



# Nonsharp Travelling Wave Fronts in the Fisher Equation with Degenerate Nonlinear Diffusion

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**Abstract**—When degenerate nonlinear diffusion is introduced into the Fisher equation, giving  $u_t = (uu_x)_x + u(1 - u)$ , the travelling wave structure changes so that there is a sharp-front wave for one particular wave speed, with smooth-front waves for all faster speeds. The sharp-front solution has been studied by a number of previous authors; the present paper is concerned with the smooth-front waves. The authors use heuristic arguments to derive a relationship between initial data and the travelling wave speed to which this initial data evolves. The relationship compares very well with the results of numerical simulations. The authors go on to consider the form of smooth-front waves with speeds close to that of the sharp-front solution. Using singular perturbation theory, they derive an asymptotic approximation to the wave which gives valuable information about the structure of the smooth-front solutions.

**Keywords**—Travelling waves, Nonlinear diffusion, Degenerate diffusion, Matched asymptotic analysis, Perturbation theory, Mathematical model.

## 1. INTRODUCTION

Travelling waves are a ubiquitous feature of spatiotemporal models in biology. Waves play a key role in many areas of ecology, epidemiology, physiology, cell biology, and developmental biology. This whole area of mathematical biology was initiated by the work of Fisher [1] and Kolmogorov *et al.* [2], who studied the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u). \quad (1)$$

Here  $t$  and  $x$  denote time and position in a one-dimensional spatial domain; throughout this paper, we restrict attention to one-dimensional domains. All the parameters in (1) have been removed by appropriate rescaling. The Fisher equation (1) remains an invaluable caricature for the study of waves in systems of reaction-diffusion equations (see the book by Murray [3])

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for review), and has been used explicitly to model a range of applications including ecological invasion [4,5] and wound healing [6,7].

In the Fisher equation, the dispersal term is based on ‘‘Fick’s Law,’’ in which population flux is taken as proportional to the local gradient in population density. In a number of ecological situations (e.g., [8,9]), field studies have shown that this simple formulation is inadequate, and that flux depends on the actual population density, as well as its gradient. Based on this evidence, Gurney and Nisbet [10,11] proposed amending (1) to the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ u \frac{\partial u}{\partial x} \right] + u(1 - u), \quad (2)$$

so that the flux is now proportional to the product  $u \frac{\partial u}{\partial x}$ . The present paper is concerned with travelling wave solutions of (2). These are solutions moving with constant shape and constant speed, and are thus, functions of the similarity variable  $z = x - ct$ , where  $c$  is the wave velocity. We take  $c$  to be positive without loss of generality; changing the sign of  $c$  is equivalent to changing the orientation of the  $x$ -axis.

Since its initial formulation, a number of authors have studied (2) and various extensions, both in terms of applications [12,13] and from a mathematical viewpoint [14–19]. Perhaps the most obvious difference between equations (1) and (2) is that in the nonlinear diffusion case, there is a travelling wave solution of sharp-front type (Figure 1a), that is, the solution decreases to  $u = 0$  at a finite spatial position, at which  $\frac{\partial u}{\partial x}$  is discontinuous in space;  $u \equiv 0$  for all subsequent values of  $x$ . In contrast, in the Fisher equation, all travelling wave solutions are of smooth-front type (Figure 1b), that is, the solution approaches 0 and 1 asymptotically as  $x \rightarrow \pm\infty$ .

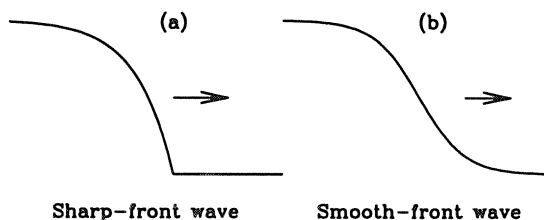


Figure 1. A schematic illustration of the qualitative form of (a) a sharp-front travelling wave, and (b) a smooth-front travelling wave.

