Defect Dynamics in an Oscillatory Reaction-Diffusion Equation

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This talk can be downloaded from my web site
www.ma.hw.ac.uk/~jas
This work is in collaboration with:

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Outline

1. Ecological Motivation
2. A Generic Mathematical Model
3. Periodic Travelling Wave Stability
4. Source-Sink Dynamics
5. Conclusions
Outline

1. Ecological Motivation
2. A Generic Mathematical Model
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Habitat Boundaries in Ecology

- Ecological habitats are often surrounded by unfavourable environments
- Examples: a wood surrounded by open terrain
  moorland surrounded by farmland
  marsh surrounded by dry ground
- An appropriate boundary condition is “population density=0”
Red grouse is a cyclic population (period 4-6 years)
The study site is moorland, with farmland at its Northern edge
Farmland is very hostile for red grouse
Second Example: Field Voles in Kielder Forest

Field voles in Kielder Forest are also cyclic (period 4 years)
Boundary Condition at the Reservoir Edge

- Voles are an important prey species for owls and kestrels
- The open expanse of Kielder Water will greatly facilitate hunting at its edge

Short eared owl

Common kestrel
Boundary Condition at the Reservoir Edge

- Voles are an important prey species for owls and kestrels.
- The open expanse of Kielder Water will greatly facilitate hunting at its edge.
- Therefore we expect very high vole loss at the reservoir edge, implying that a suitable boundary condition is “vole density=0”.
Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a periodic travelling wave (wavetrain).
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![Red Grouse](image1.jpg)  ![Field Vole](image2.jpg)
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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
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![Image of red grouse](image1.png)

![Image of field vole](image2.png)
Question

Does the Dirichlet condition at the habitat boundary play a role in generating the periodic travelling waves?
Outline

1. Ecological Motivation
2. A Generic Mathematical Model
3. Periodic Travelling Wave Stability
4. Source-Sink Dynamics
5. Conclusions
I consider a generic oscillator model ("$\lambda - \omega$ equations")

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \lambda(r)u - \omega(r)v \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \omega(r)u + \lambda(r)v \\
\end{align*}
\]

\[
\begin{align*}
r &= \sqrt{u^2 + v^2} \\
\lambda(r) &= 1 - r^2 \\
\omega(r) &= \omega_0 + \omega_1 r^2 \\
\end{align*}
\]

This is the normal form of an oscillatory reaction-diffusion system with scalar diffusion close to a supercritical Hopf bifurcation.
Typical Model Solutions

Mathematical Model

Amplitude and Phase Equations
Equilibrium Equations and the Wavetrain Amplitude

Eqns:
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \lambda(r)u - \omega(r)v \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \omega(r)u + \lambda(r)v \\

r &= \sqrt{u^2 + v^2} \\
\lambda(r) &= 1 - r^2 \\
\omega(r) &= \omega_0 + \omega_1 r^2
\end{align*}
\]

Bcs:
\[
\begin{align*}
u(0, t) = v(0, t) &= 0 \quad \text{at} \quad x = 0 \\
u_x(50, t) = v_x(50, t) &= 0 \quad \text{at} \quad x = 50
\end{align*}
\]

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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
Typical Model Solutions

### Conclusion

Dirichlet boundary conditions generate a periodic travelling wave

### Question

What is the amplitude, speed and wavelength of the periodic travelling wave?
Amplitude and Phase Equations

To study the $\lambda-\omega$ equations, it is helpful to replace $u$ and $v$ by $r = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1}(v/u)$, giving

\[
\begin{align*}
    r_t &= r_{xx} - r\theta_x^2 + r(1 - r^2) \\
    \theta_t &= \theta_{xx} + \frac{2r_x\theta_x}{r} + \omega_0 - \omega_1 r^2
\end{align*}
\]

Family of periodic travelling wave solutions ($0 < r^* < 1$):

\[
\begin{align*}
    \left\{ 
    r &= r^* \\
    \theta &= [\omega(r^*)t \pm \sqrt{\lambda(r^*)}x]
    \right\} & \leftrightarrow & \left\{ 
    u &= r^* \cos \left[ \omega(r^*)t \pm \sqrt{\lambda(r^*)}x \right] \\
    v &= r^* \sin \left[ \omega(r^*)t \pm \sqrt{\lambda(r^*)}x \right]
    \right\}
\end{align*}
\]

Our question: what $r^*$ is selected by the Dirichlet boundary condition?
Replotting the solutions in terms of $r$ and $\theta_x$ shows that the long-term solutions for $r$ and $\theta_x$ are independent of time.
There is an exact solution for
\( r = R(x), \ \theta_x = \Psi(x) \) on \( 0 < x < \infty \):

\[
R(x) = R^* \tanh \left( \frac{x}{\sqrt{2}} \right) \quad \Psi(x) = \Psi^* \tanh \left( \frac{x}{\sqrt{2}} \right)
\]

\[
R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2} \quad \Psi^* = -\text{sign}(\omega_1) \left( 1 - R^* \right)^{1/2}
\]
Wave amplitude $R^*$ is in very good agreement with that found in numerical simulations.

There is an exact solution for $r = R(x), \theta_x = \psi(x)$ on $0 < x < \infty$:

$$R(x) = R^* \tanh \left( \frac{x}{\sqrt{2}} \right) \quad \psi(x) = \psi^* \tanh \left( \frac{x}{\sqrt{2}} \right)$$

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2} \quad \psi^* = -\text{sign}(\omega_1)(1-R^2)^{1/2}$$
Outline

1. Ecological Motivation
2. A Generic Mathematical Model
3. Periodic Travelling Wave Stability
4. Source-Sink Dynamics
5. Conclusions
In any oscillatory reaction-diffusion system, some members of the periodic travelling wave family are stable as solutions of the partial differential equations, while others are unstable.

For our $\lambda-\omega$ system, the stability condition is

$$r^* > \left(\frac{2 + 2\omega_1^2}{3 + 2\omega_1^2}\right)^{1/2}$$
The stability of the selected wave depends on $\omega_1$.

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}$$

This is stable

$$\iff R^* > \left( \frac{2 + 2\omega_1^2}{3 + 2\omega_1^2} \right)^{1/2}$$

$$\iff |\omega_1| < 1.110468 \ldots$$
Stability of the Selected Wave

The stability of the selected wave depends on $\omega_1$.

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}$$

This is stable

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$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}$$

This is stable

$$\Leftrightarrow R^* > \left( \frac{2 + 2 \omega_1^2}{3 + 2 \omega_1^2} \right)^{1/2}$$

$$\Leftrightarrow |\omega_1| < 1.110468 \ldots$$
The stability of the selected wave depends on $\omega_1$.

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}$$

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$$\Leftrightarrow \quad R^* > \left( \frac{2 + 2 \omega_1^2}{3 + 2 \omega_1^2} \right)^{1/2}$$

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The stability of the selected wave depends on $\omega_1$.

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}$$

This is stable

$$\Leftrightarrow R^* > \left( \frac{2 + 2 \omega_1^2}{3 + 2 \omega_1^2} \right)^{1/2}$$

$$\Leftrightarrow |\omega_1| < 1.110468 \ldots$$
Typical Solution in an Unstable Case

\[ u \text{ (with incr time)} \]

\[ 0 \quad 50 \quad 100 \quad 150 \]

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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
Convective and Absolute Stability

There are a variety of different solution forms for $|\omega_1| > 1.110468 \ldots$ (unstable waves).
Convective and Absolute Stability

- There are a variety of different solution forms for $|\omega_1| > 1.110468 \ldots$ (unstable waves).
- The key concept for distinguishing these is “absolute stability”.
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is “convective instability”.

![Diagram illustrating convective and absolute stability]

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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
Convective and Absolute Stability

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- The key concept for distinguishing these is “absolute stability”.
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is “convective instability”.
- Alternatively, a solution can be unstable with perturbations growing without moving. This is “absolute instability”.

\[\text{Defect Dynamics in an Oscillatory Reaction-Diffusion Equation}\]
Convective and Absolute Stability

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- The key concept for distinguishing these is “absolute stability”.

- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is “convective instability”.

- Alternatively, a solution can be unstable with perturbations growing without moving. This is “absolute instability”.

- Absolute instability implies instability irrespective of boundary conditions.
Absolute Stability of Wavetrains

- Absolute stability is much harder to calculate than stability.
- For wavetrain solutions of $\lambda-\omega$ reaction-diffusion equations, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)
Absolute stability is much harder to calculate than stability.

For wavetrain solutions of $\lambda-\omega$ reaction-diffusion equations, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)

Our calculation shows that the stability of the selected wavetrain is:

\[
\begin{array}{c}
0.0 & 1.110468 & 1.576465 \\
\downarrow \text{STABLE} & \downarrow \text{UNSTABLE} & \downarrow \text{ABSOLUTELY BUT ABSOLUTELY}
\end{array}
\]

\[
\begin{array}{c}
\downarrow \text{STABLE} & \downarrow \text{UNSTABLE} & |\omega| \\
\end{array}
\]
Generation of Absolutely Stable and Unstable Wavetrains by Dirichlet Boundary Conditions

Numerical simulations show distinct behaviours in the absolutely stable and unstable parameter regimes.

Convectively unstable, absolutely stable

Absolutely unstable
Outline

1. Ecological Motivation
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I focus on the convectively unstable but absolutely stable case.

This solution is a pattern of “sources and sinks”. 
I focus on the convectively unstable but absolutely stable case.

This solution is a pattern of “sources and sinks”.

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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
I focus on the convectively unstable but absolutely stable case. This solution is a pattern of “sources and sinks”. The wavetrain between the defects has (approximately) amplitude

\[ R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} \omega_1^2} \right] \right\}^{-1/2}. \]
Sources, Sinks, and Convective Instability

**Question:** How can an unstable wavetrain persist between the sources and sinks?
Question: How can an unstable wavetrain persist between the sources and sinks?

Answer: Any growing perturbations moves, and is absorbed when it reaches a sink.
Movement of Sources and Sinks

The sources and sinks appear to be stationary.......

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Defect Dynamics in an Oscillatory Reaction-Diffusion Equation
The sources and sinks appear to be stationary........

........but very long simulations show that they move.
The source-sink patterns are of travelling wave form in amplitude.

Substitute \( r(x, t) = \hat{r}(z) \), \( \theta_x(x, t) = \hat{\psi}(z) \), \( z = x - ct \)

\[
\begin{align*}
\frac{d^2\hat{r}}{dz^2} + c \frac{d\hat{r}}{dz} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) &= 0 \\
\frac{d\hat{\psi}}{dz} + c \hat{\psi} + K - \omega_1 \hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) / \hat{r} &= 0
\end{align*}
\]

\( K \) is a constant of integration.)
Solution Structure

Between the source and the neighbouring sinks,

\[-c(1 - R_1^2)^{1/2} + \omega_1 R_1^2 = K = +c(1 - R_2^2)^{1/2} + \omega_1 R_2^2\]

\[
\implies c \text{ has the same sign as } R_1 - R_2.
\]
Stationary sources and sinks satisfy

\[
\begin{align*}
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) &= 0 \\
\frac{d \hat{\psi}}{dz} + K - \omega_1 \hat{r}^2 + 2 \hat{\psi} \left( \frac{d \hat{r}}{dz} \right) / \hat{r} &= 0.
\end{align*}
\]
Stationary sources and sinks satisfy

\[ \frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) = 0 \]

\[ \frac{d\hat{\psi}}{dz} + K - \omega_1 \hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) \hat{r} = 0. \]

Linearise about the wavetrain

⇒ stationary sources decay to the wavetrain at rate \( \sqrt{2} \)

& stationary sinks decay to the wavetrain at rate \( 1/\sqrt{2} \pm i\delta/4 \)

\( (\delta = \sqrt{11 - 12R^*} \in \mathbb{R}) \)

⇒ the effect of the moving sinks on the sources dominates

the effect of the moving sources on the sinks

⇒ when \( c \) is small, we can just consider the correction

to a stationary source
Eigenvalue Structure of Stationary Sources and Sinks

Stationary sources and sinks satisfy
\[
\begin{align*}
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) &= 0 \\
\frac{d\hat{\psi}}{dz} + K - \omega_1 \hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) / \hat{r} &= 0.
\end{align*}
\]

Linearise about the wavetrain

\( \Rightarrow \) stationary sources decay to the wavetrain at rate \( \sqrt{2} \)

& stationary sinks decay to the wavetrain at rate \( 1/\sqrt{2} \pm i\delta/4 \)

\( \Rightarrow \) the effect of the moving sinks on the sources dominates

the effect of the moving sources on the sinks

\( \Rightarrow \) when \( c \) is small, we can just consider the correction
to a stationary source
Stationary sources and sinks satisfy

\[
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) = 0
\]

\[
d\hat{\psi}/dz + K - \omega_1 \hat{r}^2 + 2\hat{\psi} \left( d\hat{r}/dz \right)/\hat{r} = 0.
\]

Linearise about the wavetrain

\( \Rightarrow \) stationary sources decay to the wavetrain at rate \( \sqrt{2} \)

and stationary sinks decay to the wavetrain at rate \( 1/\sqrt{2} \pm i\delta/4 \)

\[
(\delta = \sqrt{11 - 12R^*^2} \in \mathbb{R})
\]

\( \Rightarrow \) the effect of the moving sinks on the sources dominates

the effect of the moving sources on the sinks

\( \Rightarrow \) when \( c \) is small, we can just consider the correction

to a stationary source
Eigenvalue Structure of Stationary Sources and Sinks

Stationary sources and sinks satisfy

\[
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) = 0
\]

\[
d\hat{\psi}/dz + K - \omega_1 \hat{r}^2 + 2\hat{\psi} \left( d\hat{r}/dz \right)/\hat{r} = 0.
\]

Linearise about the wavetrain

⇒ stationary sources decay to the wavetrain at rate \(\sqrt{2}\)

& stationary sinks decay to the wavetrain at rate \(1/\sqrt{2} \pm i\delta/4\)

\[
(\delta = \sqrt{11 - 12R^*} \in \mathbb{R})
\]

⇒ the effect of the moving sinks on the sources dominates the effect of the moving sources on the sinks

⇒ when \(c\) is small, we can just consider the correction to a stationary source: \(r = R^* |\tanh(z/\sqrt{2})|\)
Perturbation Theory Calculation

The wave speed is a natural small parameter........
The wave speed is a natural small parameter....
The wave speed is a natural small parameter........

..........but
\[ \epsilon = \left[ \frac{1}{2} (R_1 + R_2) - R^* \right] \cdot (\text{constant}) \]
is a better choice, where \( R^* \) is the amplitude of the stationary source.

\[ \frac{1}{2} (R_1 + R_2) - R^* \]
Perturbation Theory Calculation

For $\epsilon = 0$:

\[ c = 0 \]
\[ K = \frac{(9 - \sqrt{81 + 72\omega_1^2})}{4\omega_1} \]
\[ \hat{r} = R^* \left| \tanh \left( \frac{z}{\sqrt{2}} \right) \right| \]
\[ \hat{\psi} = -\left(1 - R^*^2\right)^{1/2} \tanh \left( \frac{z}{\sqrt{2}} \right) \]
Ecological Motivation
A Generic Mathematical Model
Periodic Travelling Wave Stability
Source-Sink Dynamics
Conclusions

Sources, Sinks, and Convective Instability
Movement of Sources and Sinks
Travelling Waves of Amplitude
Solution Structure
Perturbation Theory Calculation

Perturbation Theory Calculation

![Diagram of amplitude and transition layers](diagram.png)

For $\epsilon \neq 0$:

$$c = \epsilon c_1 + O(\epsilon^2)$$

$$K = (9 - \sqrt{81 + 72\omega_1^2})/(4\omega_1) + \epsilon K_1 + O(\epsilon^2)$$

$$\hat{r} = R^*|tanh(z/\sqrt{2})| + \epsilon \hat{r}_1(z) + O(\epsilon^2)$$

$$\hat{\psi} = -(1 - R^*^2)^{1/2}tanh(z/\sqrt{2}) + \epsilon \hat{\psi}_1(z) + O(\epsilon^2)$$
Perturbation Theory Calculation

Results:

\[ L_\pm(\epsilon) = -\sqrt{2} \log |\epsilon| - \sqrt{2} \log \kappa_\pm + o(1) \]

where

\[ \kappa_- \exp\{i\delta \log \kappa_-\} + \kappa_+ \exp\{i\delta \log \kappa_+\} = A \]

A is a (complex) constant, independent of \( c_1 \), \( O(1) \) as \( \epsilon \to 0 \)

(recall that \( \delta = \sqrt{11 - 12R^*} \in \mathbb{R} \))
Outline

1. Ecological Motivation
2. A Generic Mathematical Model
3. Periodic Travelling Wave Stability
4. Source-Sink Dynamics
5. Conclusions
A Family of Moving Sources and Sinks

There is a three parameter family of moving sources and sinks:

Parameter 1: $\epsilon$, which reflects the difference in wavetrain amplitudes

Parameter 2: $c_1$, which reflects the speed of movement

Parameter 3: $\kappa_{\pm}$, which reflects the $O(1)$ contribution to the source-sink separation
Source-sink separations are variable. This corresponds to different values of $\kappa_\pm$ associated with different sources.
Implications for the PDE Solutions

- Source-sink separations are variable. This corresponds to different values of $\kappa_\pm$ associated with different sources.

- The sources and sinks all stop moving at (approximately) the same time. This is expected: all are part of a single travelling wave solution.
Implications for the PDE Solutions

- Source-sink separations are variable. This corresponds to different values of $\kappa_{\pm}$ associated with different sources.
- The sources and sinks all stop moving at (approximately) the same time. This is expected: all are part of a single travelling wave solution.
- There is (approximately) no change in the source-sink separation when the sources and sinks stop moving. This is because the equation for $\kappa_{\pm}$ does not involve the parameter $c_1$. 
What is the distance between the leading sink and the $x = 0$ boundary when the sources and sinks stop moving? A minor adaptation of the calculation shows that the separation $L_{bdy}(\epsilon)$ is

$$-\sqrt{2} \log |\epsilon| - \sqrt{2} \log \kappa_{bdy} + o(1)$$

where $\text{Im} \left[ \sigma_2 \kappa_{bdy} \exp \left\{ +i\delta (\log |\epsilon| + \log \kappa_{bdy}) \right\} \right] = \text{sign}(\epsilon)$

($\sigma_2$ is a (complex) constant; recall $\delta = \sqrt{11 - 12R^2 \in \mathbb{R}}$)
Overall Conclusion

\[ |\omega| \]

0.0 \rightarrow 1.110468 \rightarrow 1.576465

\begin{align*}
\text{STABLE} & \quad \text{STABLE} \\
\text{UNSTABLE} & \quad \text{BUT ABSOLUTELY UNSTABLE} \\
\text{ABSOLUTELY UNSTABLE} & \quad \text{WAVETRAIN}
\end{align*}

\begin{align*}
\text{PATTERNS OF SOURCES AND SINKS} & \quad \text{DISORDERED SPATIOTEMPORAL OSCILLATIONS}
\end{align*}
Objectives for Future Work

- A better understanding of how a particular member of the family of moving sources and sinks is selected.

- A better understanding of the disordered solutions in the absolutely unstable parameter regime.