Modelling Vegetation Patterns in Semi-Arid Environments

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In collaboration with
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Outline

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Mathematical Models for Vegetation Pattern Formation

There are a variety of models for vegetation pattern formation in semi-arid environments.

Mathematical Models for Vegetation Pattern Formation

There are a variety of models for vegetation pattern formation in semi-arid environments.

- **Klausmeier**: Competition for water and nonlinear uptake
- **Rietkirk**: Klausmeier model with diffusion of water in the soil
- **van de Koppel**: Klausmeier model with grazing
- **Maron**: two variable model (plant density and water in the soil) with water transport based on porous media theory
- **Lejeune**: short range activation (shading) and long range inhibition (competition for water)

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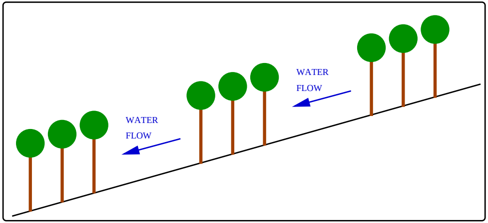

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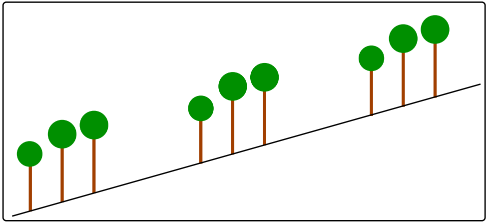

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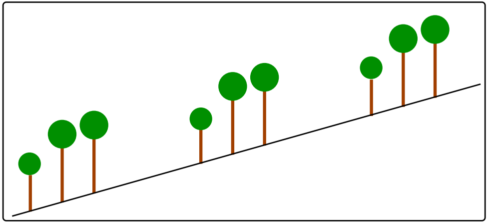

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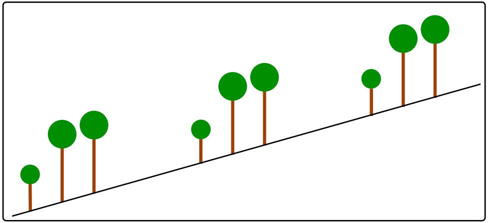

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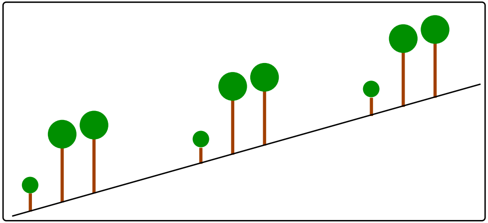

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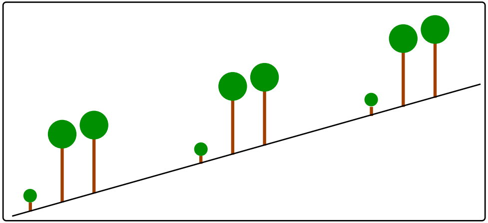

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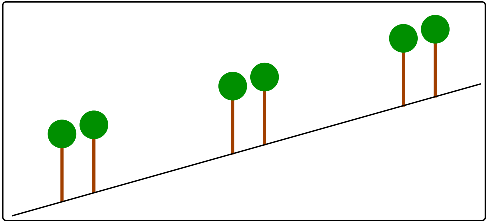

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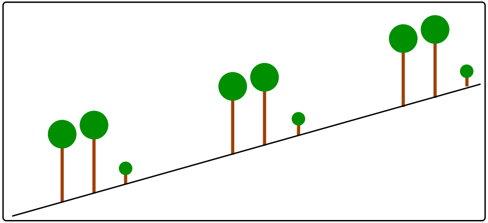

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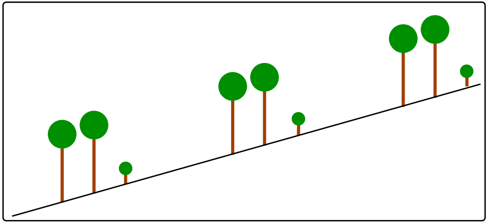

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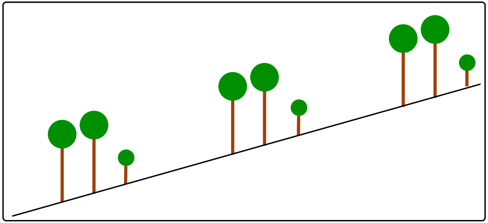

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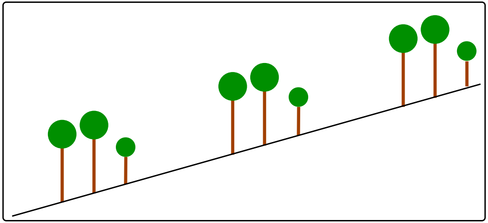

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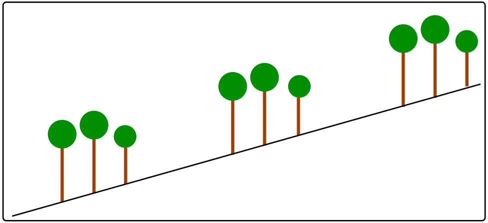

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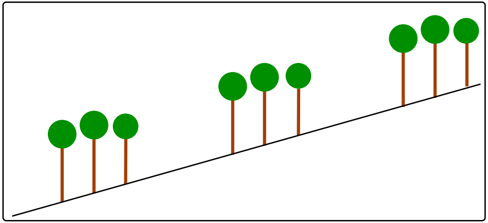

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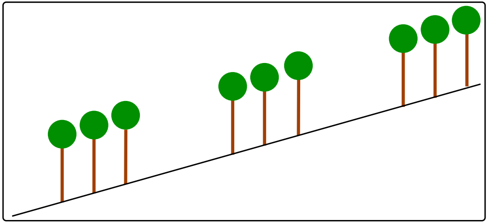

- This mechanism suggests that the stripes would move uphill; this remains controversial.

Mathematical Model of Klausmeier

Rate of change of water = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change plant biomass = Growth, proportional to water uptake – Mortality + Random dispersal

$$\partial w/\partial t = A - w - wu^2 + \nu \partial w/\partial x$$

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\end{aligned}
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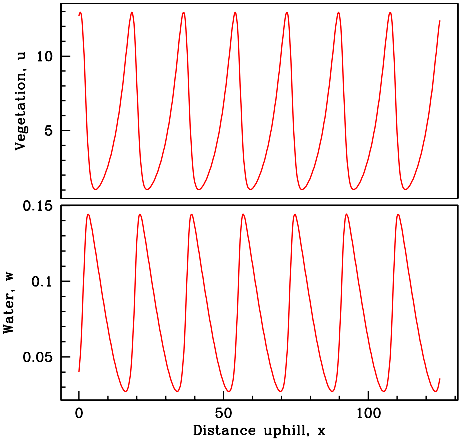
Water uptake increases nonlinearly with plant density because the presence of roots increases water infiltration into the soil.

Typical Solution of the Model



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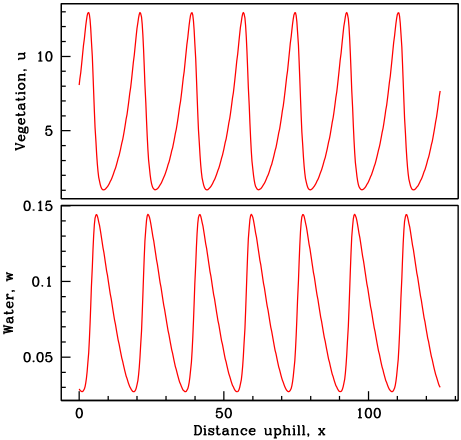

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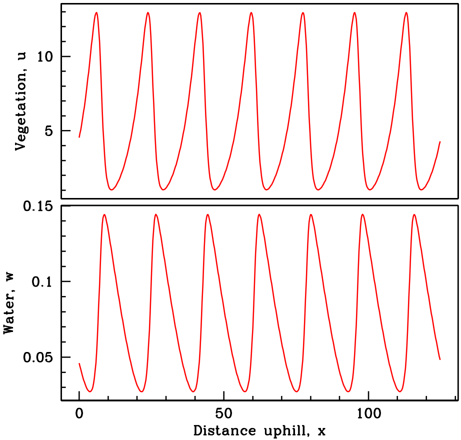

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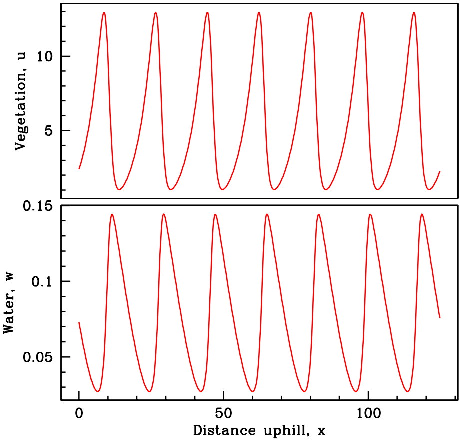

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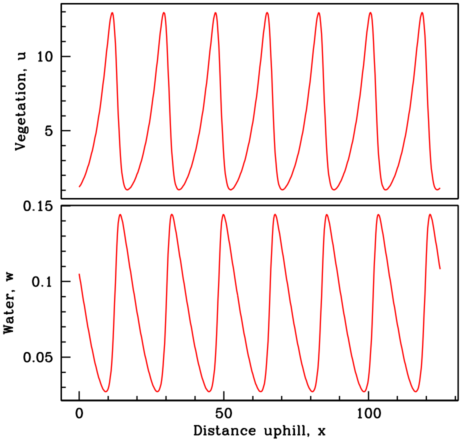

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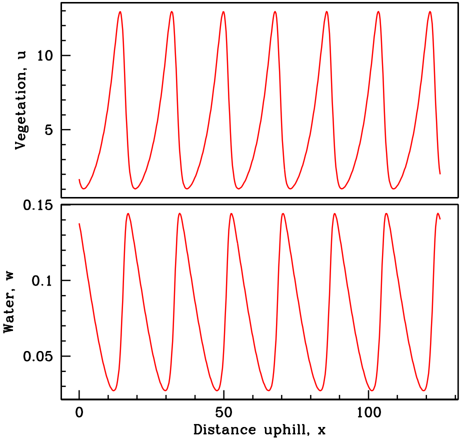

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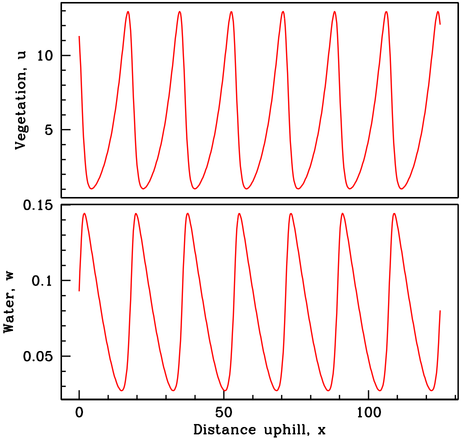

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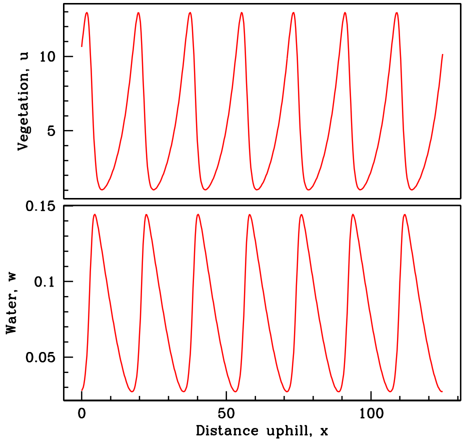

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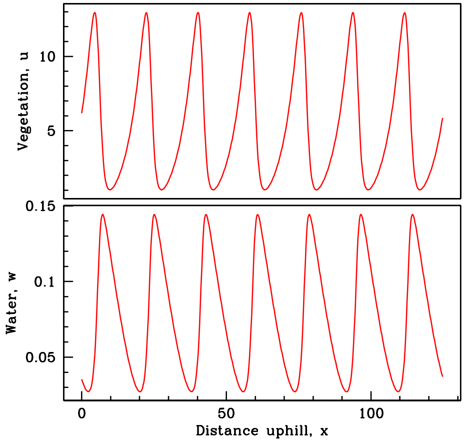

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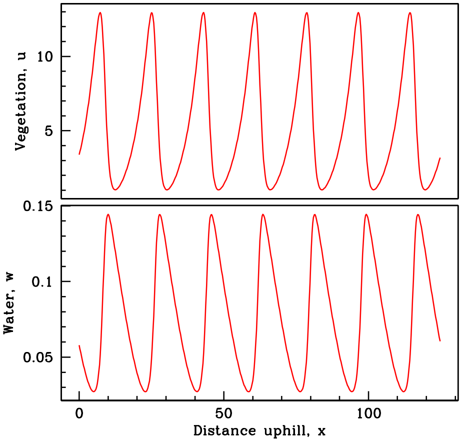

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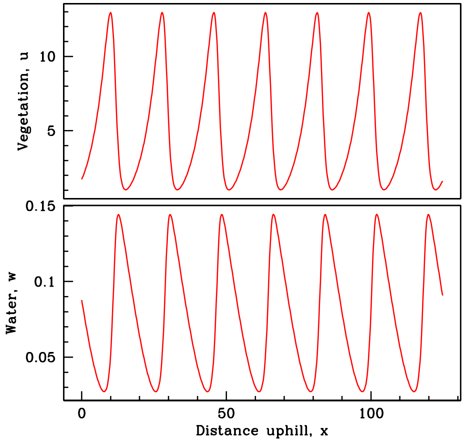

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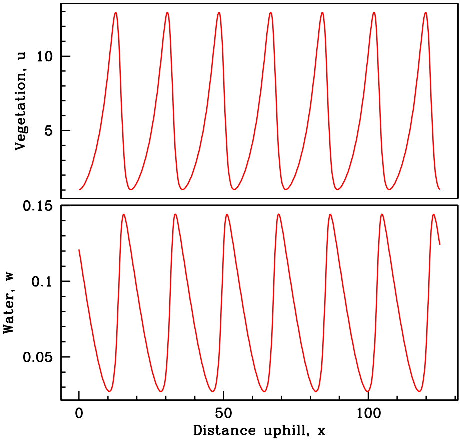

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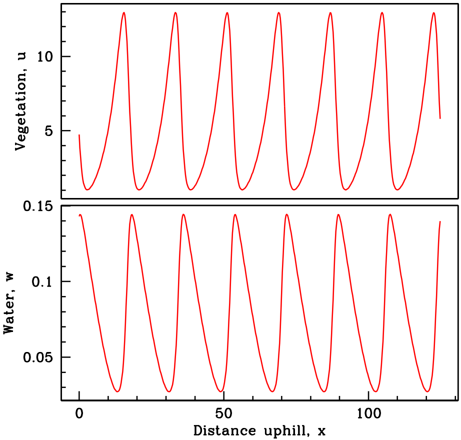

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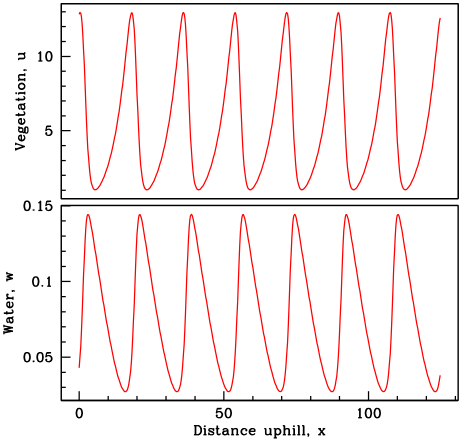

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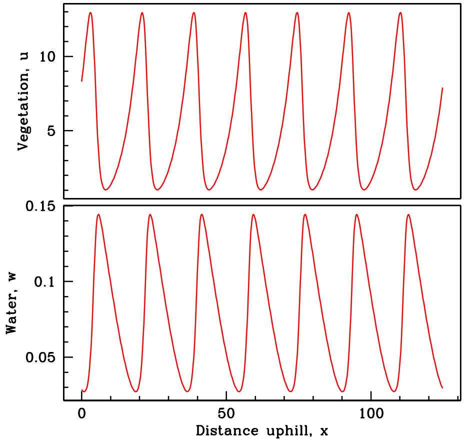

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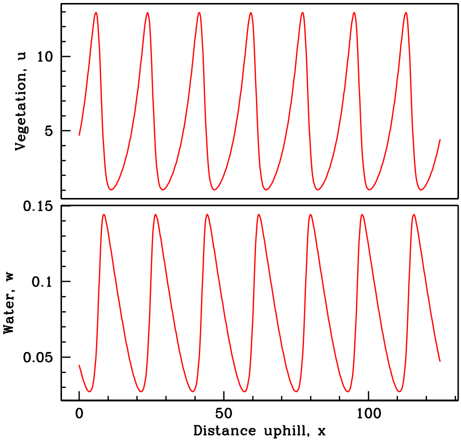

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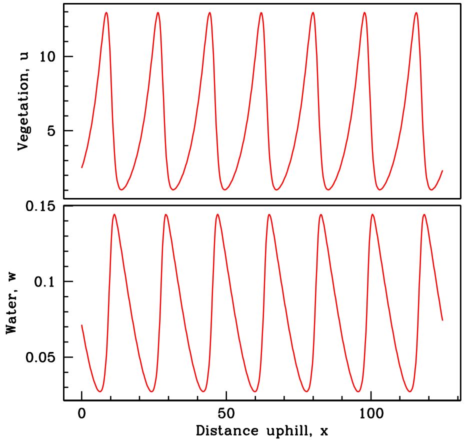

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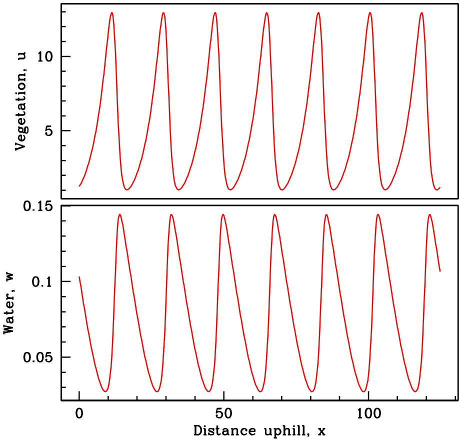

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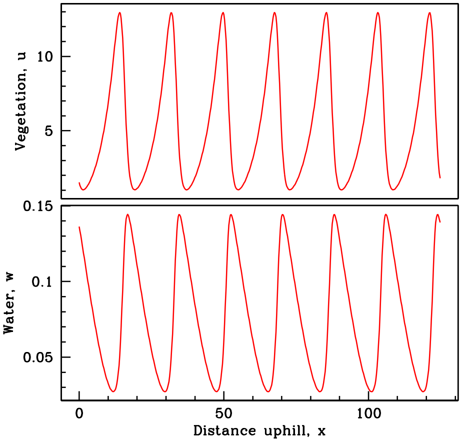

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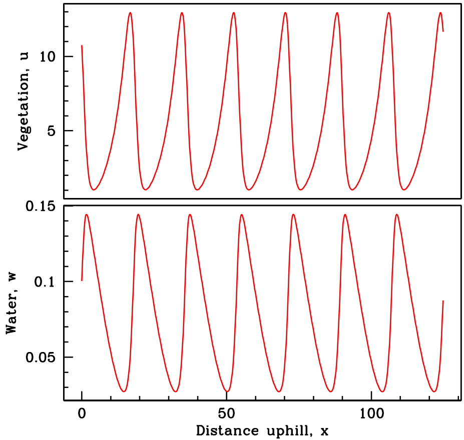

