1 Existence of Travelling Wave Fronts for a Reaction-Diffusion Equation with Quadratic-Type Kinetics

**Theorem.** Consider the equation $u_t = D u_{xx} + f(u)$ with $f(0) = f(1) = 0$, $f(u) > 0$ on $0 < u < 1$, $f'(0) > 0$, and $f'(u) < f'(0)$ on $0 < u < 1$. There is a positive travelling wave solution for all wave speeds $\geq 2 \sqrt{D f'(0)}$, and no positive travelling waves for speeds less than this critical value.

**Proof.** The travelling wave solution of speed $c$ is $u(x,t) = U(z), z = x - ct$, satisfying $DU'' + cU' + f(U) = 0$, i.e.

\[
\begin{align*}
U' &= V \\
V' &= -\frac{1}{D} [cV + f(U)],
\end{align*}
\]

This has steady states: $(0, 0), (1, 0)$. Consider the behaviour near $(0, 0)$. The stability matrix is:

\[
\begin{pmatrix}
0 & 1 \\
-f'(0)/D & -c/D
\end{pmatrix}
\]

which has eigenvalues:

\[
\frac{1}{2D} \left( -c \pm \sqrt{c^2 - 4f'(0)D} \right).
\]

Therefore, $(0, 0)$ is a stable node if $c \geq 2 \sqrt{f'(0)D}$, and a stable focus otherwise. If it is a focus then $U < 0$ at some points, whereas we require $U \geq 0$. Therefore, there are no positive waves if $c < 2 \sqrt{f'(0)D}$.

Now consider the behaviour near $(1, 0)$. The stability matrix is:

\[
\begin{pmatrix}
0 & 1 \\
-f'(1)/D & -c/D
\end{pmatrix}
\]

which has eigenvalues:

\[
\frac{1}{2D} \left( -c \pm \sqrt{c^2 - 4f'(1)D} \right).
\]

Therefore, $(1, 0)$ is a saddle point. (Note that $f'(1) < 0$). Therefore, there is exactly one trajectory starting at $(1, 0)$ with $V$ becoming negative. If there is a travelling wave, it must correspond to this trajectory. Where does the trajectory go? There are three possibilities (see figure 1.1):
Figure 1.1: The three possible paths of the (unique) trajectory starting at (1,0) with V becoming negative.

- In case 1: At the point P, \( f(U) > 0 \) and \( V = 0 \) ⇒ \( V' < 0 \), while in fact \( V \) is changing from negative to positive ⇒ \( V' > 0 \). Therefore, case 1 cannot occur.

- In case 3: Let \( \lambda \) be one of the eigenvalues at (0,0) (real and negative, since \( c \geq 2 \sqrt{f'(0)D} \)). Consider the line \( V = \lambda U \) in the phase plane. Then,

\[
\frac{dV}{dU} = \frac{V'}{U'} = -\frac{1}{D} \left[ cV + f(U) \right] \]

\[
= -\frac{c}{D} + \frac{f(U)}{D(-V)}
\]

\[
< -\frac{c}{D} + \frac{f'(0)U}{D(-V)} \text{ using } f(U) < f'(0)U
\]

\[
= -\frac{c}{D} + \frac{f'(0)}{-\lambda D} \text{ using } V = \lambda U
\]

\[
= \lambda,
\]

since \( D\lambda^2 + c\lambda + f'(0) = 0 \) is the eigenvalue equation. At point Q, \( dV/dU < \lambda \). But \( dV/dU > \lambda \) (see figure 1.2). Hence, case 3 cannot occur.

