SMSTC Mathematical Modelling Stream
Assignment for the Lecture on Biological Waves

(a) Some researchers modelling insect populations, and also some studying cell movement, have used equations with "density-dependent diffusion". A simple example of such a model, in one space dimension and after suitable nondimensionalisation, is:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\,\frac{\partial u}{\partial x}\right) + u(1-u)\,.$$

Suggest a biological justification for the nonlinearity in the diffusion coefficient. Why does the new factor of u appear inside the derivative (as opposed to $u\,\partial^2 u/\partial x^2$)?

What is the ordinary differential equation satisfied by travelling wave solutions $U(z)$ of this equation? Here z is the travelling wave coordinate. Write this equation as a system of two coupled first order ODEs by substituting $V = dU/dz$. Show that these equations are satisfied by a solution of the form $V = c(U-1)$ for exactly one value of the wave speed c. For this solution, calculate U as a function of z and sketch the solution. Why is this type of wave known as a "sharp front wave"?

(b) Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\gamma) \qquad (*)$$

with $0 < \gamma < 1$. This is the prototype for a reaction-diffusion equation with bistable kinetics. By looking for solutions of the form

$$u(x,t) = \left[1 + k\exp\left(\alpha x - \beta t\right)\right]^{-1},$$

determine the unique value of the wave speed. Why is the constant k undetermined?

A swarm of locusts moves as a travelling wave satisfying the equation

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2} + Cw(A-w)(w-B)\,.$$

Here $w(x,t)$ is the population density, and A, B, C and D are dimensional parameters. Nondimensionalise this model to give an equation of the form $(*)$, and hence determine the (dimensional) speed at which the swarm moves.

Please send your solutions to: Professor Jonathan A. Sherratt, Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK.