Electronic Supplementary Material for the Paper "Generation of Periodic Travelling Waves in Cyclic Populations by Hostile Boundaries"

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In this supplementary material, I provide convergence tables for the numerical schemes used to solve the two reaction-diffusion systems used in the paper: the Rosenzweig-MacArthur model for predator-prey interactions, and the λ-ω equations that give the leading order form of that model for C close to C_{hopf}. For both systems I used a semi-implicit finite difference (Crank-Nicolson) scheme with a uniform spatial grid (spacing δx) and a constant time step δt.

Rosenzweig-MacArthur model

For the Rosenzweig-MacArthur model my solution measures were the wavelength and prey (h) amplitude for the periodic travelling wave generated by the Dirichlet boundary conditions $p = h = 0$, and the corresponding values for the boundary condition $p = p_s$, $h = h_s$. Here (p_s, h_s) is the coexistence steady state for predators and prey. Details of the method used to estimate the wavelength and amplitude at the end of a numerical simulation are given in the Appendix of the main paper. For numerical convergence tests, I used $A = 3$, $B = 4$ and $C = 5$, and I solved on a domain of length 800 for a time of 6000. This time is sufficient for a comprehensive decay of transients, and changes in the amplitudes and wavelengths at later times are significantly smaller than the errors due to numerical discretisation. Note however that longer solution times are necessary for smaller values of C.

Tables S.1–S.4 show the convergence of the two wavelengths and two amplitudes as the space and time step are decreased. Calculation of the ratios of successive differences shows that convergence is linear in δt and quadratic in δx.

δt	$\delta x = 1.0000$	$\delta x = 0.5000$	$\delta x = 0.2500$	$\delta x = 0.1250$	$\delta x = 0.0625$
0.1000000	42.431336	42.869371	42.998659	43.032168	43.040646
0.0500000	42.319937	42.769275	42.899288	42.932961	42.941469
0.0250000	42.269318	42.718590	42.850032	42.883625	42.892212
0.0125000	42.245299	42.694205	42.825534	42.859095	42.867636
0.0062500	42.231507	42.680744	42.812886	42.846878	42.855361
0.0031250	42.228744	42.675181	42.806575	42.840658	42.849251
0.0015625	42.220630	42.671024	42.803672	42.837617	42.846181
0.0007813	42.224532	42.670773	42.801829	42.836062	42.844639
0.0003906	42.217846	42.670078	42.801492	42.835232	42.843865
0.0001953	42.219894	42.669418	42.800712	42.834962	42.843483

Table S.1: Convergence of the wavelength of the periodic travelling wave generated by the boundary condition $p = h = 0$ for the Rosenzweig-MacArthur model. Solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper.

δt	$\delta x = 1.0000$	$\delta x = 0.5000$	$\delta x = 0.2500$	$\delta x = 0.1250$	$\delta x = 0.0625$
0.1000000	0.664213	0.669863	0.671471	0.671887	0.671991
0.0500000	0.657871	0.663556	0.665177	0.665593	0.665697
0.0250000	0.654704	0.660381	0.661999	0.662415	0.662520
0.0125000	0.653091	0.658785	0.660402	0.660819	0.660924
0.0062500	0.652278	0.657978	0.659602	0.660019	0.660124
0.0031250	0.651894	0.657578	0.659202	0.659618	0.659723
0.0015625	0.651693	0.657376	0.659001	0.659418	0.659523
0.0007813	0.651595	0.657274	0.658901	0.659318	0.659423
0.0003906	0.651547	0.657223	0.658852	0.659268	0.659373
0.0001953	0.651499	0.657198	0.658826	0.659243	0.659347

Table S.2: Convergence of the amplitude of the prey density h in the periodic travelling wave generated by the boundary condition $p = h = 0$ for the Rosenzweig-MacArthur model. Solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper.

