The Dynamics of Vegetation Patterning in Semi-Arid Environments

Jonathan A. Sherratt

Department of Mathematics
Heriot-Watt University

University of Edinburgh, 19 January 2007

In collaboration with
Gabriel Lord

Outline

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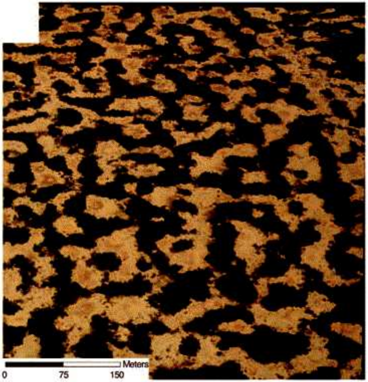
Vegetation Pattern Formation

- Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico (rainfall 100-700 mm/year)
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees

More Pictures of Vegetation Patterns



Labyrinth of bushy vegetation in Niger

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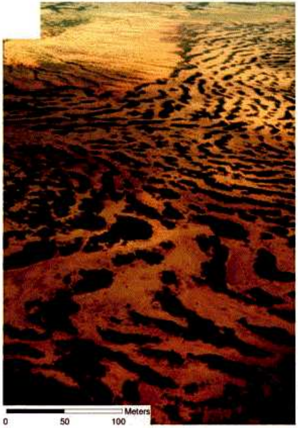
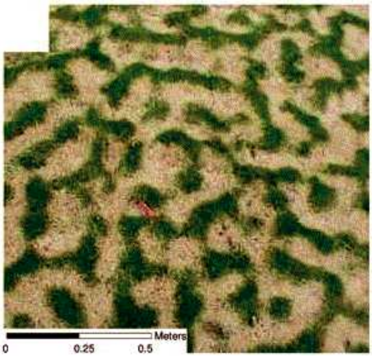


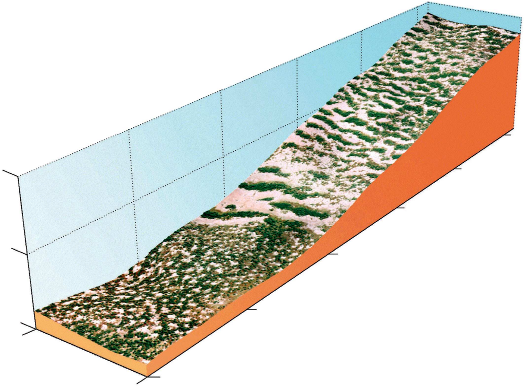
Striped pattern of bushy vegetation in Niger

More Pictures of Vegetation Patterns



Labyrinth of grass in Israel

Vegetation Pattern Formation (contd)



- On flat ground, irregular mosaics of vegetation are typical
- On slopes, the patterns are stripes, parallel to contours (“Tiger bush”)

Mechanisms for Vegetation Patterning

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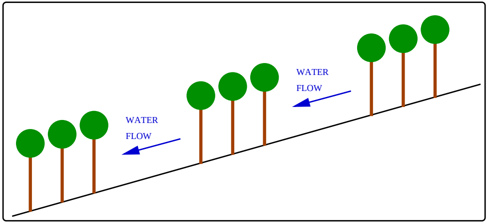

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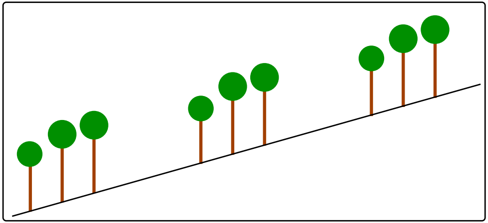

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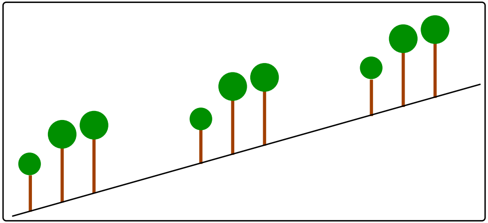

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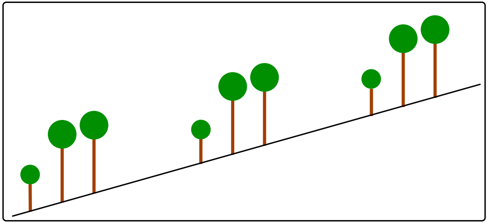