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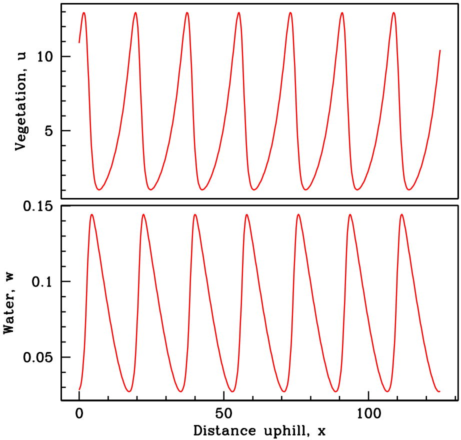

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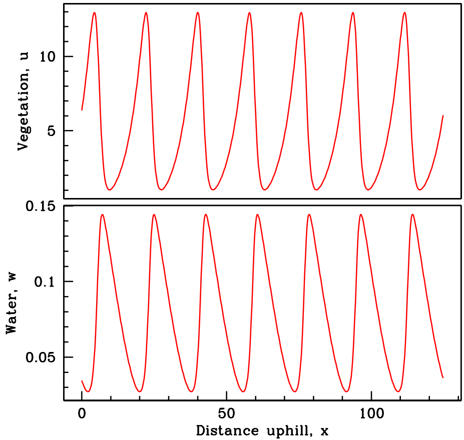

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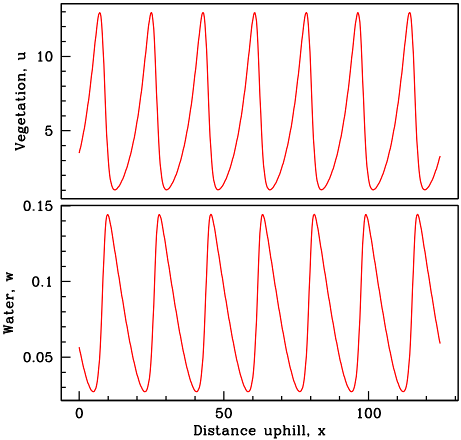

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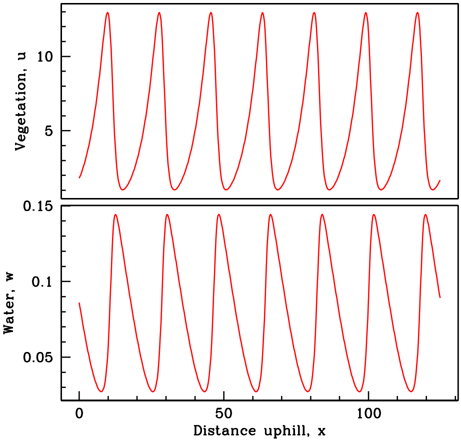

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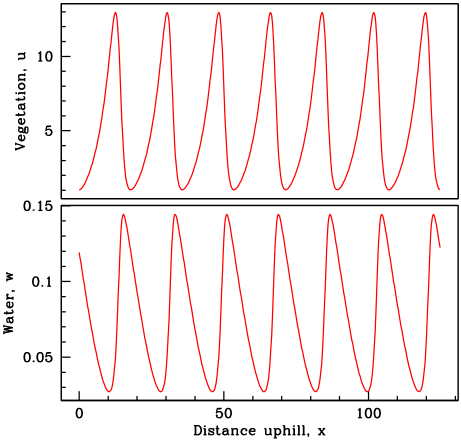

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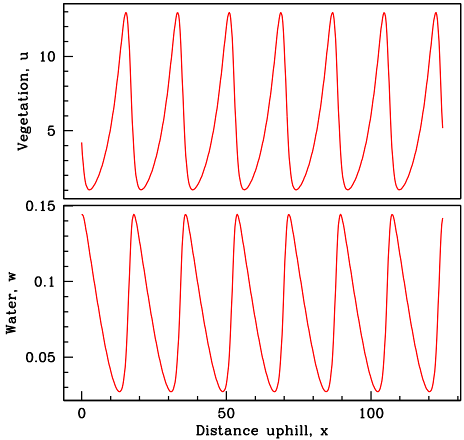

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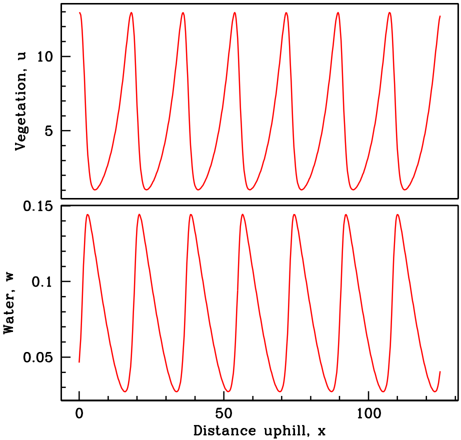

