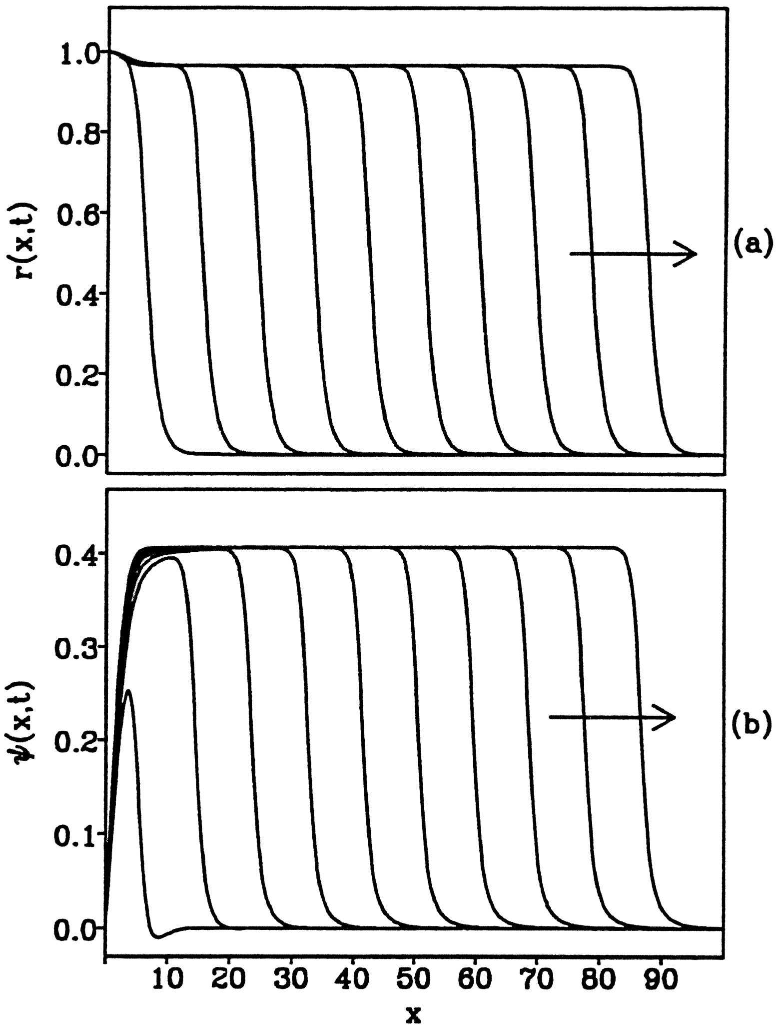

FIG. 2. *A typical solution for* (a) $r(x,t)$ *and* (b) $\psi(x,t) = \partial\theta/\partial x$, *plotted as a function of* x *at equally spaced times* t. *The solution appears to evolve to a transition wave, moving with constant shape and speed in the positive* x *direction. Behind the wave front,* r *and* ψ *are constant, corresponding to a periodic plane wave. The parameter values are* $\xi = 0.8$, $\lambda_0 = 1$, $\omega_0 = 0.5$, $p = 5$, *and the time interval between solutions is* 4.4.

affects the direction of movement of the periodic plane waves, and I determined this from plots of solutions in the form illustrated in Fig. 1. For a wide range of parameter values, the periodic plane waves calculated in this way compare extremely well with the regular spatiotemporal oscillations in the partial differential equations.

3. The wave speeds. If one plots the numerically calculated solution amplitude $r(x,t)$ as a function of x at equally spaced times, one observes a sharp wave front moving in the positive x direction with constant shape and constant speed (Fig. 2a). This is a transition wave, which changes the amplitude from zero to that of the periodic plane waves, and this transition wave will form the basis of my analysis. To facilitate this analysis, it is convenient to work in polar coordinates in u–v space, so that one changes dependent variables to $r = (u^2 + v^2)^{1/2}$ and $\theta = \tan^{-1}(v/u)$. This is a standard transformation for λ–ω systems, and in terms of these variables, the equations (1) are

$$r_t = r\lambda(r) + r_{xx} - r\theta_x^2, \tag{7a}$$

$$\theta_t = \omega(r) + \theta_{xx} + 2r_x\theta_x/r, \tag{7b}$$

where the subscripts denote partial derivatives.

