Vegetation Patterning in Semi-Arid Environments

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In collaboration with
Gabriel Lord

Outline

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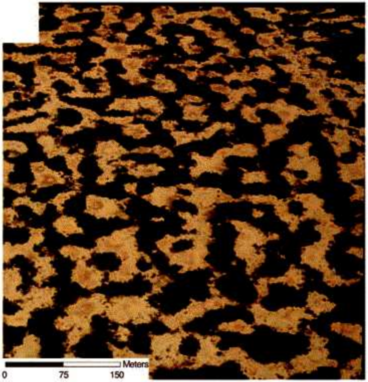
Vegetation Pattern Formation

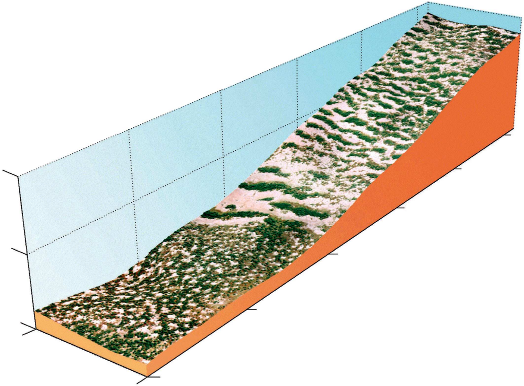
- Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees

Mosaic and Striped Patterns



Labyrinth of bushy vegetation in Niger

Mosaic and Striped Patterns



- On flat ground, irregular mosaics of vegetation are typical
- On slopes, the patterns are stripes, parallel to contours (“Tiger bush”)

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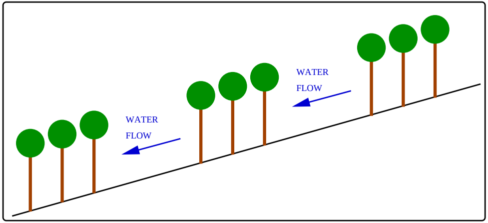

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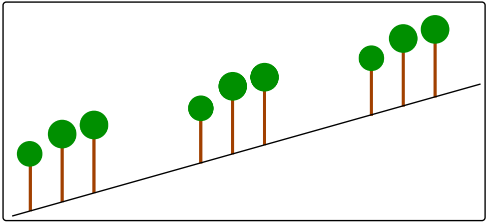

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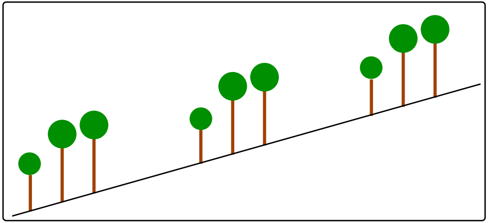

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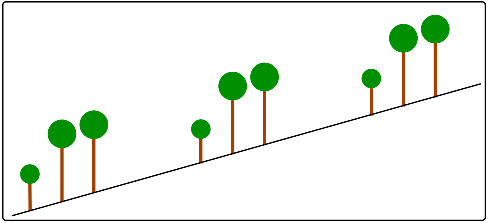

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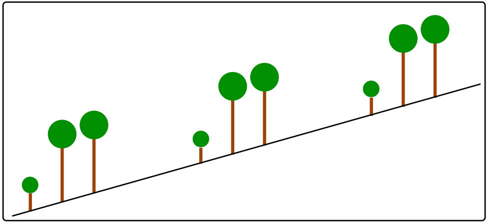

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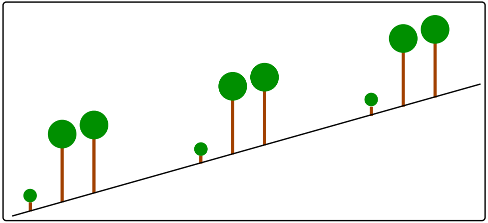

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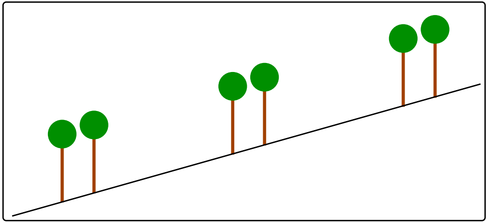

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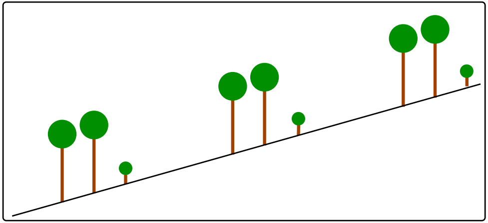

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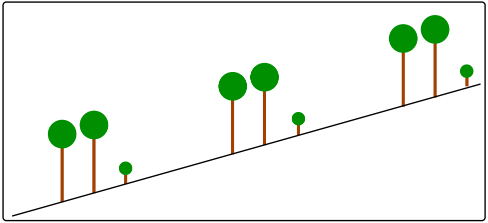

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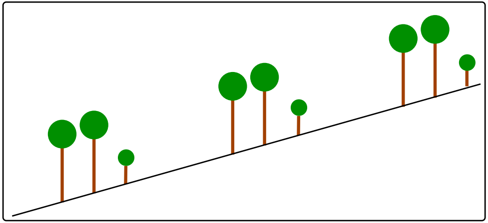

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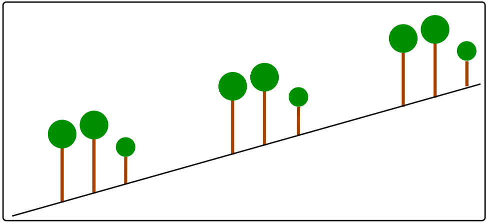

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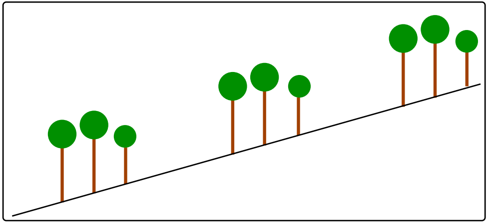

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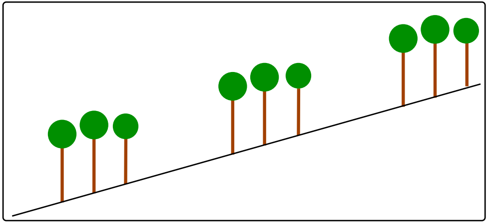