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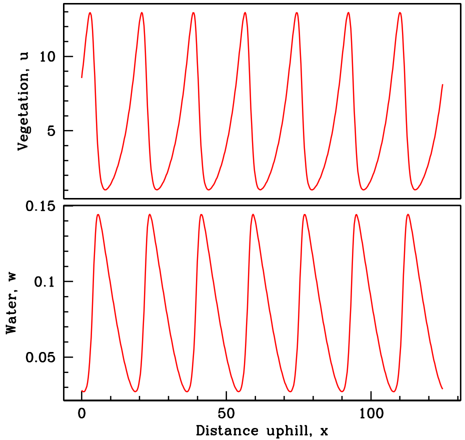

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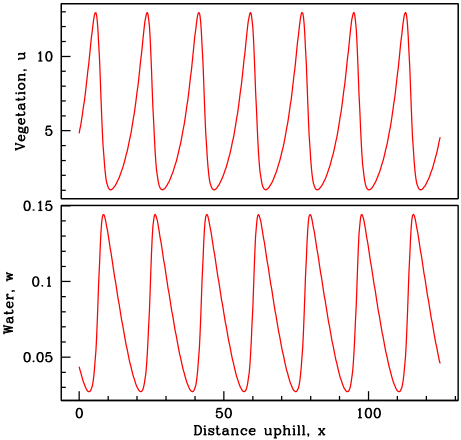

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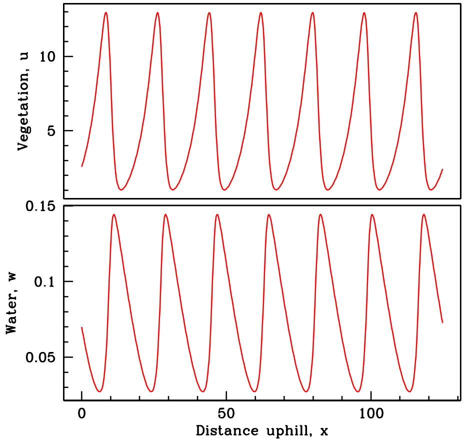

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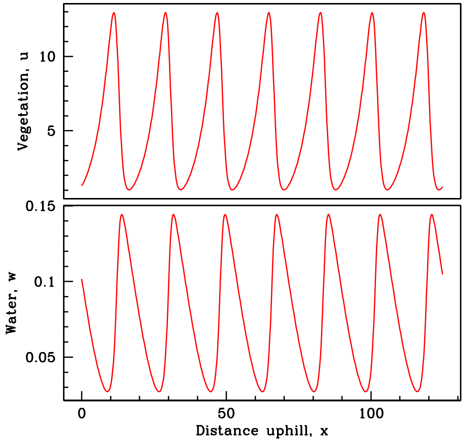

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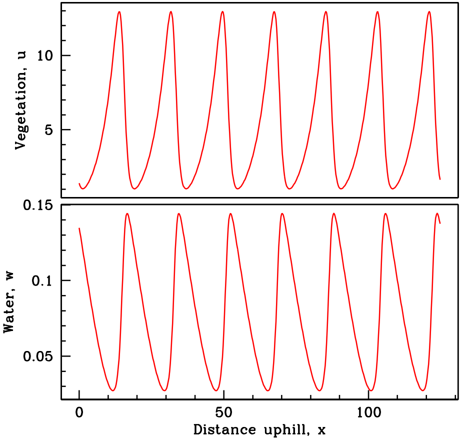

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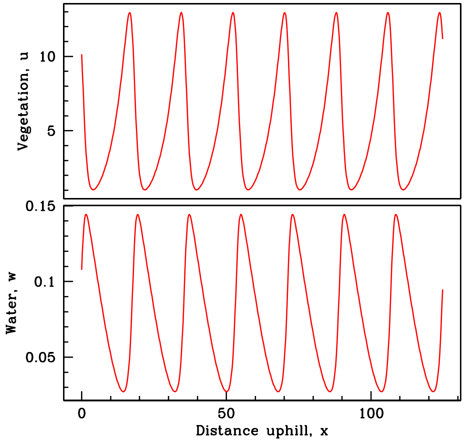

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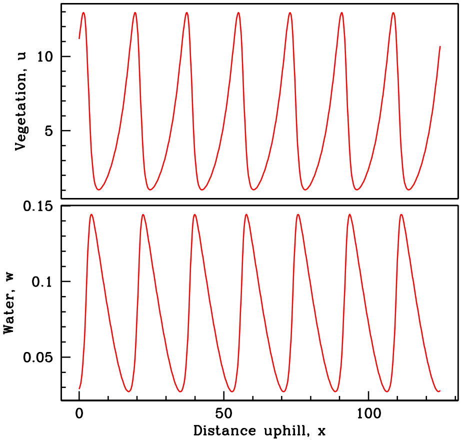