δt	$\delta x = 1.0000$	$\delta x = 0.5000$	$\delta x = 0.2500$	$\delta x = 0.1250$	$\delta x = 0.0625$
0.1000000	50.866994	50.616224	50.551130	50.535206	50.531123
0.0500000	51.191410	50.935757	50.870906	50.854875	50.850870
0.0250000	51.356027	51.098655	51.036336	51.020175	51.016214
0.0125000	51.430908	51.184376	51.119934	51.104287	51.100249
0.0062500	51.469706	51.226173	51.163033	51.146681	51.142634
0.0031250	51.489385	51.247536	51.183807	51.167831	51.163926
0.0015625	51.499286	51.258325	51.194294	51.178576	51.174544
0.0007813	51.514056	51.263745	51.199562	51.184016	51.179919
0.0003906	51.515885	51.265867	51.202201	51.186658	51.182603
0.0001953	51.512312	51.268182	51.204051	51.187981	51.183931

Table S.3: Convergence of the wavelength of the periodic travelling wave generated by the boundary condition $p = p_s$, $h = h_s$ for the Rosenzweig-MacArthur model. Solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper.

δt	$\delta x = 1.0000$	$\delta x = 0.5000$	$\delta x = 0.2500$	$\delta x = 0.1250$	$\delta x = 0.0625$
0.1000000	0.735550	0.735395	0.735359	0.735349	0.735347
0.0500000	0.732643	0.732458	0.732419	0.732410	0.732407
0.0250000	0.731109	0.730974	0.730937	0.730928	0.730926
0.0125000	0.730361	0.730234	0.730193	0.730184	0.730182
0.0062500	0.730012	0.729857	0.729822	0.729812	0.729809
0.0031250	0.729805	0.729664	0.729634	0.729625	0.729623
0.0015625	0.729713	0.729573	0.729541	0.729532	0.729529
0.0007813	0.729670	0.729528	0.729494	0.729485	0.729483
0.0003906	0.729732	0.729507	0.729471	0.729461	0.729459
0.0001953	0.729665	0.729496	0.729460	0.729450	0.729448

Table S.4: Convergence of the amplitude of the prey density h in the periodic travelling wave generated by the boundary condition $p = p_s$, $h = h_s$ for the Rosenzweig-MacArthur model. Solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper.

λ–ω equations

For the λ–ω equations I performed convergence tests with $u_{\text{bdy}} = v_{\text{bdy}} = 0$. For this special case there is an exact expression for the amplitude of the periodic travelling wave that develops away from the boundaries (see the main paper for details) and I used the calculated value of this amplitude to assess numerical convergence. I used $\alpha = 1.5$, $\beta = 0$, $\omega_1 = 0.8$ and $\mu = 1$, and I solved on a domain of length 100 for a time of 1000. This time is sufficient for a comprehensive decay of transients, and changes in the amplitude at later times are significantly smaller than the errors due to numerical discretisation. Table S.5 shows the convergence of the wave amplitude as the space and time steps are decreased. As for the Rosenzweig-MacArthur model, calculation of the ratios of successive differences shows that convergence is linear in δt and quadratic in δx.

Numerical accuracy is strongly dependent on the value of μ, because a decrease in μ implies an increase in ω_0. Therefore maintainance of accuracy requires that δt be decreased with μ (see the main paper for a fuller discussion). To illustrate this, Table S.6 shows the convergence of the wave amplitude as the time step is decreased, with a fixed space step, for four different values of μ. These results show that δt must change roughly in proportion to μ^2 in order to maintain the level of numerical error.

The convergence tests described thus far have used $u_{\text{bdy}} = v_{\text{bdy}} = 0$, which has the advantage of having an exact solution for the periodic travelling wave amplitude. As a final step, it is necessary to test whether the convergence properties are the same for $u_{\text{bdy}} = v_{\text{bdy}} = 0$ and $u_{\text{bdy}}, v_{\text{bdy}} \neq 0$. This is particularly important because of the spatiotemporal oscillations near the left hand boundary in the latter case, which are high frequency and highly localised for small μ, and which might therefore influence numerical convergence. To do this I selected a relatively small value of μ ($= 1/64$) and performed a high accuracy numerical calculation of the PTW amplitude for $u_{\text{bdy}} = v_{\text{bdy}} = 1$. I then performed a series of runs for the same domain length and solution time, but with larger space and time steps. I estimated the percentage errors by comparison with the result of my high accuracy calculation, and I compared these with the errors occurring for the same space and time steps when $u_{\text{bdy}} = v_{\text{bdy}} = 0$. The results are shown in Table S.7: the errors for zero and non-zero u_{bdy} are very similar, and follow parallel convergence trends. This confirms the validity of using $u_{\text{bdy}} = v_{\text{bdy}} = 0$ to estimate numerical errors.