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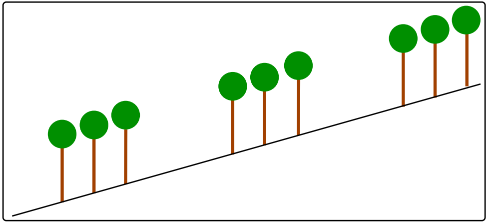

- This mechanism suggests that the stripes move uphill

Outline

Mathematical Model of Klausmeier

Rate of change of water = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change plant biomass = Growth, proportional to water uptake – Mortality + Random dispersal

$$\partial w/\partial t = A - w - wu^2 + \nu \partial w/\partial x$$

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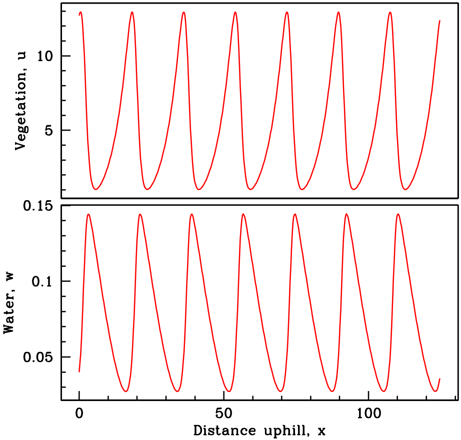
The nonlinearity in wu^2 arises because the presence of roots increases water infiltration into the soil.

Typical Solution of the Model



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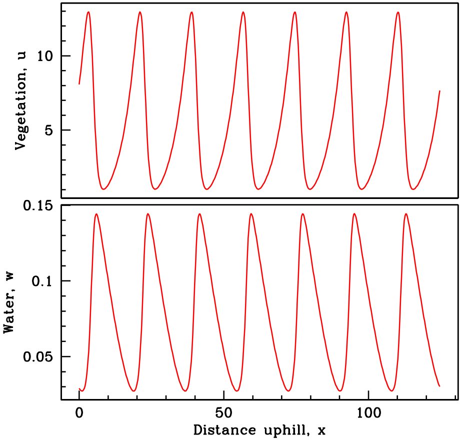

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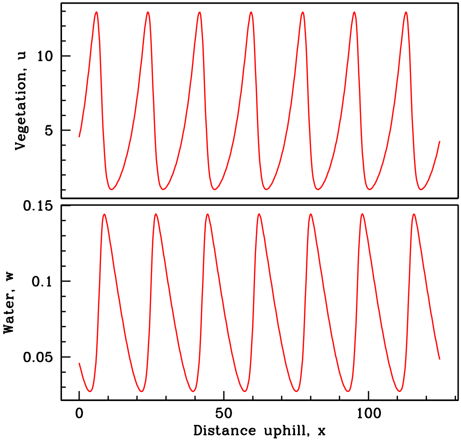

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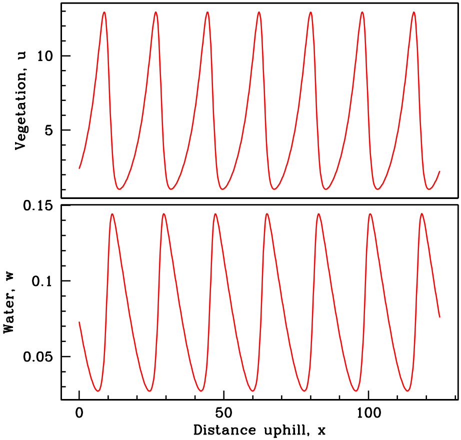

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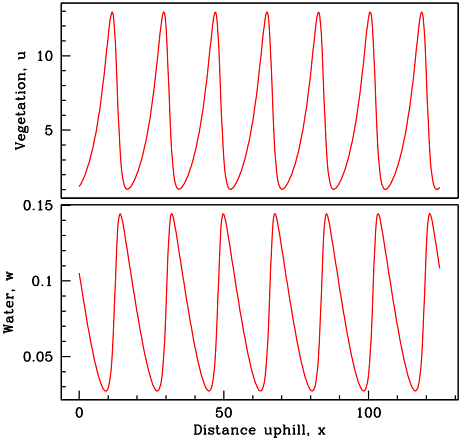

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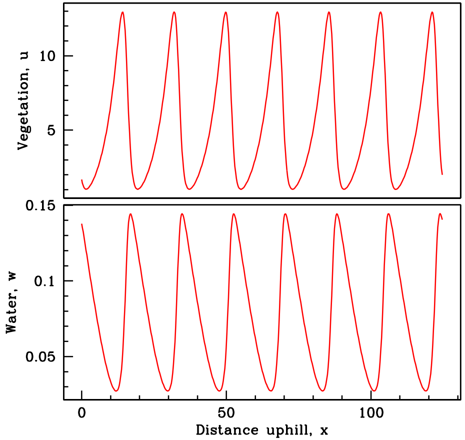

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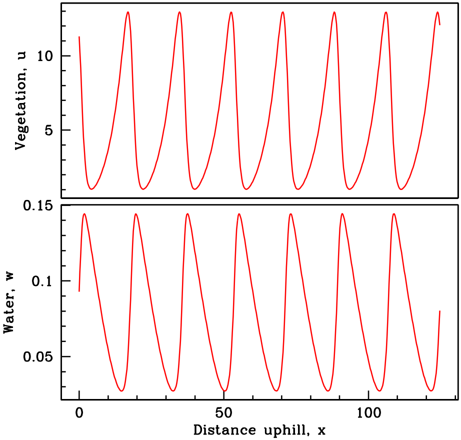

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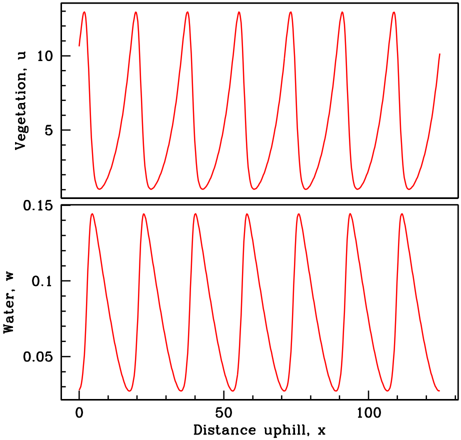