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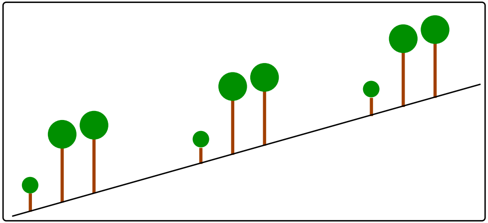

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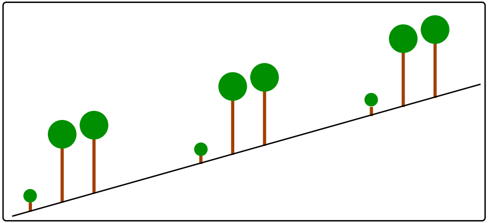

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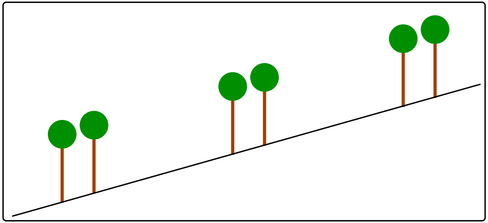

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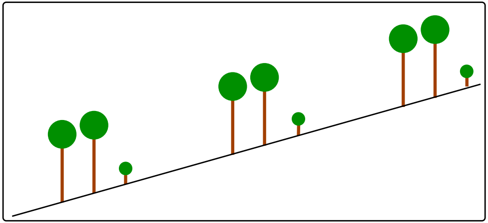

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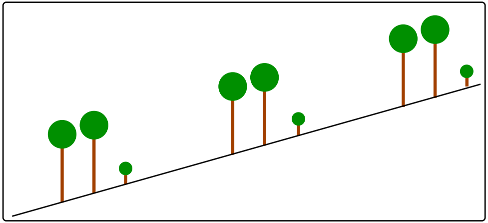

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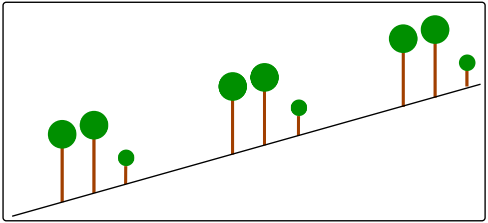

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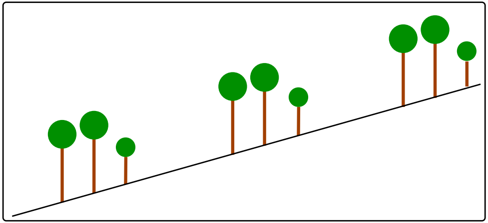

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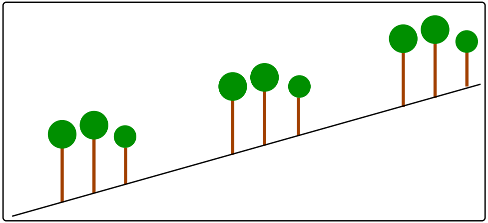

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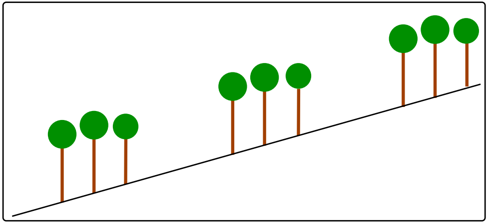

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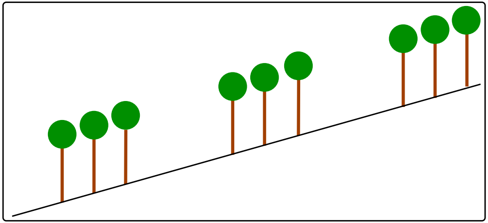

- This mechanism suggests that the stripes would move uphill; this remains controversial.