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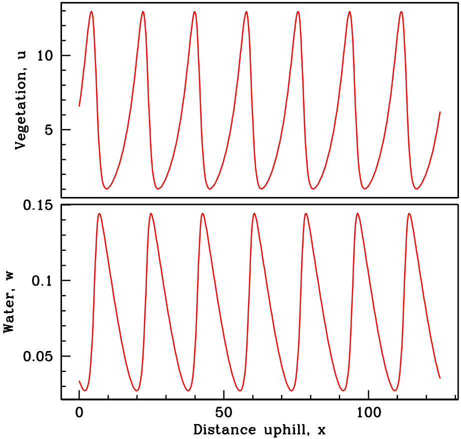

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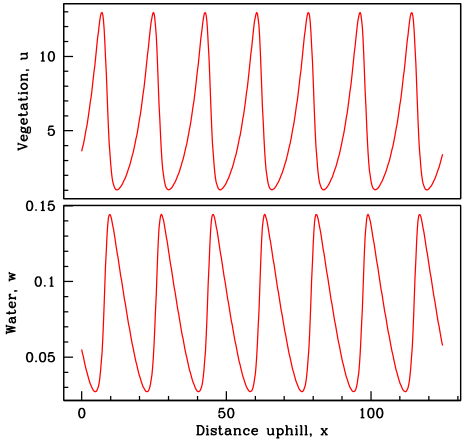

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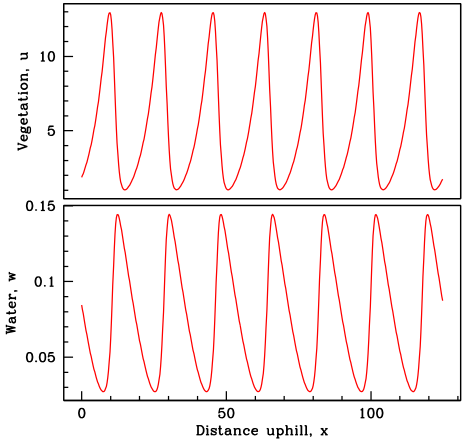

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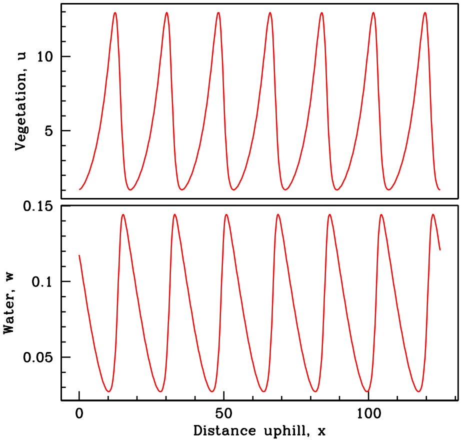

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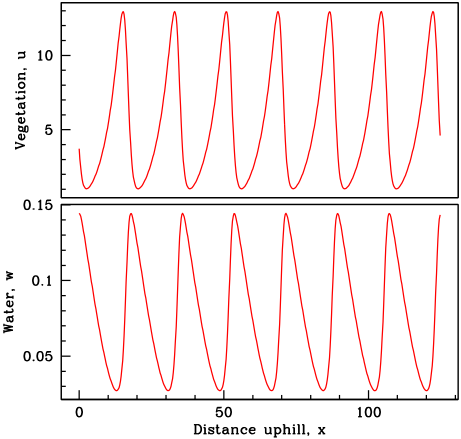

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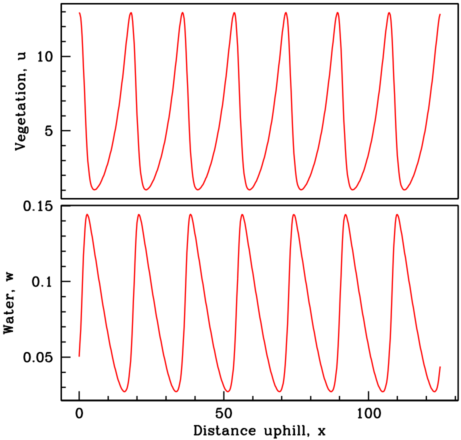

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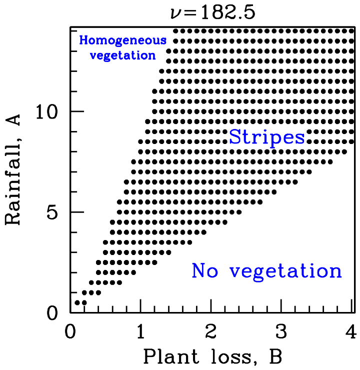
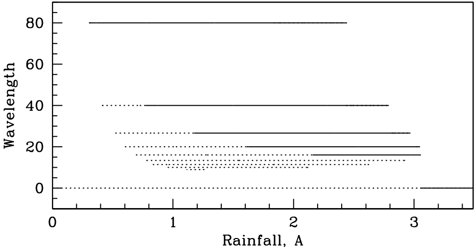
An Illustration of Conditions for Patterning



Predicting Pattern Wavelength

Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography.