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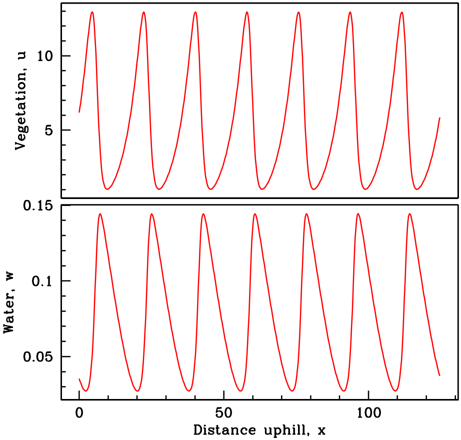

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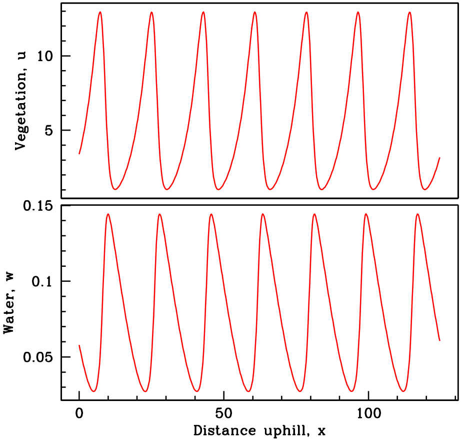

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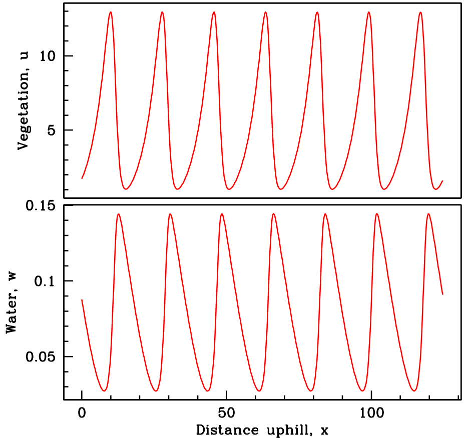

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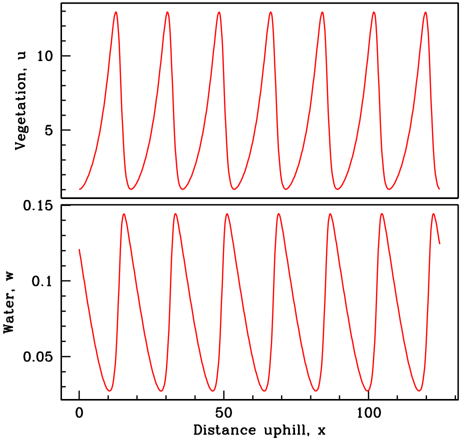

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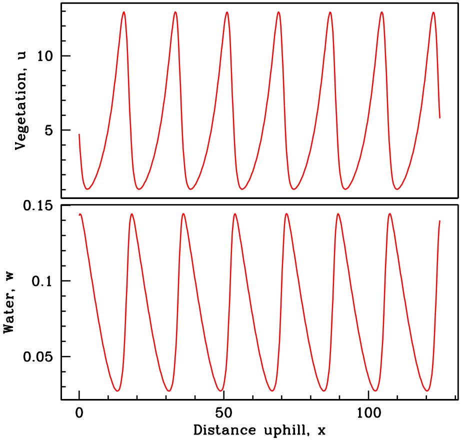

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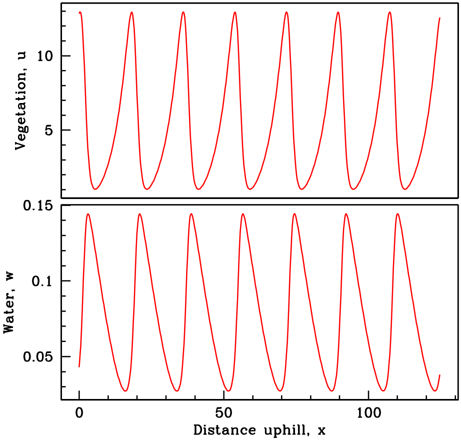

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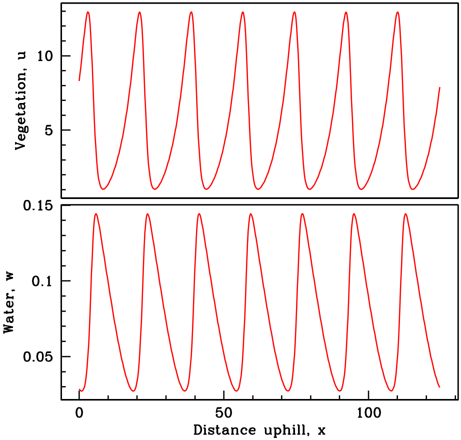

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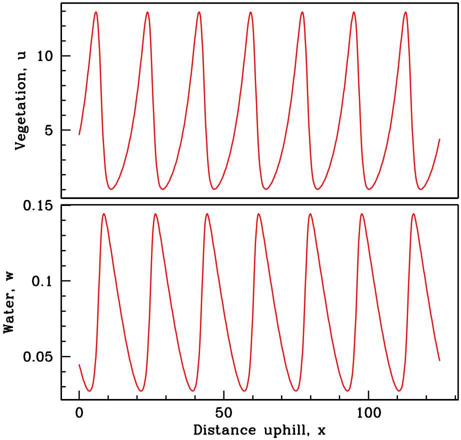

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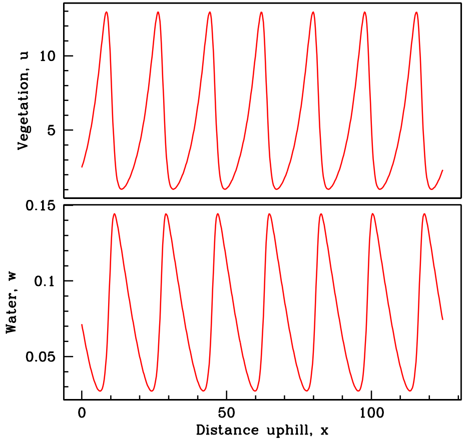

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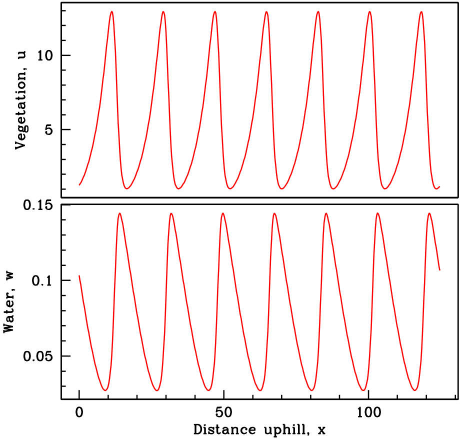

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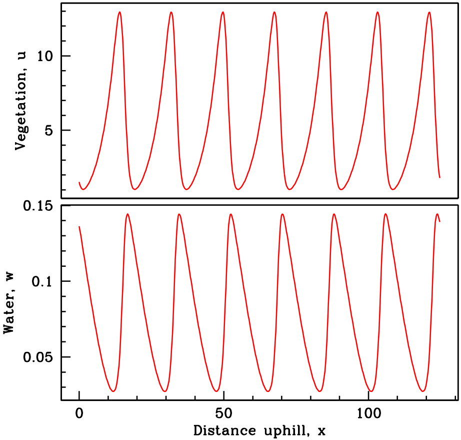

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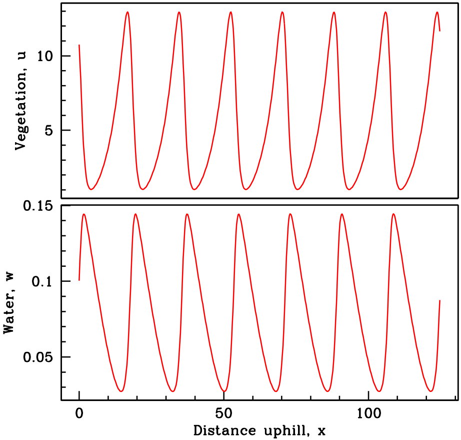

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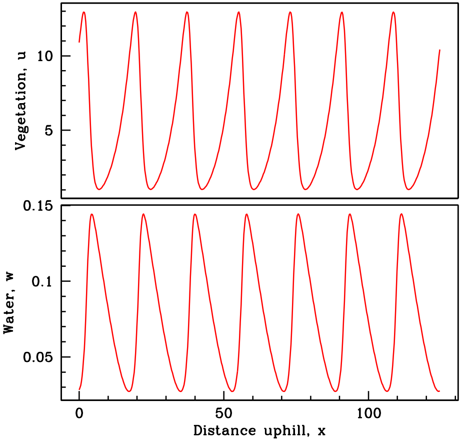

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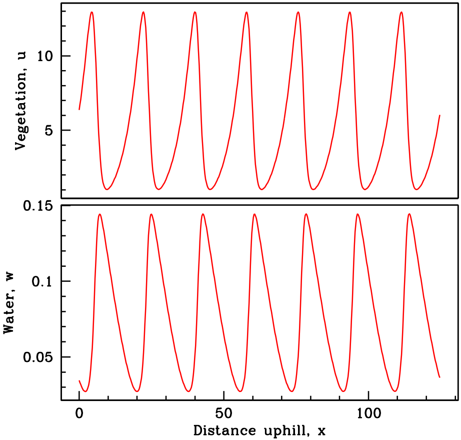

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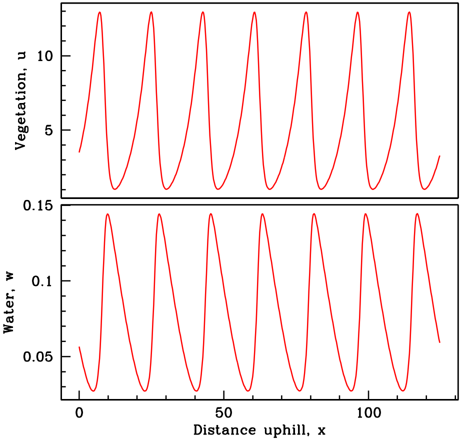

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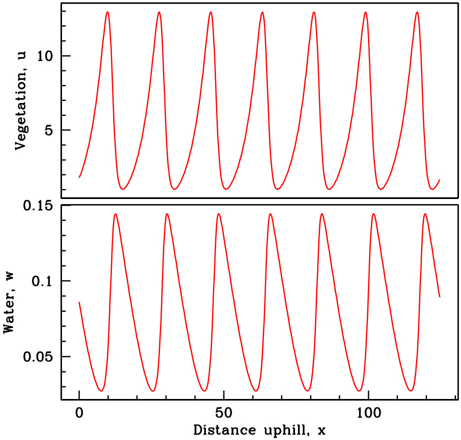

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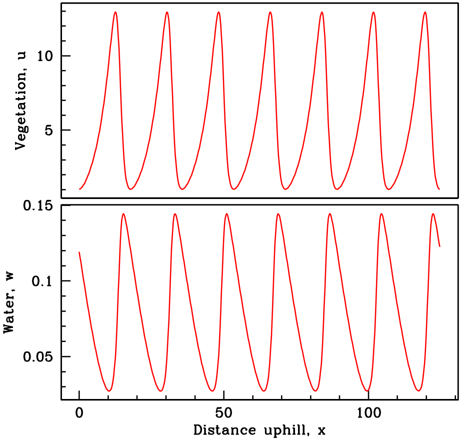

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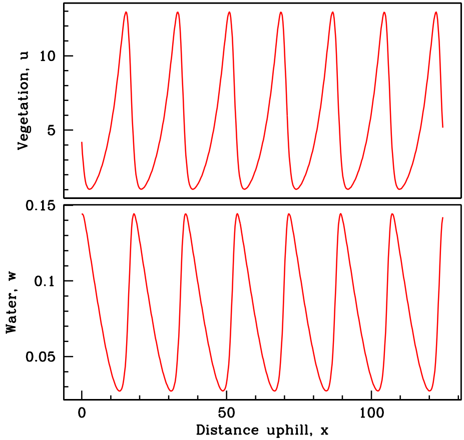