Hence, case 2 must occur, i.e. whenever \( c \geq 2 \sqrt{f'(0)D} \), the trajectory leaving (1,0) ends at (0,0), corresponding to a travelling wave solution. \( \square \)
Figure 1.2: An illustration of the intersection that would occur in case 3, between the travelling wave trajectory and the line $V = \lambda U$. 

\[ \text{Diagram showing intersection at point (1,0) with line } V = \lambda U. \]
2 Existence of Travelling Wave Fronts for a Reaction-Diffusion Equation with Bistable Kinetics

Theorem. Consider the equation $u_t = Du_{xx} + f(u)$ with $f(u_1) = f(u_2) = f(u_3) = 0$, $f'(u_1) < 0$, $f'(u_2) > 0$, $f'(u_3) < 0$. There is a positive travelling wave solution $u(x,t) = U(x-ct)$ with $U(-\infty) = u_1$ and $U(+\infty) = u_3$ for exactly one value of the wave speed $c$.

Proof. : Write $u(x,t) = U(z)$, $z = x - ct$ (travelling wave solution), and $V(z) = dU/dz$. Without loss of generality assume that $f$ is such that the wave moves from $u_1$ to $u_3$. The travelling wave ODEs are:

$$
U' = V,
$$
$$
V' = -\frac{c}{D}V - \frac{f(U)}{D}.
$$

The eigenvalues at the steady states are:

$$
\lambda = \frac{1}{2D} \left[ -c \pm \sqrt{c^2 - 4f'(u_i)} \right].
$$

Now $f' < 0$ at $(u_1,0)$ and $(u_3,0)$ ⇒ both are saddle points (two real eigenvalues, one positive, one negative).

A travelling wave corresponds to a trajectory leaving $(u_1,0)$ and ending at $(u_3,0)$ with $V > 0$ ($U$ is increasing). Therefore, the trajectory must leave $(u_1,0)$ along $T_1$ and enter $(u_3,0)$ along $T_3$ (see figure (2.1)).

Figure 2.1: An illustration of the trajectories $T_1$ and $T_3$.

Therefore, there is a travelling wave ⇔ $T_1$ and $T_3$ are the same trajectory.
Fix $\xi \in (u_1, u_3)$ and let $V_1, V_3$ be the values of $V$ at which $T_1, T_3$ hit the line $U = \xi$. Then

$$\frac{dV}{dU} = -\frac{c}{D} - \frac{f(U)}{D},$$

which decreases as $c$ increases.

Hence, $V_1$ decreases and $V_3$ increases as $c$ increases (explained below). Moreover

as $c \to +\infty$, $dV/dU \to -\infty$, everywhere $\Rightarrow V_3 \to +\infty, V_1 \to -\infty$

and

as $c \to -\infty$, $dV/dU \to +\infty$, everywhere $\Rightarrow V_3 \to -\infty, V_1 \to +\infty$.

Therefore, $V_1$ and $V_3$ are the same for exactly one value of $c \Rightarrow$ there is a travelling wave for this speed only.

**Why does $V_1 \downarrow$ as $c \uparrow$?**

The eigenvector for $T_1$ at $(u_1, 0)$ is $\left(1, \frac{1}{2D} \left[-c + \sqrt{c^2 - 4f'(u_1)}\right]\right)$, which becomes shallower as $c$ increases. Therefore, if $V_1(c = c_B) > V_1(c = c_A)$ with $c_B > c_A$, then at $P$:

$$\left.\frac{dV}{dU}\right|_{c=c_B} > \left.\frac{dV}{dU}\right|_{c=c_A}.$$  

But, $dV/dU \downarrow$ as $c \uparrow$. (Similarly, for $V_3 \uparrow$ as $c \downarrow$.)
3 Condition for Stability of Periodic Travelling Waves in $\lambda-\omega$ Equations with $\omega(.)$ constant

Work with the equations for $r$ and $\theta$, which are:

\[
\begin{align*}
    r_t &= r\lambda(r) + r_{xx} - r\theta_x^2 \\
    \theta_t &= \omega_0 + \theta_{xx} + \frac{2}{r}r_x\theta_x
\end{align*}
\]

where $\omega(.) \equiv \omega_0$. A periodic travelling wave solution is $r = R, \theta = \sqrt{\lambda(R)}x + \omega_0t$. Consider a small perturbation:

\[
\begin{align*}
    r &= R + \tilde{r}(x,t) \\
    \theta &= \sqrt{\lambda(R)}x + \omega_0t + \tilde{\theta}(x,t)
\end{align*}
\]