Mathematical prediction of wavelength as a function of parameters (rainfall, plant loss, slope) is difficult because there are multiple pattern solutions.



Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation

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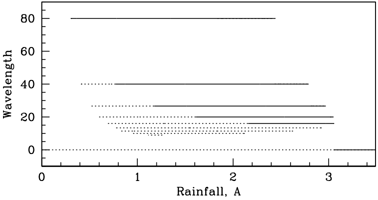

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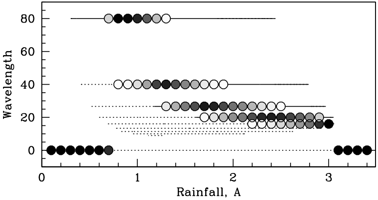

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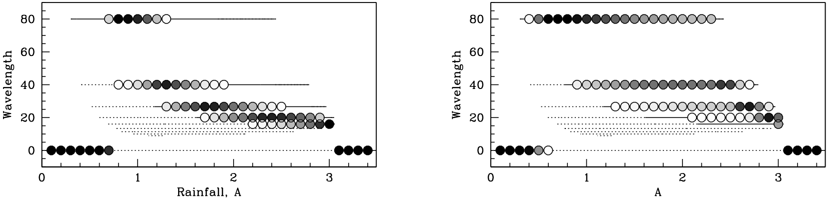
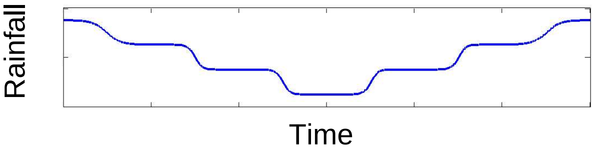


Key Result

For a wide range of rainfall levels, there are multiple stable patterns.

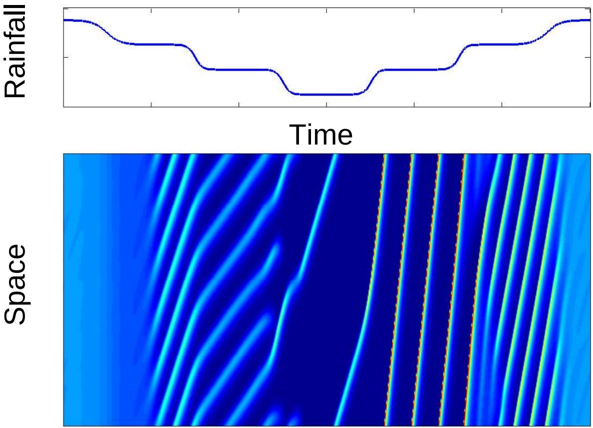
Outline

Hysteresis

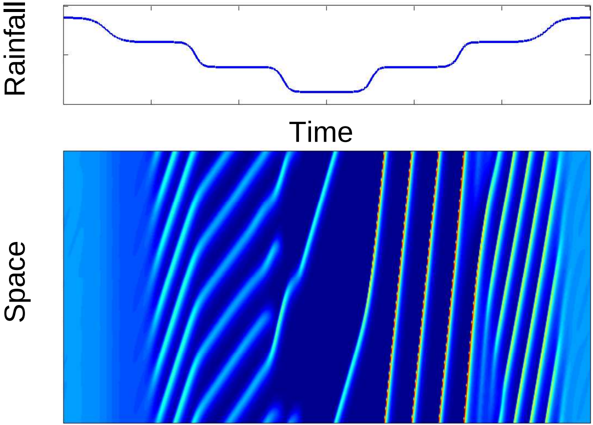


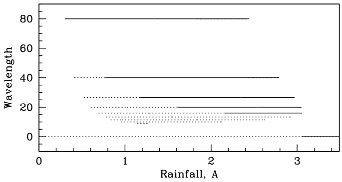
- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A
- Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year

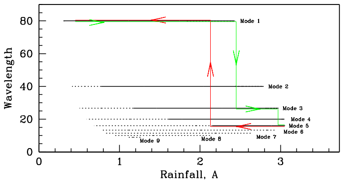
Hysteresis



Hysteresis


Rainfall
Time
Space

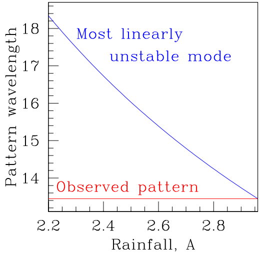

Wavelength
80
60
40
20
0
0
1
2
3
Rainfall, A


Wavelength
Rainfall, A
Mode 1
Mode 2
Mode 3
Mode 4
Mode 5
Mode 6
Mode 7
Mode 8
Mode 9

Conclusion

- In general, pattern wavelength depends on initial conditions
- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at a constant value.

$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



List of Frames

Bifurcation Diagram for Discretized PDEs