Outline

Mathematical Model of Klausmeier

Rate of change of water	=	Rainfall – Evaporation	–	Uptake by plants	+	Flow downhill
Rate of change plant biomass	=	Growth, proportional to water uptake	–	Mortality	+	Random dispersal

$$
\begin{aligned}
\partial w/\partial t &= A - w - wu^2 + \nu \partial w/\partial x \\
\partial u/\partial t &= wu^2 - Bu + \partial^2 u/\partial x^2
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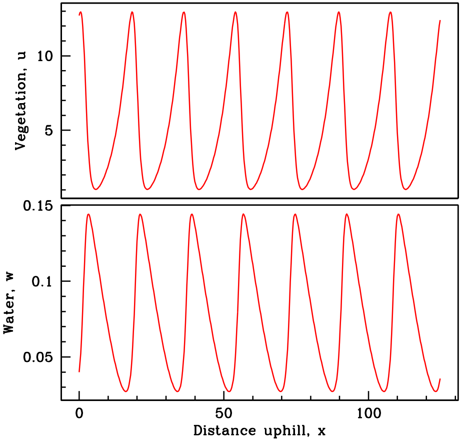
The nonlinearity in wu^2 arises because the presence of roots increases water infiltration into the soil.

Typical Solution of the Model



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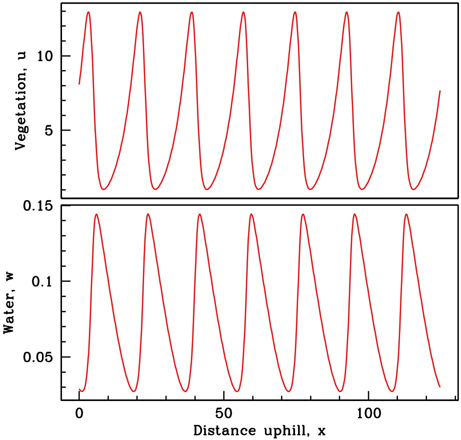

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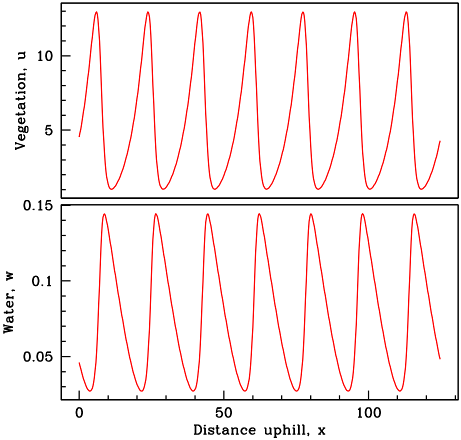

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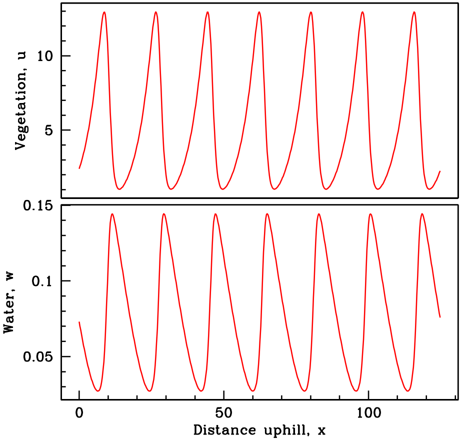

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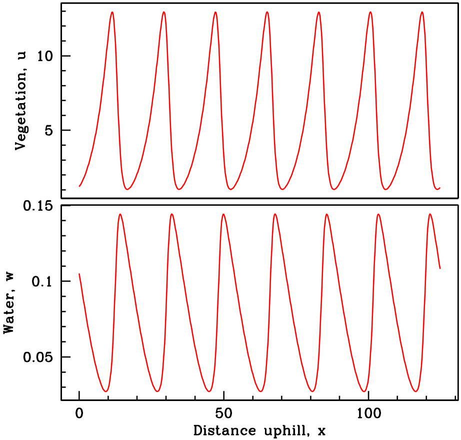

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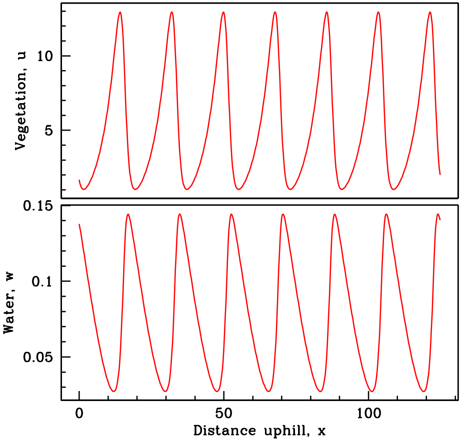