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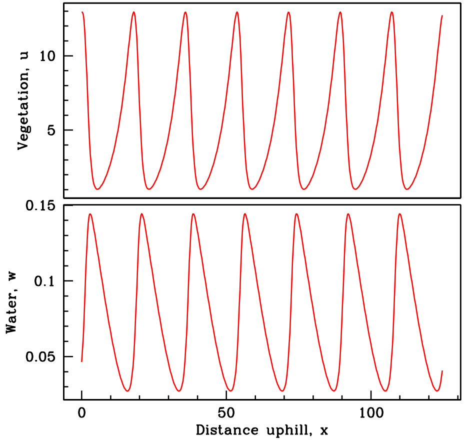

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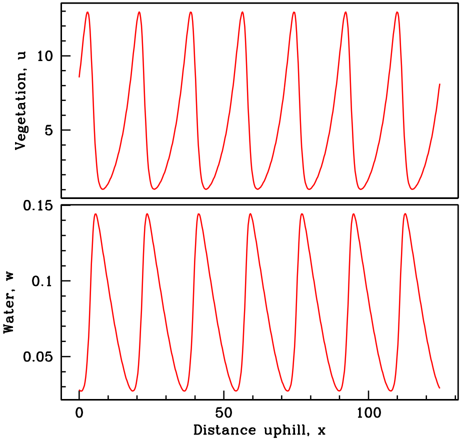

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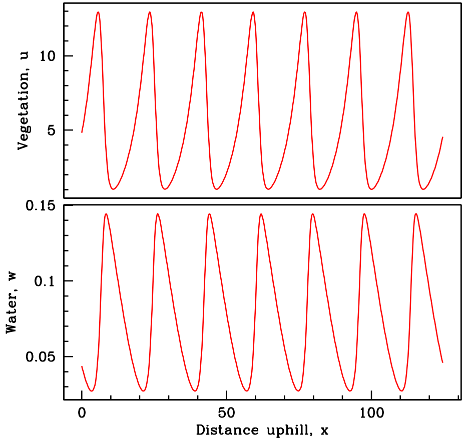

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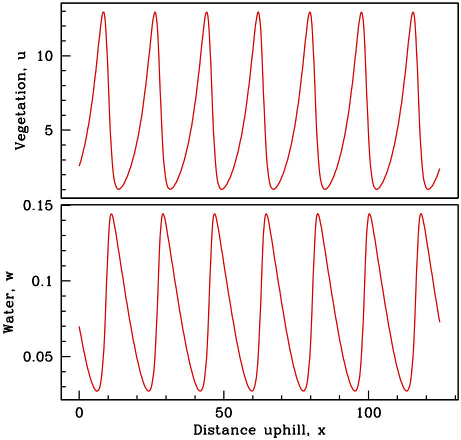

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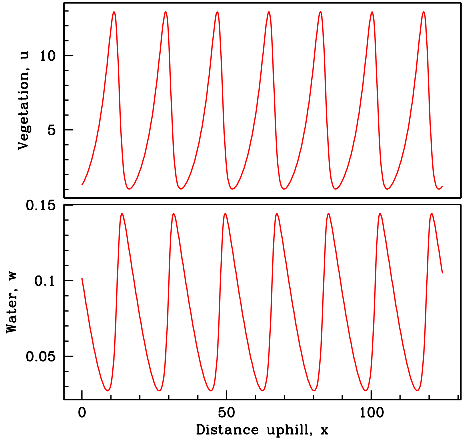

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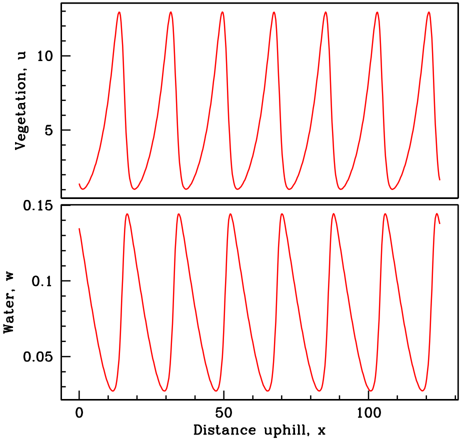

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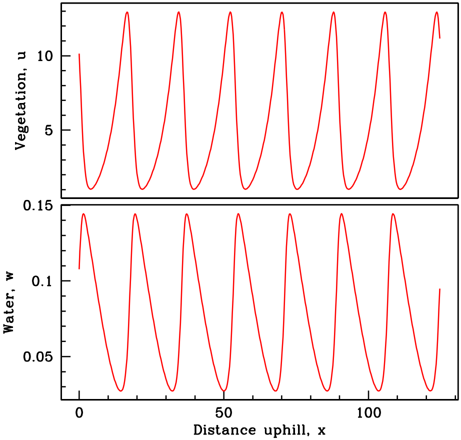

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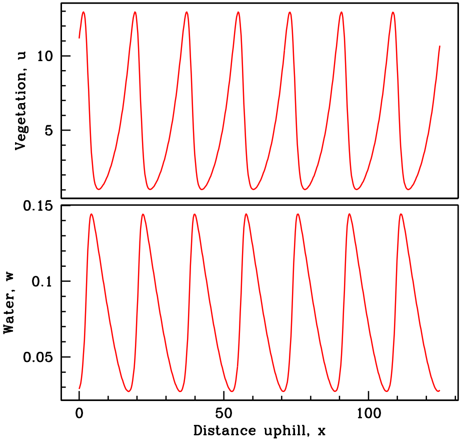

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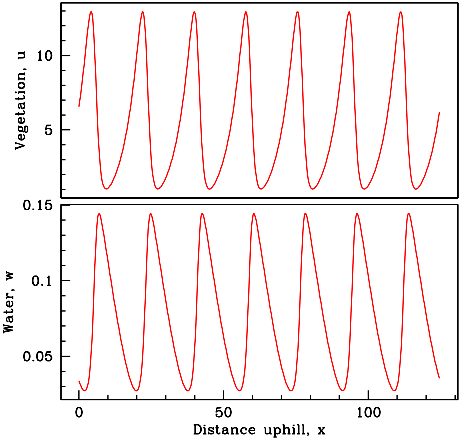

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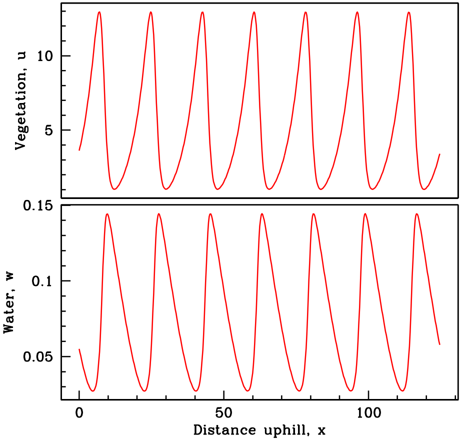