As $x \to \infty$, one does not expect that θ will tend towards a constant value, since $\omega(0) \neq 0$. However, at $t = 0$, $\theta_x \to 0$ as $x \to \infty$, and, moreover, $(\partial/\partial t)\theta_x = 0$ when $\theta_x = r = 0$, so that one does expect $\theta_x \to 0$ as $x \to \infty$ at all times t, and this is confirmed in numerical solutions. Therefore, (7a) decouples from (7b) to leading order at the leading edge of the front in r, giving $r_t = \lambda_0 r + r_{xx}$. This is identical to the linearization of the scalar equation

$$y_t = \lambda_0 y[1 - f(y)] + y_{xx}\,, \tag{8}$$

with $f(0) = 0$, f strictly increasing and $f(y_0) = 1$. For this family of single reaction-diffusion equations, which includes the Fisher equation, it has been shown that with $x \in (-\infty, \infty)$ and for a wide range of initial conditions satisfying $y(x, 0) = O_s(e^{-\xi x})$ as $x \to \infty$, $y(x, t)$ evolves towards a travelling front, connecting the steady states $y = 0$ and $y = y_0$. The speed c of the front is given by

$$c = \begin{cases} \xi + \lambda_0/\xi, & 0 < \xi \leq \sqrt{\lambda_0}, \\ 2\sqrt{\lambda_0}, & \xi \geq \sqrt{\lambda_0}. \end{cases} \tag{9}$$

(see [11], [9], [13], and [10]). The fact that the linearization of (8) and (7a) are the same ahead of the front suggests that the speed of the travelling fronts of r in the solution of (7) subject to (5) and (6) might also be given by (9), and we have confirmed this numerically for a wide range of parameter values.

The speed and functional form of the periodic plane waves into which the solution of (1) evolves is determined entirely by the steady state amplitude r_0 behind the transition front in r. The key step in the analytical determination of r_0 was to consider the form of $\psi(x, t) \equiv \theta_x$. As we have already discussed, $\psi \to 0$ as $x \to \infty$, and since $\theta = [\omega(r_0)t \pm \lambda(r_0)^{1/2}x]$ for a periodic plane wave, one expects $\psi = \pm\lambda(r_0)^{1/2}$ behind the transition wave in r. Numerical calculation of the form of ψ reveals a transition wave front between these two steady state values, moving in parallel with the r wave, with speed c given by (9) (Fig. 2b).

Based on this numerical observation, I look for a solution of (7) of the form $r = \tilde{r}(x - ct)$, $\psi \equiv \theta_x = \tilde{\psi}(x - ct)$, with c given by (9). This implies that $\theta(x, t) = \tilde{\Psi}(x - ct) + f(t)$, where $\tilde{\Psi}(.)$ is an indefinite integral of $\tilde{\psi}(.)$, and $f(.)$ is an arbitrary function. As $x \to \infty$, $\tilde{r} \to 0$ and $\tilde{\psi} \to 0$, and thus substituting this solution form into (7b) and letting $x \to \infty$ implies that $f'(t) = \omega_0$. Therefore we have the travelling front solution

$$r = \tilde{r}(x - ct), \qquad \theta = \tilde{\Psi}(x - ct) + \omega_0 t + \theta_0,$$

where θ_0 is an arbitrary constant.

Substituting this solution form into (7) gives a third-order system of ordinary differential equations

$$\tilde{r}'' + c\tilde{r}' + \tilde{r}\lambda(\tilde{r}) - \tilde{r}\tilde{\psi}^2 = 0, \tag{10a}$$

$$\tilde{\psi}' + (c + 2\tilde{r}'/\tilde{r})\tilde{\psi} + \omega(\tilde{r}) - \omega_0 = 0. \tag{10b}$$

The steady states $r = r_s$, $\psi = \psi_s$ of (10) must satisfy $c\psi_s = \omega_0 - \omega(r_s)$ and $r_s[\lambda(r_s) - \psi_s^2] = 0$, so that either $r_s = \psi_s = 0$ or $[\omega_0 - \omega(r_s)]^2 = c^2\lambda(r_s)$. The first of these is the steady state ahead of the r–ψ transition front, and the second is an equation for the amplitude r_0 of the periodic plane waves behind this front. Substituting the functional forms (2) gives an equation for r_0 with a unique real positive solution:

$$r_0 = \left[\tfrac{1}{2}\left(\left\{c^4 + 4c^2\lambda_0\right\}^{1/2} - c^2\right)\right]^{1/p}. \tag{11}$$

We denote the corresponding value of ψ_s by ψ_0. This prediction of the amplitude of the periodic plane waves agrees extremely well with the numerically calculated amplitude for a wide range of parameter values. It is straightforward to show that the amplitude given by (11) is a strictly increasing function of c, and is thus a decreasing function of ξ, constant on $\xi > \lambda_0^{1/2}$. This constant maximum value is $[2(\sqrt{2}-1)\lambda_0]^{1/p}$, while as $\xi \to 0$, $r_0 = \lambda_0^{1/p} + O(\xi^2)$. This last result is to be expected intuitively, since as ξ decreases towards 0, the initial conditions become more and more spatially homogeneous, and thus one expects the resulting solution to tend towards the limit cycle of the kinetics, which does indeed have amplitude $\lambda_0^{1/p}$.

The velocity of the periodic plane waves is $c_{ppw} = -\omega(r_0)/\psi_0$; this is signed so that a positive velocity corresponds to a wave moving in the positive x direction. Therefore

$$c_{ppw} = \frac{-\omega(r_0)c}{\omega_0 - \omega(r_0)} = c\left(1 - \frac{\omega_0}{r_0^p}\right).$$

Thus ω_0 does not affect the amplitude of the periodic plane waves, but it does affect their velocity. In particular, the waves move in the positive or negative x direction according to whether ω_0 is less than or greater than r_0^p; if $\omega_0 = r_0^p$, standing waves form behind the transition front (Fig. 3).

For simplicity, we have assumed that $u(x,0)$ and $v(x,0)$ are the same, but this is not essential. In particular, if $u(x,0) = A_u \exp[-\xi_u x]$ and $v(x,0) = A_v \exp[-\xi_v x]$ (with ξ_u, $\xi_v > 0$), one would expect intuitively that the smallest of ξ_u and ξ_v would determine the behavior, and this is confirmed in numerical solutions. That is, for a given value of ξ_u, the behavior is essentially the same for all $\xi_v \geq \xi_u$, including the case $v(x,0) \equiv 0$, and vice-versa.

4. Asymptotic solution of the wave forms. Both the leading transition wave and the periodic plane waves behind it are reflected in the solution of (10) connecting the two steady states $(0,0)$ and (r_0, ψ_0). We now use perturbation theory to determine an analytic approximation to this heteroclinic connection. The analysis is reminiscent of the asymptotic solution of the plane wave equations corresponding to Fisher's equation [2]. We use the following rescalings:

$$\epsilon = \lambda_0/c^2, \quad \zeta = \epsilon^{1/2}\lambda_0^{1/2}(x-ct), \quad \hat{r}(\zeta) = \tilde{r}(x-ct)/\lambda_0^{1/p}, \quad \hat{\psi}(\zeta) = \tilde{\psi}(x-ct)/(\epsilon^{1/2}\lambda_0^{1/2}).$$

Note that this implies that $\epsilon \leq 1/4$. With these rescalings, (10) becomes

$$\epsilon\hat{r}'' + \hat{r}' + \hat{r}(1 - \hat{r}^p) - \epsilon\hat{r}\hat{\psi}^2 = 0, \tag{12a}$$

$$\epsilon(\hat{\psi}' + 2\hat{\psi}\hat{r}'/\hat{r}) + \hat{\psi} - \hat{r}^p = 0. \tag{12b}$$

Here and throughout this section, prime denotes $d/d\zeta$. I look for a solution of the form

$$\hat{r} = \hat{r}_0 + \epsilon\hat{r}_1 + \epsilon^2\hat{r}_2 + \cdots, \tag{13a}$$