The discontinuity in the sharp-front solution of (2) is possible because the (nonlinear) diffusion coefficient  $u$  is zero when  $u = 0$ ; this is sometimes referred to as ‘‘degenerate diffusion.’’ Sánchez-Garduño and Maini [17] showed that for a large family of reaction-diffusion equations with degenerate nonlinear diffusion, including (2), there is a travelling wave of sharp-front type for exactly one value of the wave speed,  $c^*$  say. For  $c < c^*$ , there are no travelling waves, while for each  $c > c^*$ , there is a travelling wave of smooth-front type. In general, there is no known expression for the speed  $c^*$ , but in the particular case (2), Aronson [14] showed that  $c^* = 1/\sqrt{2}$ , and moreover derived a simple formula for the form of the sharp-front wave

$$u^*(z) = \begin{cases} 1 - \exp\left(\frac{z - z_c}{\sqrt{2}}\right), & \text{if } z < z_c, \\ 0, & \text{if } z > z_c. \end{cases} \quad (3)$$

The derivation was based on the simple observation that the wave solution is linear when plotted in a phase plane of the travelling wave ODEs, enabling the form of this solution to be ‘‘guessed,’’ and thus, the method does not generalise at all. Nevertheless, the knowledge of an exact solution form for the sharp-front wave is very valuable, and will be heavily exploited in the present paper. By way of comparison, it has been known for many years that in the Fisher equation (1), there is a (smooth-front) travelling wave solution for all wave speeds  $c \geq 2$  (see [2,20]), and in fact,

exact wave forms have been found for certain speeds [20, Chapter 11], but crucially not for the minimum speed  $c = 2$ .

In view of this rather detailed knowledge of travelling wave solutions of equation (2), it is perhaps surprising that there has been essentially no work on the evolution of different initial conditions towards travelling waves. In the first part of this paper, we use a combination of analytical and numerical techniques to address this issue, and derive a formula relating the behaviour of the initial conditions as  $x \rightarrow \infty$  to the speed of the travelling wave to which these conditions evolve. In the second part of the paper, we discuss the use of singular perturbation theory and the formula (3) to derive an approximate analytic form for smooth-front waves of speed  $c$  close to  $c^*$ .

## 2. EVOLUTION OF WAVES AND INITIAL DATA

For the Fisher equation (1), the evolution of different classes of initial data to waves of different speeds has been extensively studied. There are two key results. In their initial paper, Kolmogorov *et al.* [2] showed that all initial conditions with compact support evolve to the travelling wave with the minimum speed of 2. More recently, Rothe [20] showed that if  $u(x, 0)$  decreases with  $x$ , with  $u(x, 0) \rightarrow 1$  as  $x \rightarrow -\infty$  and  $u(x, 0) = \mathcal{O}_s(e^{-\xi x})$  as  $x \rightarrow \infty$ , then the solution approaches a travelling wave as  $t \rightarrow \infty$ ; the speed of the wave is related to the initial condition as follows:

$$c = \begin{cases} 2, & \text{if } \xi > 1, \\ \xi + \frac{1}{\xi}, & \text{if } \xi < 1. \end{cases}$$

Here we are using the notation  $\mathcal{O}_s$  in a standard way, to mean “of the same order as”; formally  $f = \mathcal{O}_s(g) \Leftrightarrow f = \mathcal{O}(g)$  and  $f \neq o(g)$ .

With this in mind, we investigated numerically the solution of the nonlinear diffusion equation (2) with initial conditions

$$u(x, 0) = \begin{cases} 1, & \text{if } x < 0, \\ e^{-\xi x}, & \text{if } x > 0. \end{cases} \quad (4)$$

For each of the large number of  $\xi$  values we tried, the numerical results showed that the solution did evolve to a travelling wave (e.g., Figure 2). The numerically calculated speed of the wave decreases as  $\xi$  increases up to  $\xi \approx \sqrt{2}$ , when it becomes constant at the value  $c \approx 1/\sqrt{2}$  (Figure 3). Recall that  $1/\sqrt{2}$  is the minimum speed for travelling wave solutions, and corresponds to a sharp-front wave.

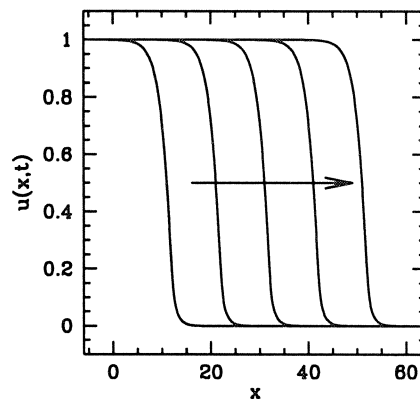


Figure 2. An illustration of the evolution of the numerical solution of (2) to a travelling wave. The solution is plotted against  $x$  at equally spaced times (spacing is 1), with initial conditions given by (4) with  $\xi = 1$ . For these initial conditions, the solution evolves to a smooth front travelling wave of speed  $c = 1$ . The equation was solved numerically using the method of lines and Gear’s method; the spatial domain used was considerably larger than that shown in the figure.