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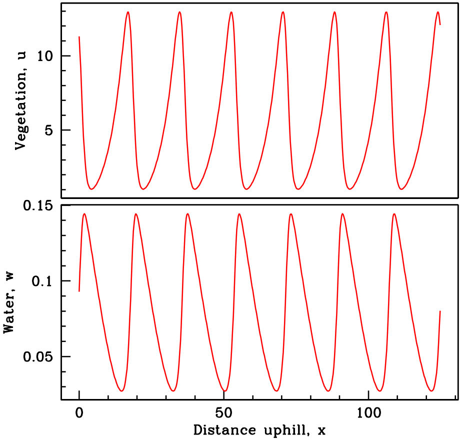

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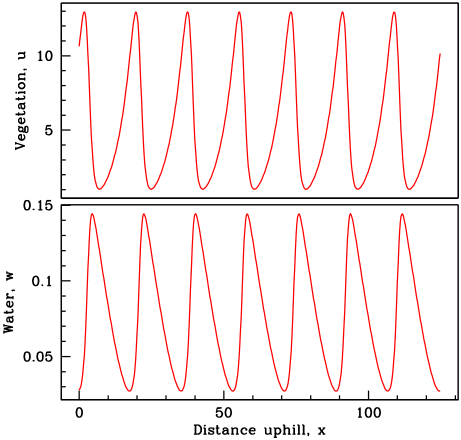

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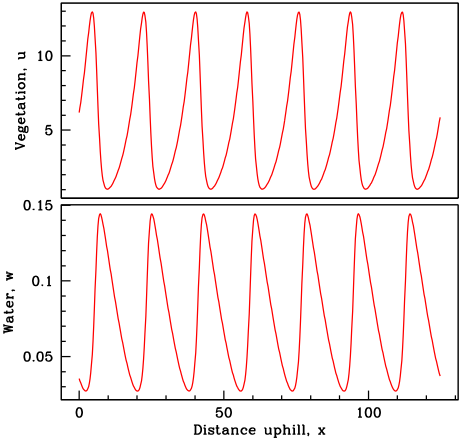

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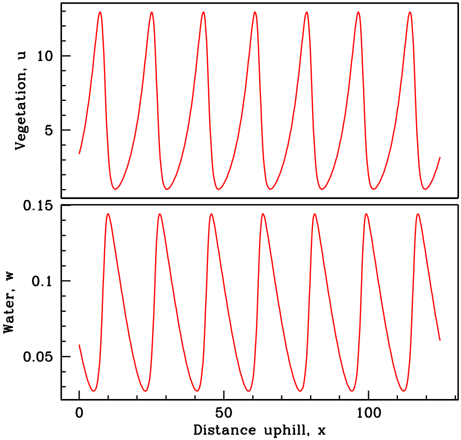

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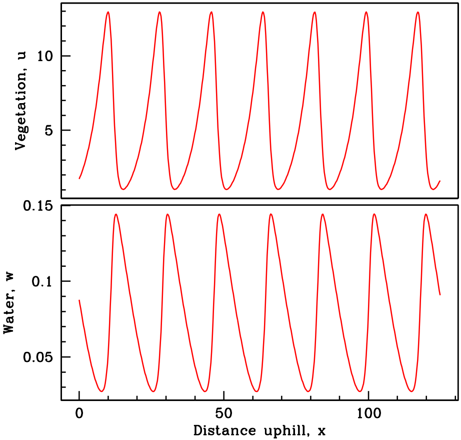

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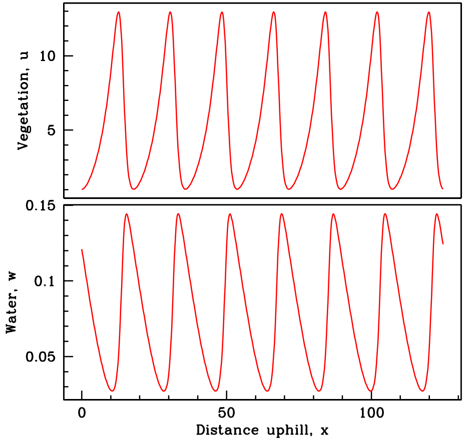