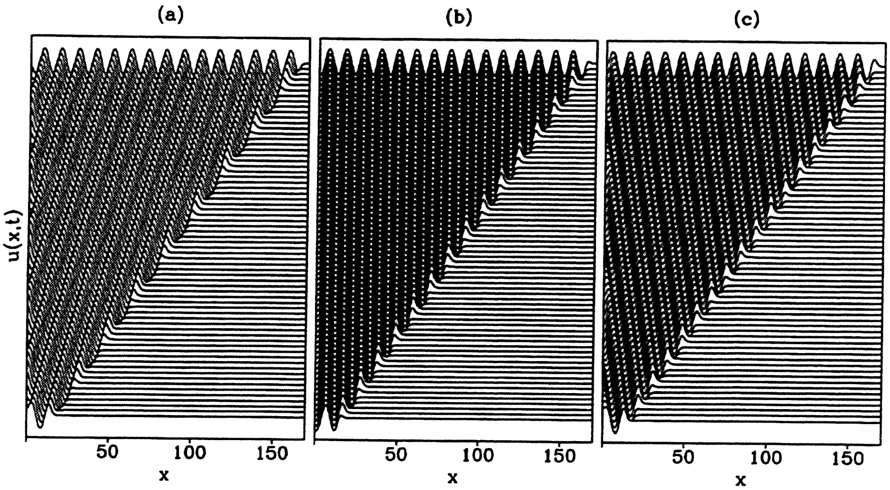

FIG. 3. *An illustration of the effect of ω_0 on the velocity of the periodic plane waves. The parameter values are $\xi = \lambda_0 = p = 2$, which imply $r_0^p \approx 1.66$. In* (a), *$\omega_0 = 1$ and the periodic plane waves move in the positive x direction. In* (b), *$\omega_0 = r_0^p$, and the periodic plane waves are stationary, while in* (c), *$\omega_0 = 2$ and they move in the negative x direction. The solution for u is plotted as a function of space x at successive times t, with the vertical separation of solutions proportional to the time interval; the solution for v is qualitatively very similar.*

(13b) $$\hat{\psi} = \hat{\psi}_0 + \epsilon\hat{\psi}_1 + \epsilon^2\hat{\psi}_2 + \cdots .$$

The boundary condition $\hat{r}_i(\infty) = \hat{\psi}_i(\infty) = 0$ must be satisfied at each $i \geq 0$, and to fix the origin of ζ we impose the condition $\hat{\psi}_i(0) = \hat{\psi}_i(-\infty)/2$ for all $i \geq 0$. Substituting (13) into (12) and equating powers of ϵ gives to leading order

(14a) $$\hat{r}_0' + \hat{r}_0(1 - \hat{r}_0^p) = 0 \Longrightarrow \hat{r}_0 = (1 + e^{p\zeta})^{-1/p},$$

(14b) $$\hat{\psi}_0 = \hat{r}_0^p \Longrightarrow \hat{\psi}_0 = 1/(1 + e^{p\zeta}) .$$

It is straightforward to integrate $\hat{\psi}_0$ with respect to ζ, to obtain the leading order expression for θ. This gives the following solutions for u and v at leading order:

(15a) $$u = \left(\frac{\lambda_0}{1 + e^{\lambda_0 p(x/c-t)}}\right)^{1/p} \cos\left[\omega_0 t + \theta_0 - (1/p)\log\left(1 + e^{-\lambda_0 p(x/c-t)}\right)\right],$$

(15b) $$v = \left(\frac{\lambda_0}{1 + e^{\lambda_0 p(x/c-t)}}\right)^{1/p} \sin\left[\omega_0 t + \theta_0 - (1/p)\log\left(1 + e^{-\lambda_0 p(x/c-t)}\right)\right] .$$

This approximation compares quite well with the numerical solution of the partial differential equations, especially when $\xi \ll \sqrt{\lambda_0}$, so that $\epsilon \ll 1/4$ (Fig. 4).