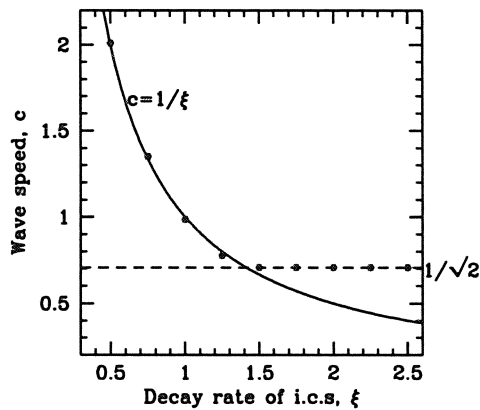


Figure 3. A comparison of the formula (6) for the dependence of wave speed on initial data, with the results of numerical solutions of (2). The points indicate numerically calculated wave speeds, the solid line is the curve  $c = 1/\xi$ , and the dashed line is  $c = 1/\sqrt{2}$ . The comparison is extremely good. Again the method of lines and Gear's method were used for numerical solution.

To investigate this further, we consider the ODE satisfied by travelling wave solutions  $u(x, t) = U(z)$ ; recall that the travelling wave variable  $z = x - ct$ . This second order ODE is highly singular at the equilibrium point  $U = U' = 0$ , and we use a standard change of variables [14] to remove this singularity

$$\zeta = \int_{s=0}^z \frac{ds}{U(s)}.$$

In the case of a smooth-front wave solution, this transformation maps the real line onto itself; in the case of the sharp-front solution (3),  $(-\infty, z_c)$  is mapped to  $\mathbb{R}$ . More detailed discussions of the validity of this transformation are given by Aronson [14] and Sánchez-Garduño and Maini [19]. In terms of  $\zeta$ , the travelling wave equations are

$$\frac{dU}{d\zeta} = UV, \quad (5a)$$

$$\frac{dV}{d\zeta} = -cV - V^2 - U(1 - U). \quad (5b)$$

There are three steady states in this case,  $(0, 0)$ ,  $(0, -c)$ , and  $(1, 0)$ . Straightforward linear analysis shows that  $(1, 0)$  is a saddle point, so that there is exactly one trajectory,  $\mathcal{T}$  say, leaving it and entering the fourth quadrant. Sánchez-Garduño and Maini [17] showed that if  $c < 1/\sqrt{2}$ ,  $\mathcal{T}$  terminates  $(0, -\infty)$ , so that there is no travelling wave solution, while if  $c > 1/\sqrt{2}$ ,  $\mathcal{T}$  terminates at  $(0, 0)$ , corresponding to a smooth-front travelling wave. If  $c = \sqrt{2}$ , then  $\mathcal{T}$  terminates at  $(0, -c)$ ; thus, at  $\zeta = \infty$ , corresponding to  $z = z_c$ ,  $V = \frac{dU}{dz}$  is nonzero, so that the solution is a sharp-front wave.

Our aim in studying the travelling wave phase plane is to find a relationship between the decay rate of the initial conditions (4) and speed of the travelling waves into which these initial conditions evolve. To do this, we consider the equilibrium point  $(0, 0)$ , which has eigenvalues  $-c$  and  $0$ , corresponding to eigenvectors  $(0, 1)$  and  $(-c, 1)$ . Intuitively one would not expect a travelling wave trajectory to approach its terminal steady state along the  $V$  axis, and in fact, Sánchez-Garduño and Maini [17] showed that approach is along the other eigenvector  $(-c, 1)$ . Standard application of centre manifold theory shows that this eigenvector does indeed have a stable centre manifold. On this eigenvector,  $\frac{dU}{dz} = V = -U/c$ ,  $\Rightarrow U(z) = \mathcal{O}_s[\exp(-z/c)]$  as  $z \rightarrow \infty$ . In the Fisher equation, the dependence of wave speed on slowly decaying initial data is such that the decay rates of the initial data and of the resultant travelling wave as  $x \rightarrow \infty$  are the same. If this were also true in the case of nonlinear diffusion, it would imply that  $c = 1/\xi$ . Comparison of this relationship with the results of our numerical study (Figure 3) show very

close agreement provided  $\xi < \sqrt{2}$ , so that  $c$  is greater than the minimum value of  $1/\sqrt{2}$ . Thus, we conclude that the initial conditions (4) for equation (2) evolve to a travelling wave solution whose speed is related to  $\xi$  as follows:

$$c = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \geq \sqrt{2} \quad (\text{sharp-front wave}), \\ \frac{1}{\xi}, & \text{if } \xi < \sqrt{2} \quad (\text{smooth-front wave}). \end{cases} \quad (6)$$

### 3. THE FORM OF SMOOTH-FRONT WAVES

We have already explained that a rather unusual feature of equation (2) is the existence of a simple analytic expression (3) for the sharp-front travelling wave solution. This was exploited previously by Sánchez-Garduño and Maini [18], who used regular perturbation theory to calculate the form of the sharp-front wave when small changes are made to the nonlinear diffusion coefficient. Here we consider smooth-front waves. Intuitively, one would expect the form of smooth-front waves of speeds just above  $1/\sqrt{2}$  to be similar to that of the sharp front wave, and based on this, we attempt to derive an asymptotic expansion for the smooth-front wave of speed  $\epsilon + 1/\sqrt{2}$ .

We are looking for a solution of the travelling wave equation

$$U \frac{d^2 U}{dz^2} + \left( \frac{1}{\sqrt{2}} + \epsilon \right) \frac{dU}{dz} + \left( \frac{dU}{dz} \right)^2 + U(1 - U) = 0, \quad (7)$$

subject to the boundary conditions  $U(-\infty) = 1$  and  $U(+\infty) = 0$ . This boundary value problem does of course have a solution when  $\epsilon = 0$ , namely the sharp-front travelling wave (3). Nevertheless, this is not a regular perturbation problem, and attempts to solve it using regular perturbation theory give unmatchable terms even at first order. The reason for this is simply that the  $\epsilon = 0$  solution (3) is nonsmooth at  $z = z_c$ , and thus, any asymptotic expansion with this as leading term will also be nonsmooth. This is incompatible both with the equation (7) and with the known qualitative form of the smooth-front travelling waves. Therefore we expect the solution to have the form of two outer expansions on either side of  $z = z_c$ , with a corner layer centred on  $z = z_c$  to give a smooth transition between these outer solutions.

We have determined the solution to (7) for  $\epsilon \ll 1$  in precisely this form. The calculation is essentially a standard application of singular perturbation theory, and rather than give the full details, we will just summarise the solution structure and its implications. In the corner layer, the appropriate rescaled variables are  $\tilde{U} = u/\epsilon$  and  $\tilde{z} = (z - z_c)/\epsilon$ , and one proceeds in the usual way by expanding  $\tilde{U}(\tilde{z}; \epsilon)$ ,  $U_{\text{left}}(\tilde{z}; \epsilon)$ , and  $U_{\text{right}}(\tilde{z}; \epsilon)$  as power series in  $\epsilon$ ; here  $U_{\text{left}}$  and  $U_{\text{right}}$  denote the outer solutions in the regions  $z < z_c$  and  $z > z_c$ , respectively. Matching at the right of the corner layer ( $\tilde{z} \rightarrow \infty$ ,  $z \rightarrow z_c^+$ ) determines constants of integration as expected. However, at the left of the layer ( $\tilde{z} \rightarrow -\infty$ ,  $z \rightarrow z_c^-$ ), matching to order  $\epsilon$  is not possible because there is an unbalanced term of order  $\epsilon \log \epsilon$  in the layer solution. To deal with this, we introduce an intermediate term of this order in the expansion of  $U_{\text{left}}$ ; matching to order  $\epsilon$  is then possible.

This procedure gives a first order matched asymptotic expansion for the smooth-front wave form. In fact, the solution contains an undetermined constant at order  $\epsilon$ . This arises because the solution is invariant to a translation in  $z$ , and fixing  $z_c$  only eliminates such translations at leading order. Thus, the undetermined constant is genuinely arbitrary.

Although we omit illustration for brevity, the order  $\epsilon$  composite expansion of the solution does compare very well with the wave form in numerical solutions of (2) for suitable exponentially decaying initial data (see Section 2), provided  $\epsilon$  is reasonably small. Moreover, the presence of the  $\epsilon \log \epsilon$  term in  $U_{\text{left}}$  implies the rather unexpected result that the maximum difference between the smooth-front and sharp-front waves is  $\mathcal{O}_s[(c - c^*) \log(c - c^*)]$  as  $c \rightarrow c^{*+}$ .

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