δt	$\delta x = 0.800$	$\delta x = 0.400$	$\delta x = 0.200$	$\delta x = 0.100$	$\delta x = 0.050$	$\delta x = 0.025$
0.0400000	0.639872	0.801554	0.839572	0.848852	0.851158	0.851734
0.0200000	0.214536	0.380893	0.419941	0.429476	0.431846	0.432438
0.0100000	−0.003160	0.165593	0.205172	0.214837	0.217240	0.217840
0.0050000	−0.113312	0.056654	0.096501	0.106233	0.108652	0.109256
0.0025000	−0.168721	0.001855	0.041838	0.051604	0.054031	0.054637
0.0012500	−0.196509	−0.025627	0.014424	0.024206	0.026638	0.027245
0.0006250	−0.210425	−0.039389	0.000696	0.010487	0.012921	0.013528
0.0003125	−0.217387	−0.046275	−0.006173	0.003622	0.006057	0.006664
0.0001563	−0.220870	−0.049719	−0.009609	0.000188	0.002623	0.003231

Table S.5: Convergence of the amplitude of the periodic travelling wave generated by the boundary condition $u = v = 0$ for the λ–ω system. The parameter values were $\alpha = 1.5$, $\beta = 0$, $\omega_1 = 0.8$ and $\mu = 1$. The tabulated values are the percentage error in the periodic travelling wave amplitude in the centre of the domain at the end of the simulation. Solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper. Note that in the lower left hand corner of the table, errors from the spatial discretisation dominate, and changing the time step has no significant effect; the reverse is true in the top right hand corner.

δt	$\mu = 1.0000$	$\mu = 0.5000$	$\mu = 0.2500$	$\mu = 0.1250$
0.04000000	0.848852			
0.02000000	0.429476			
0.01000000	0.214837	1.459479		
0.00500000	0.106233	0.736301		
0.00250000	0.051604	0.368621	1.821193	
0.00125000	0.024206	0.183217	0.916700	
0.00062500	0.010487	0.090118	0.458694	2.016563
0.00031250	0.003622	0.043469	0.228220	1.013744
0.00015625	0.000188	0.020120	0.112611	0.507049
0.00007813		0.008439	0.054714	0.252355
0.00003906		0.002596	0.025741	0.124668
0.00001953			0.011249	0.060739
0.00000977			0.004002	0.028753
0.00000488				0.012755
0.00000244				0.004755

Table S.6: An illustration of the dependence on μ of the periodic travelling wave generated by the boundary condition $u = v = 0$ for the λ–ω system. The tabulated values are the percentage error in the periodic travelling wave amplitude in the centre of the domain at the end of the simulation. The parameter values were $\alpha = 1.5$, $\beta = 0$ and $\omega_1 = 0.8$, and I solved on a domain of length 100 for a time of 1000, with a grid spacing $\delta x = 0.1$. Other solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper.

δt	0.0002000	0.0001000	0.0000500	0.0000250	0.0000125
$u_{\text{bdy}} = v_{\text{bdy}} = 0$	37.868890	20.525322	10.760519	5.518518	2.791802
$u_{\text{bdy}} = v_{\text{bdy}} = 1$	38.195840	20.687557	10.841607	5.559907	2.813670

Table S.7: A comparison of percentage errors in the amplitude of the periodic travelling waves generated by the boundary conditions $u = v = 0$ and $u = v = \mu^{-1/2}$ for the λ–ω system. The tabulated values are the percentage error in the periodic travelling wave amplitude in the centre of the domain at the end of the simulation. These errors are calculated relative to the exact solution for $u = v = 0$, and relative to a higher accuracy numerical estimate for $u = v = \mu^{-1/2}$. The parameter values were $\alpha = 1.5$, $\beta = 0$, $\mu = 0.015625$ and $\omega_1 = 0.8$. I solved on a domain of length 100 for a time of 1000, with a grid spacing $\delta x = 0.2$. For the higher accuracy solution for $u = v = \mu^{-1/2}$, I solved for a time of 2000 with a grid spacing of 0.1, again on a domain of length 100, using the three time steps 9.76×10^{-6}, 4.88×10^{-6} and 2.44×10^{-6}; I then estimated the periodic travelling wave amplitude using the Aitken acceleration formula. Other solution details were as described in the main text of this supplementary material, and in the Appendix of the main paper. Note that the percentage errors in this table are significantly higher than those for similar space and time steps in other tables; this is due to the smaller value of μ.