Equating the order ϵ terms in (12) gives the following equations for the first-order correction:

(16a) $$\hat{r}_1' + [1 - (1+p)\hat{r}_0^p]\,\hat{r}_1 = \hat{r}_0\hat{\psi}^2 - \hat{r}_0'',$$

(16b) $$\hat{\psi}_1 = p\hat{r}_0^{p-1}\hat{r}_1 - \hat{\psi}_0' - 2\hat{\psi}_0\hat{r}_0'/\hat{r}_0 .$$

Equation (16a) can be simplified by making the substitution $w(\hat{\psi}_0) = \hat{r}_1(\zeta)/\hat{r}_0(\zeta)$. This gives

$$(\hat{\psi}_0 - 1)\frac{dw}{d\hat{\psi}_0} - w = \frac{\hat{\psi}_0}{p} - \frac{(\hat{\psi}_0 - 1)^2}{p\hat{\psi}_0} - (\hat{\psi}_0 - 1),$$

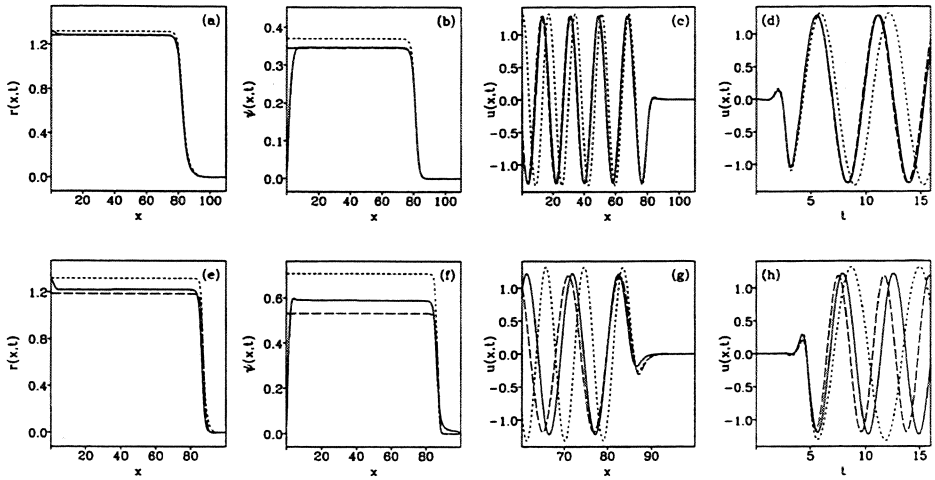

FIG. 4. *An illustration of the asymptotic approximations obtained in* § 4 *to the partial differential equation solutions. The numerical solutions for* r, ψ *and* u *are shown as a function of* x *at* (a)–(c) $t = 16$ *and* (e)–(g) $t = 32$, *and the solution for* u *is also shown as a function of* t *at* (d) $x = 10$ *and (h)* $x = 11$. *The solutions are denoted as follows:* ——— *partial differential equation solution;* - - - - - - - - - $O(1)$ *asymptotic approximation, given in* (14) *and* (15); — — — $O(\epsilon)$ *asymptotic approximation, given in* (17) *and* (18). *In* (a)–(d), *the* $O(\epsilon)$ *approximation is barely distinguishable from the partial differential equation solution. The parameter values are* $\lambda_0 = 2$, $\omega_0 = 3$, $p = 2.5$, *with* (a)–(d) $\xi = 0.4 \Rightarrow \epsilon \approx 0.07$ *and* (e)–(h) $\xi = 2 \Rightarrow \epsilon = 0.25$. *The comparison between the partial differential equation solution and the asymptotic approximation for* v *is qualitatively very similar, and is omitted for brevity.*

whose solution implies

(17a)

$$\hat{r}_1/\hat{r}_0 = (1 - 1/p)(1 - \hat{\psi}_0)\log(1 - \hat{\psi}_0) + (1/p)(1 - \hat{\psi}_0)\log\hat{\psi}_0 - 1/p + k(1 - \hat{\psi}_0),$$

where k is a constant of integration. Substituting this into (16b) gives

$$\hat{\psi}_1 = \hat{\psi}_0(1 - \hat{\psi}_0)\Big[(p-1)\log(1 - \hat{\psi}_0) + \log\hat{\psi}_0 + 2 + p + kp\Big] - \hat{\psi}_0\,, \tag{17b}$$

and the condition $\hat{\psi}_1(0) = \hat{\psi}_1(-\infty)/2$ implies that $k = \log 2 - 1 - 2/p$. Again this can be integrated to obtain leading-order corrections to θ, and this gives the first-order approximation to u as