Substitute this into the $\lambda-\omega$ PDEs and linearise:

\[
\begin{align*}
    \tilde{r}_t &= \tilde{r}\lambda(R) + R\lambda'(R)\tilde{r} + \tilde{r}_{xx} - 2R\sqrt{\lambda(R)}\tilde{\theta}_x - \tilde{r}\lambda(R) \\
    \tilde{\theta}_t &= \tilde{\theta}_{xx} + 2R\sqrt{\lambda(R)}\tilde{r}_x.
\end{align*}
\]

Look for solutions: $\tilde{r} = \bar{r}e^{\nu t + ikx}, \tilde{\theta} = \bar{\theta}e^{\nu t + ikx}$, where $\bar{r}, \bar{\theta}$ are constants:

\[
\begin{align*}
    \{\nu - R\lambda'(R) + k^2\} \bar{r} + 2ikR\sqrt{\lambda(R)}\bar{\theta} &= 0 \\
    \frac{2ik}{R} \sqrt{\lambda(R)}\bar{r} - (k^2 + \nu)\bar{\theta} &= 0.
\end{align*}
\]

Therefore, for non-trivial solutions we require:

\[
\begin{align*}
    \{\nu - R\lambda'(R) + k^2\} (k^2 + \nu) &= 4k^2\lambda(R) \\
    \text{i.e.} \quad \nu^2 + [2k^2 - R\lambda(R)]\nu + k^2 \left[k^2 - \{4\lambda(R) + R\lambda'(R)\}\right] &= 0.
\end{align*}
\]

We have $\lambda'(R) < 0$, so that the coefficient of $\nu$ is strictly positive. Therefore there are either:

- two real negative roots for $\nu \implies$ wave is stable
- or complex conjugate roots with -ve real part for $\nu \implies$ wave is stable
- or one real +ve and one real -ve root for $\nu \implies$ wave is unstable.
The condition for the third possibility is \( k^2 - \{4\lambda(R) + R\lambda'(R)\} < 0 \). This is true for some real \( k \iff 4\lambda(R) + R\lambda'(R) > 0 \). Therefore,

the wave is stable \iff stable to perturbations with any wavenumber \( k \)
\iff 4\lambda(R) + R\lambda'(R) < 0.
4 Generation of Periodic Waves in $\lambda - \omega$ Systems

Local disturbance of $u = v = 0$ causes travelling fronts in $r$ and $\theta_x$. Ahead of these fronts, $u$ and $v \to 0$, and behind them $u$ and $v$ approach periodic travelling waves (so that $r$ and $\theta_x$ approach constant values). The $r-\theta$ PDEs are:

$$\begin{align*}
  r_t &= r\lambda(r) + r_{xx} - r\theta_x^2 \\
  \theta_t &= \omega(r) + \theta_{xx} + 2r_x\theta_x/r.
\end{align*}$$

Look for solutions of form:

$$\begin{align*}
  r &= \bar{r}(x - st) \\
  \theta_x &= \bar{\psi}(x - st) \Rightarrow \theta = \int \bar{\psi}(x - st) + f(t) .
\end{align*}$$

As $x \to \infty$, $r \to 0$ and $\theta_x \to 0 \Rightarrow \theta \to \omega(0)t \Rightarrow f(t) = \omega(0)t + \text{constant}$. Substituting this into the $r-\theta$ PDEs gives:

$$\begin{align*}
  -s\bar{r}' &= \bar{r}\lambda(\bar{r}) + \bar{r}'' - \bar{r}\bar{\psi}^2 \\
  -s\bar{\psi} + \omega(0) &= \omega(\bar{r}) + \psi' + 2\bar{r}'\bar{\psi}/\bar{r}.
\end{align*}$$

Now consider behaviour as $x \to -\infty$ (so that $\bar{r} \to r_s$, $\bar{\psi} \to \bar{\psi}_s$):

$$\begin{align*}
  0 &= r_s\lambda(r_s) + 0 - r_s\psi_s^2 \\
  -s\psi_s + \omega(0) &= \omega(r_s) + 0 + 0
\end{align*}$$

$$\Rightarrow \begin{align*}
  \psi_s &= \pm \sqrt{\lambda(r_s)} \\
  s^2\lambda(r_s) &= \left[\omega(0) - \omega(r_s)\right]^2
\end{align*}$$

Therefore, there is a unique solution for $r_s$, dependent on the front speed $s$.