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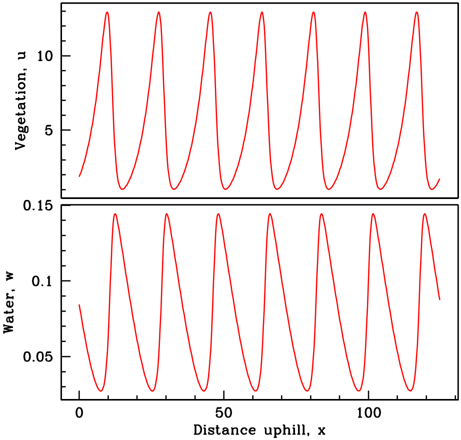

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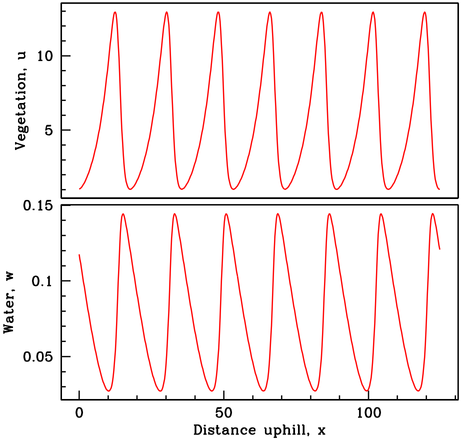

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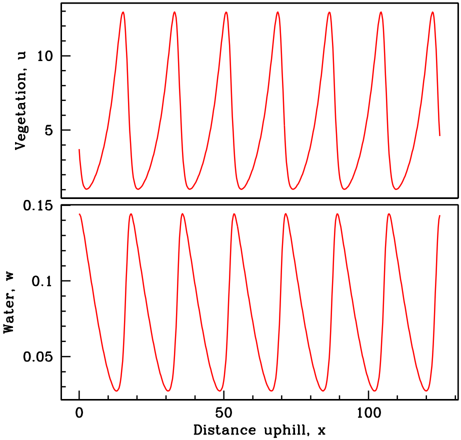

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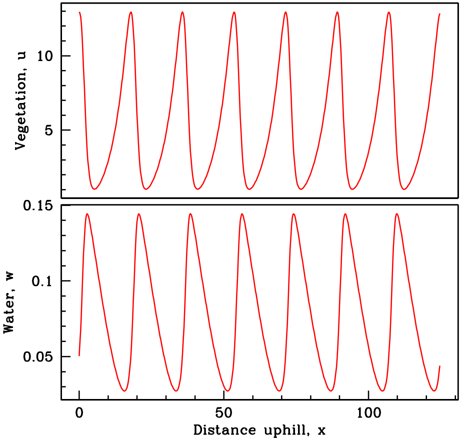

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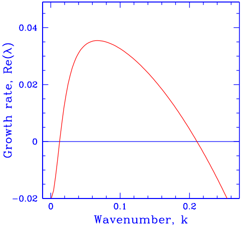
stable to homog pertns for $B < 2$

- Patterns develop when (u_s, w_s) is unstable to inhomogeneous perturbations

Approximate Conditions for Patterning

Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0)\exp\{ikx + \lambda t\}$



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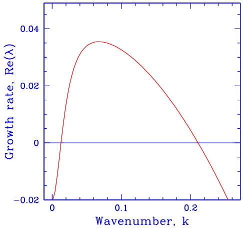

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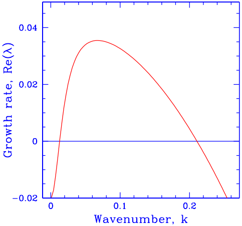

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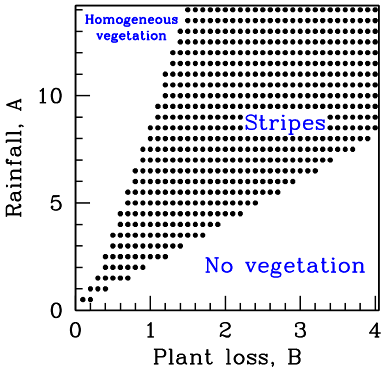
One can niavely assume that existence of (u_s, w_s) gives a second condition

An Illustration of Conditions for Patterning



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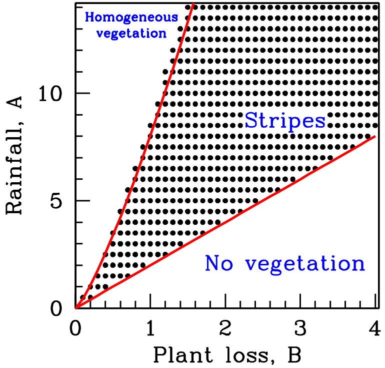

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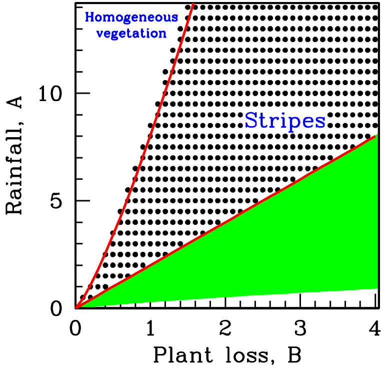
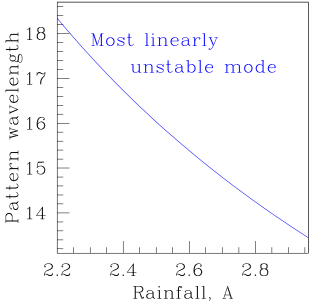


Predicting Pattern Wavelength

Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography. Wavelength can be predicted from the linear analysis



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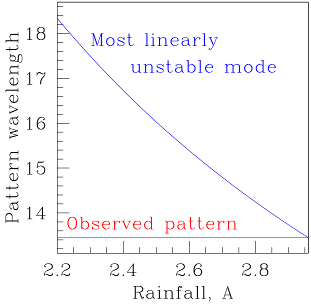


However this prediction doesn't fit the patterns seen in numerical simulations.