$$\begin{aligned} u = \Bigg[& 1 + \tfrac{\lambda_0}{pc^2}\Bigg(\Big\{(1-p)\log\left(1 + e^{-\lambda_0 p(x - c/t)}\right) \\ & - \log\left(1 + e^{\lambda_0 p(x - c/t)}\right) + \log 2 - 1 - 2/p\Big\} \\ & \Big/ \left\{1 + e^{-\lambda_0 p(x - c/t)}\right\} - 1\Bigg)\Bigg] \left(\tfrac{\lambda_0}{1 + e^{\lambda_0 p(x - c/t)}}\right)^{1/p} \\ & \cdot \cos\Bigg[\omega_0 t + \theta_0 - (1/p)\log\left(1 + e^{-\lambda_0 p(x/c - t)}\right) \\ & + \tfrac{\lambda_0}{pc^2}\left\{\left[1 + (2-p)e^{\lambda_0 p(x - c/t)}\right]\log\left(1 + e^{-\lambda_0 p(x - c/t)}\right)\right. \\ & \left. + \log\left(1 + e^{\lambda_0 p(x - c/t)}\right) + p(1 - \log 2)\right\} \Big/ \left\{1 + e^{\lambda_0 p(x - c/t)}\right\}\Bigg] \end{aligned} \tag{18}$$

and similarly for v. The first order correction significantly improves the comparison between the asymptotic approximation and the numerical solution of the partial differential equations, as illustrated in Fig. 4.

5. Stability of the periodic plane waves. For $\lambda(.)$ and $\omega(.)$ given by (2), the condition (4) for linear stability of periodic plane waves of (constant) amplitude $\hat{r}$ is just $\hat{r} > [8\lambda_0/(8+p)]^{1/p}$. Thus waves of low amplitude are unstable as solutions of the partial differential equations. Now as ξ varies, the periodic plane waves induced by the initial condition (5) all have amplitude $\geq \left[2(\sqrt{2}-1)\lambda_0\right]^{1/p}$, and thus all these waves are stable for large p. However, for $p < 4(\sqrt{2}-1) \approx 1.66$, the waves induced by some values of ξ will be linearly unstable. The expressions (9) and (11) imply after some simplification that the condition for instability is

$$\frac{\xi}{\sqrt{\lambda_0}} > \frac{1}{\sqrt{p^2+8p}}\left[4-\sqrt{16-8p-p^2}\right]. \tag{19}$$

The parameter domains in which the waves are stable and unstable is illustrated in Fig. 5.

We have explored the consequences of the waves being unstable in the numerical solutions of the partial differential equations. As before, a wave front moves in the positive x direction with periodic plane waves behind it. However at a certain distance behind the front these periodic plane waves destabilize, resulting in more irregular spatiotemporal oscillations. Thus there is a band of regular oscillations immediately behind the transition front in r, with more irregular oscillations further back (Fig. 6). The nature of these irregular oscillations is not clear. For all parameter sets I have investigated, they move in the same direction as the periodic plane waves, but with irregular speed (see Fig. 6). Their amplitude also varies in an apparently disordered way, but within a range of values whose upper limit is $\lambda_0^{1/p}$, which is the amplitude of the limit cycle in the kinetics (Fig. 7).

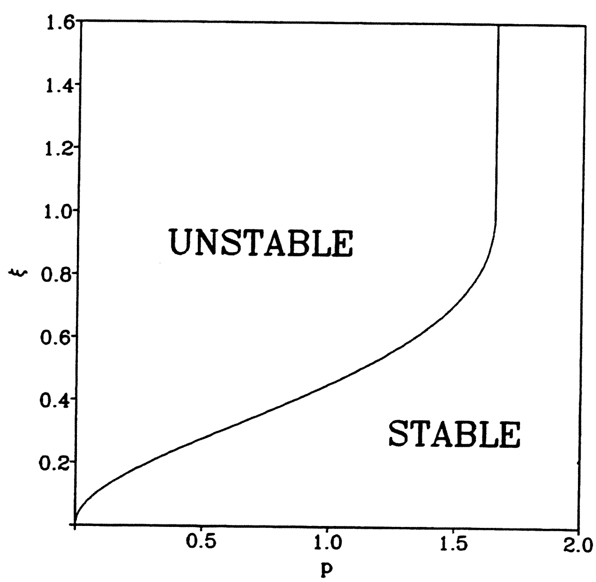

FIG. 5. *The parameter domains in which the periodic plane waves behind the transition front in* r *are stable and unstable. This is a graphical illustration of the inequality* (19).