The front speed $s$ can be expected to be $2\sqrt{\lambda(0)}$ based on results for scalar equations (linearising ahead of the front gives $r_t = \lambda(0)r + r_{xx}$). A proof of this is currently lacking (but numerical simulations provide strong evidence that it’s correct). Hence, a unique periodic wave is selected, with amplitude given by the solution of:

$$4\lambda(0)\lambda(r_s) = \left[\omega(0) - \omega(r_s)\right]^2.$$ 

Some examples of wave generation of this type are illustrated on the next page (figure 4.1).
Figure 4.1: Examples of the generation of periodic travelling waves by local disturbance of $u = v = 0$ in $\lambda - \omega$ systems. For details of functional forms and parameter values, see the legend of Figure 2 in the paper J.A. Sherratt: Periodic waves in reaction-diffusion models of oscillatory biological systems. *FORMA* 11: 61-80 (1996). which is available from [www.ma.hw.ac.uk/~jas/publications.html](http://www.ma.hw.ac.uk/~jas/publications.html).
5 Spiral Waves in $\lambda - \omega$ Systems

Look for a solution of the form:
\[
\begin{align*}
r & = r(\rho) \\
\theta & = \Omega t + m\phi + \psi(\rho)
\end{align*}
\]
\(r, \theta \equiv\) polar coordinates in \(x-y\) plane

In 2-D, \(r-\theta\) PDEs are:
\[
\begin{align*}
rt & = r\lambda(r) + \nabla^2 r - r|\nabla \theta|^2 \\
\theta_t &= \omega(r) + \nabla^2 \theta + 2r\nabla r \cdot \nabla \theta.
\end{align*}
\]

Substitute the solution form into these equations using expressions for \(\nabla\) and \(\nabla^2\) in polar coordinates:
\[
\begin{align*}
r'' + \frac{1}{\rho} r' - r\psi'^2 - \frac{1}{\rho^2} m^2 + r\lambda(r) & = 0 \\
\psi'' + \left( \frac{1}{\rho} + \frac{2m}{\rho^2} \right)\psi' & = \Omega - \omega(r)
\end{align*}
\]

We require \(r\) and \(\psi'\) \(\to\) constants as \(\rho \to \infty\) (\(\to\) periodic travelling wave):
\[
\begin{align*}
\lambda(r_\infty) & = \psi'^2_\infty \quad \text{(compatible with periodic trav. wave)} \\
0 & = \Omega - \omega(r_\infty) \quad \text{(this determines } \Omega)\).
\end{align*}
\]

Now consider solutions near \(\rho = 0\). Then, to leading order:
\[
\begin{align*}
r'' + \frac{1}{\rho} r' - \frac{m^2}{\rho^2} r & = 0 \\
\psi''(0) & = \Omega - \omega(0).
\end{align*}
\]

(Note that \(\psi'(0) = 0\) for regularity, but \(\psi(0) \neq 0\) in general)

Therefore, the form of the solution near the origin is:
\[
\frac{u}{v} \sim \rho^m \cos \left( \Omega t + m\phi + \psi(0) + \frac{1}{2}(\Omega - \omega(0))\rho^{2m} \right) \\
\quad \text{Taylor expansion of } \psi \text{ near zero}
\]

The maximum of \(u\) occurs when \(\Omega t + m\phi + \psi(0) + \frac{1}{2}(\Omega - \omega(0))\rho^{2m} = 0\). For fixed \(t\) this implies
\[
\phi = \frac{1}{2m}(\omega(0) - \omega(r_\infty))\rho^{2m} + \text{constant}.
\]

This is the polar coordinate equation of a spiral.