Outline

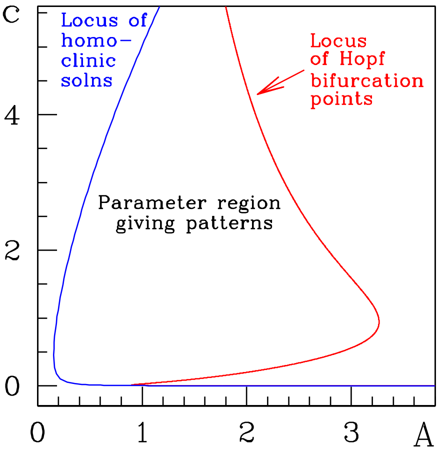
Travelling Wave Equations

The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z)$, $w(x,t) = W(z)$, $z = x - ct$

$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these ODEs

When do Patterns Form?



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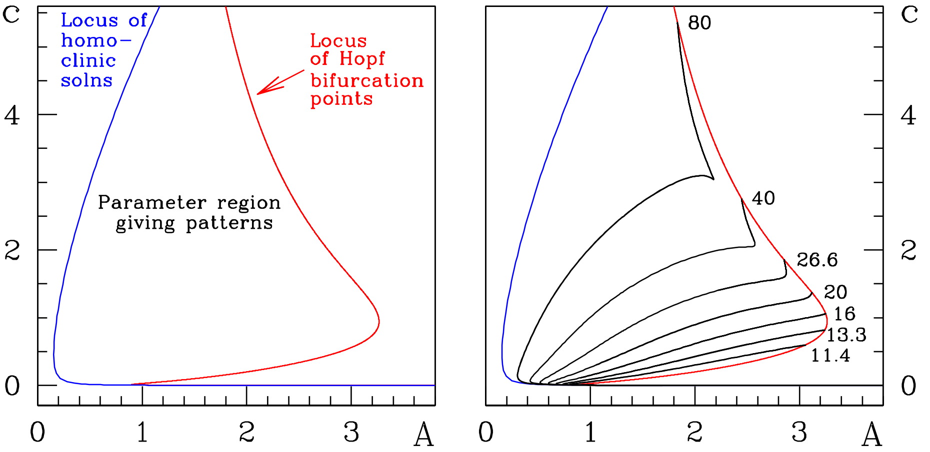
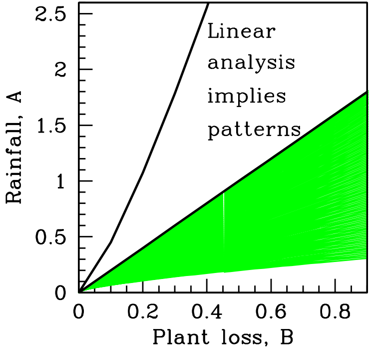


Pattern Formation for Low Rainfall



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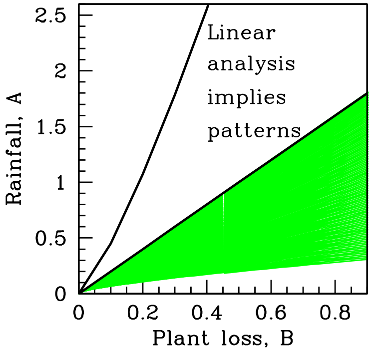

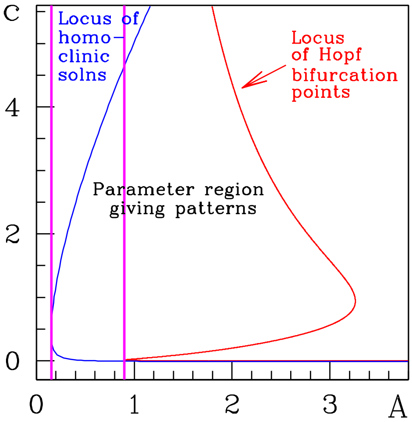

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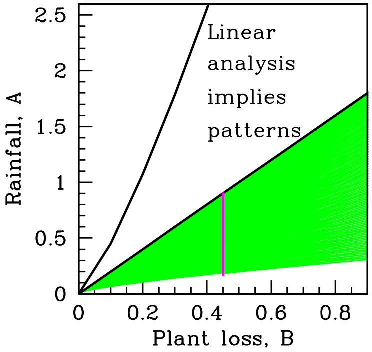
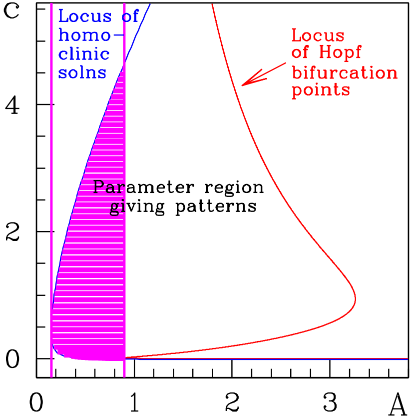




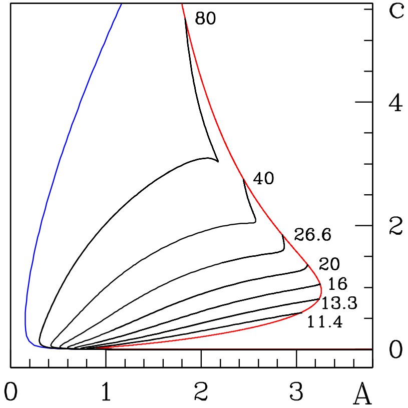
Outline

Basic Approach: Discretizing the PDEs

- We discretize the PDEs in space to give a large system of ODEs
- We then apply numerical bifurcation methods (AUTO)
- This gives the various patterns as before, but with a key piece of additional information: stability

Speed vs Rainfall for the Discretized PDEs

c vs *A* for travelling wave ODEs



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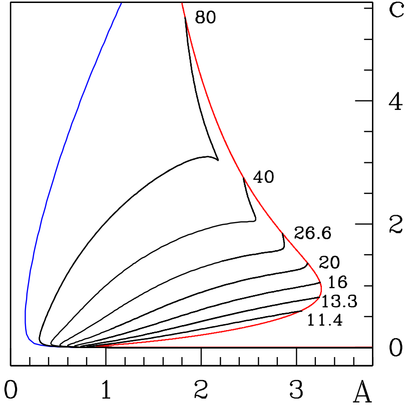