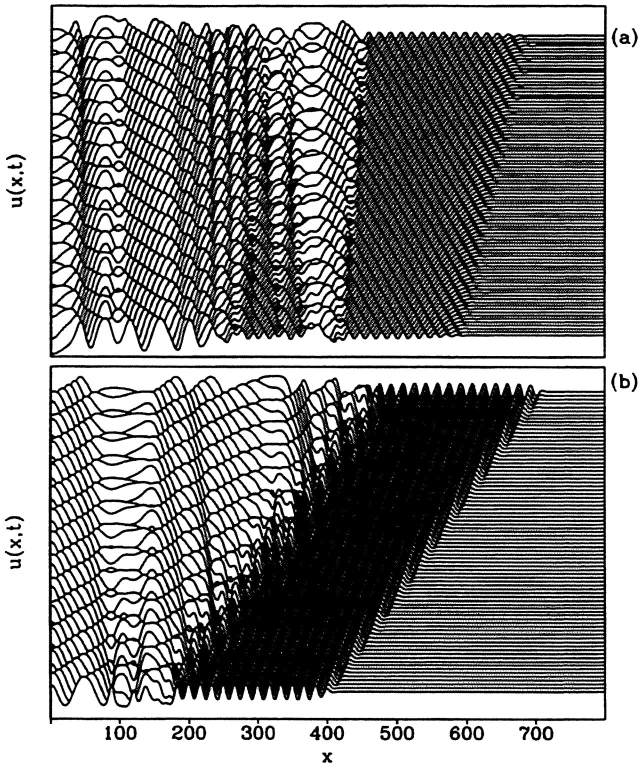

FIG. 6. *Typical numerical solutions of* (1) *with* (2) *subject to* (5) *and* (6), *for values of* p *and* ξ *such that the periodic plane waves behind the transition front are unstable as reaction-diffusion solutions. In* (a), *the parameters are such that the periodic plane waves move in the negative* x *direction, while in* (b) *they move in the positive* x *direction. In both cases, there is a band of regular spatiotemporal oscillations immediately behind the transition front, with more irregular oscillations further back. The solutions for* u *are plotted as a function of space* x *at successive times* t, *with the vertical separation of solutions proportional to the time interval. The solutions for* v *are qualitatively very similar. The parameter values are* (a) $\xi = 0.8$, $\lambda_0 = 1$, $\omega_0 = 3$, $p = 0.1$, $310 \leq t \leq 355$; (b) $\xi = 1.1$, $\lambda_0 = 0.8$, $\omega_0 = p = 0.3$, $230 \leq t \leq 400$.

As p decreases below its critical value for stability, the periodic plane waves become progressively less stable. This is reflected in the partial differential equation solutions, since as p decreases, the width of the band of regular oscillations decreases and the variation in amplitude behind this band becomes more irregular (Fig. 7).

6. Conclusions. The ability of different initial conditions to induce travelling waves of different speeds in reaction-diffusion equations has been the subject of active research for many years: see the books by Fife [4], Britton [1] and Murray [12] for review. Here we have investigated a particular aspect of this problem for a simple class of system of two coupled reaction-diffusion equations. We have presented numerical evidence suggesting that starting from exponentially decaying initial conditions, the solutions evolve to a wave front moving through the domain, with periodic plane waves behind it. We have derived expressions for the speeds of both the front and the periodic plane waves, and we have considered the possibility that these waves are unstable as reaction-diffusion solutions.

Many authors have used λ–ω systems as a prototype for reaction-diffusion equations whose kinetics have a limit cycle. It is therefore natural to ask whether simi-

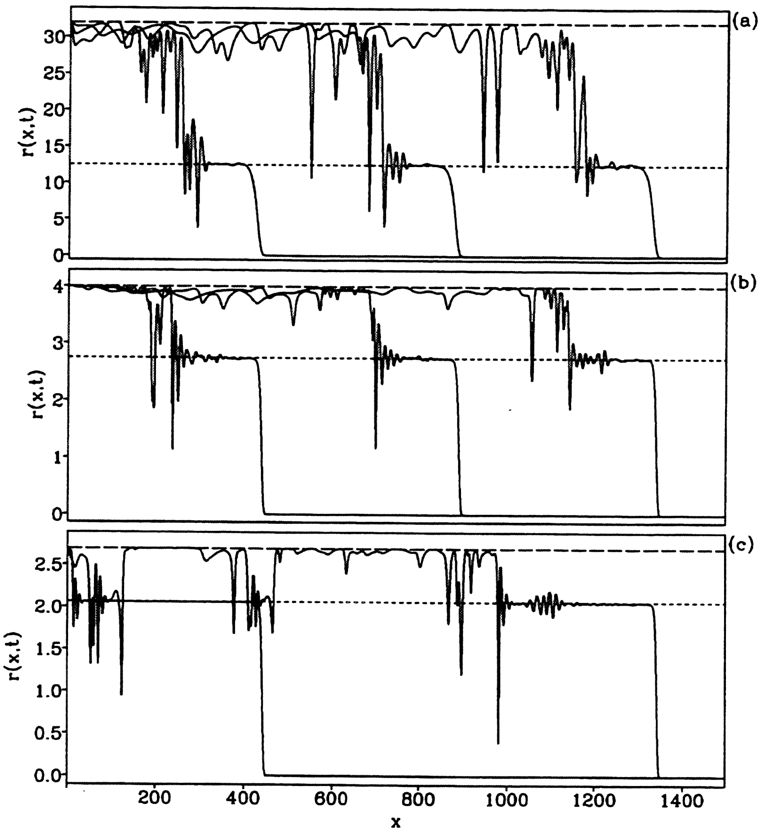

FIG. 7. *An illustration of the variation of the partial differential equation fsolutions with the parameter p, within the region of parameter space such that the periodic plane waves behind the transition front are unstable as reaction-diffusion solutions. The solution for* $r(x,t)$ *is plotted as a function of space at times with an equal spacing of* 160. *However, in* (c) *the solution at* $t = 320$ *is omitted to aid clarity. The parameter values are* $\xi = \lambda_0 = 2$, $\omega_0 = 3$, *with* (a) $p = 0.2$, (b) $p = 0.5$, (c) $p = 0.72$. *In each case the amplitude is close to the periodic plane wave value* r_0 (- - - - - - - -) *immediately behind the transition front, and further back oscillates in an apparently irregular way. However, these irregular oscillations seem to occur within a range of values of r whose upper limit is* $\lambda_0^{1/p}$ (— — —), *which is the amplitude of the limit cycle in the kinetics. As p increases, the length of the interval in which the solution is close to* r_0 *increases, and the irregularity of the oscillations behind this decreases.*

lar behavior is induced by exponentially decaying initial conditions in more general reaction-diffusion systems. We have previously investigated this numerically in one such system, and we did indeed find very similar behavior (see Sherratt [14], [15]). However, this system was not of λ–ω type, and we were unable to obtain any analytical results. In more general reaction-diffusion systems, the additional complexity of the equations can have one particularly interesting consequence. If the kinetics have a limit cycle, this will of course enclose an unstable steady state; however, there is also the possibility of a second steady state, outside the limit cycle. This is not possible in any system of λ–ω type.

The existence of two steady states immediately raises the possibility of a transition wave front. Even though the steady state inside the limit cycle is unstable in the kinetic ordinary differential equations, it may have a stable manifold in the equations governing travelling wave solutions of some speeds. This means that a transition wave could exist which moves with constant shape and speed, and which, at the trailing edge of the wave front, leaves the system in the steady state inside the limit cycle. Since the transition wave front would be given by a solution of a system of ordinary differential equations, the front would decay *exponentially* to this steady state at its trailing edge, and thus periodic plane waves could be induced simply by the passage of the transition front. We have previously discussed this mechanism in detail for a particular system of reaction-diffusion equations [14], [15], but the key point pertaining to this paper is that in general reaction-diffusion equations, it is not necessary for exponential decay to be applied artificially as an initial condition, since travelling fronts automatically leave behind exponentially decaying tails. This adds significantly to the importance of a detailed understanding of the response of reaction-diffusion systems to exponentially decaying initial conditions.

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