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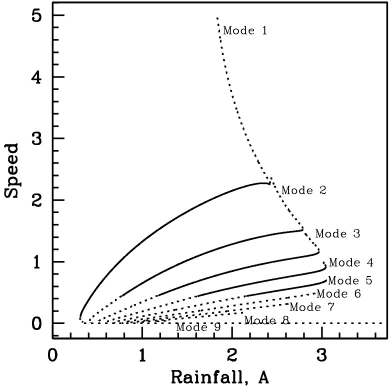
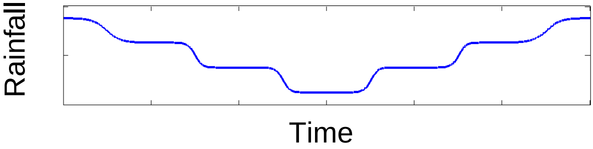


Key Result

Many of the possible patterns are unstable and thus will never be seen.

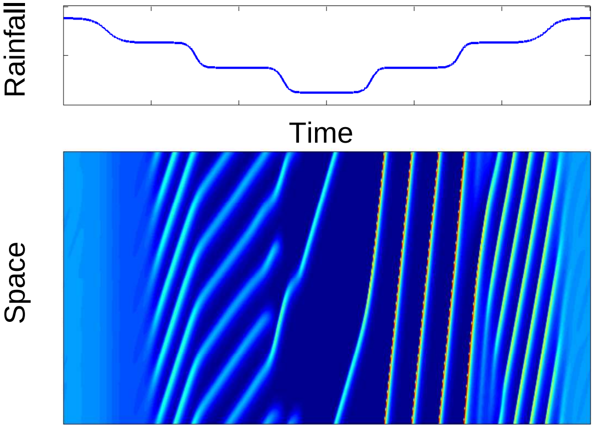
However, for a wide range of rainfall levels, there are multiple stable patterns.

Hysteresis



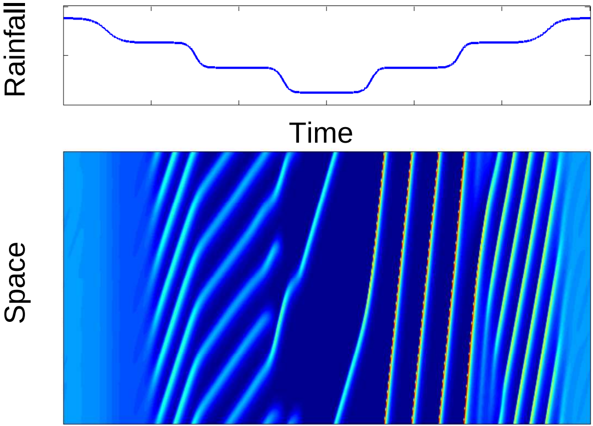
- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A
- Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year

Hysteresis



<< Mode 5 >> <<<<< Mode 1 >>>>> < Mode 3 >

Hysteresis



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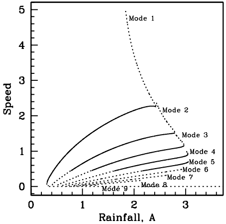
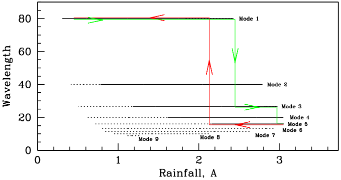
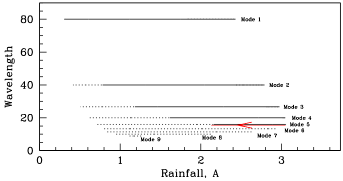




Outline

Predictions of Pattern Wavelength

- In general, pattern wavelength depends on initial conditions
- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.



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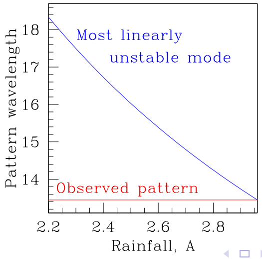
$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



Other Potential Mechanisms for Vegetation Patterns

Rietkirk Klausmeier model with diffusion of water in the soil

van de Koppel Klausmeier model with grazing

Maron two variable model (plant density and water in the soil) with water transport based on porous media theory

Lejeune short range activation (shading) and long range inhibition (competition for water)

All of these models predict patterns. To discriminate between them requires a detailed understanding of each model.

List of Frames

Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation

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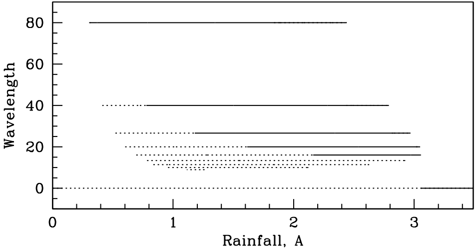

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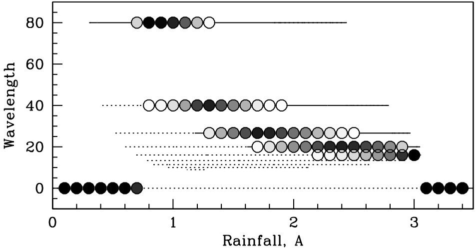

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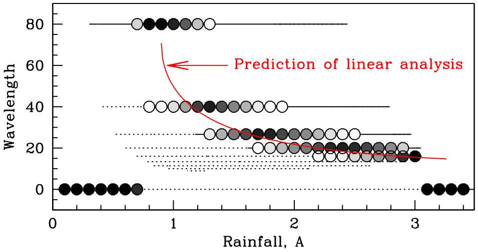
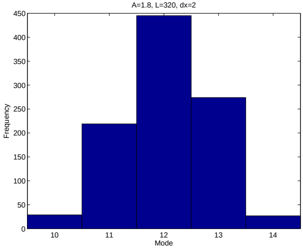


The wavelength is close to that predicted by linear stability analysis

Pattern Selection on Larger Domains

The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged



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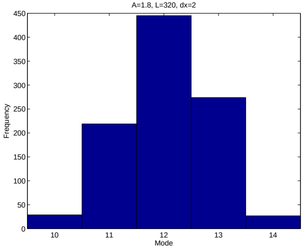

But it does not apply for other initial conditions, such as perturbations about (u_u, w_u)