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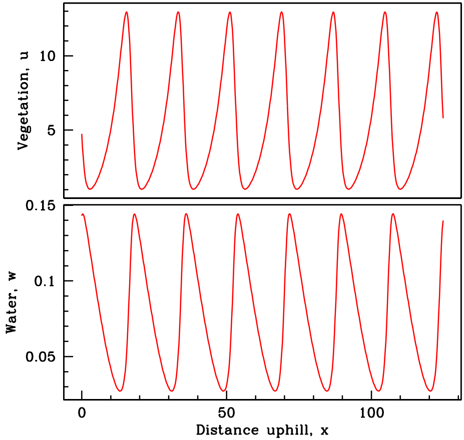

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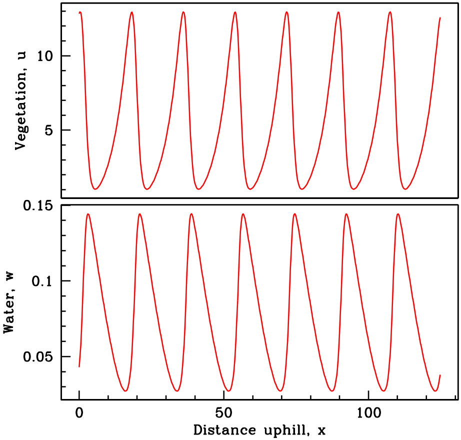

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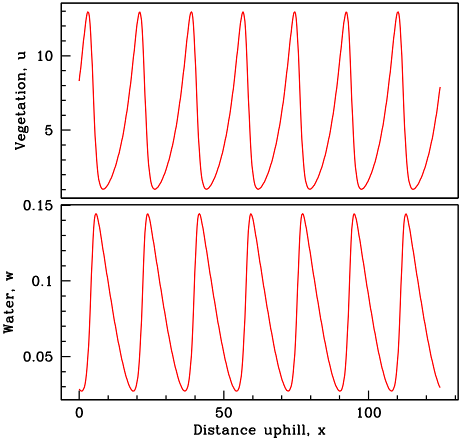

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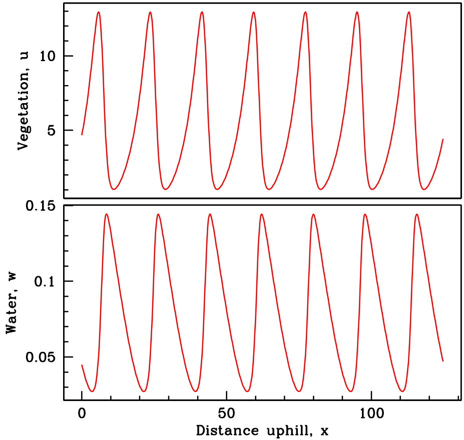

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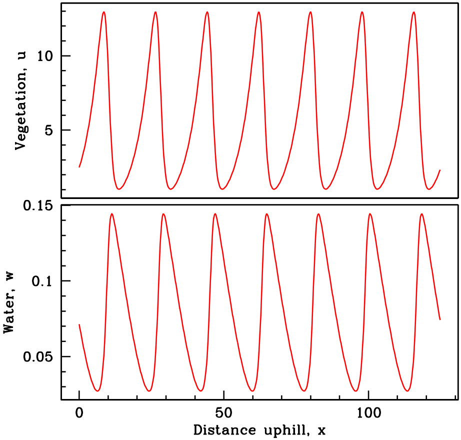

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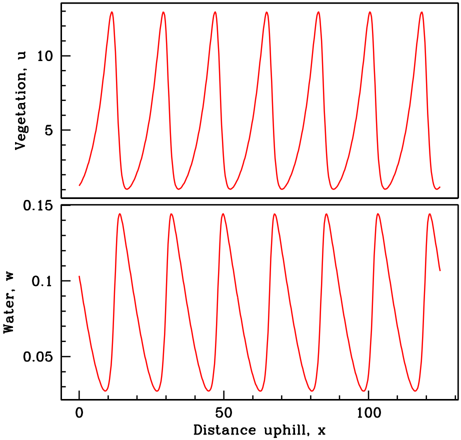

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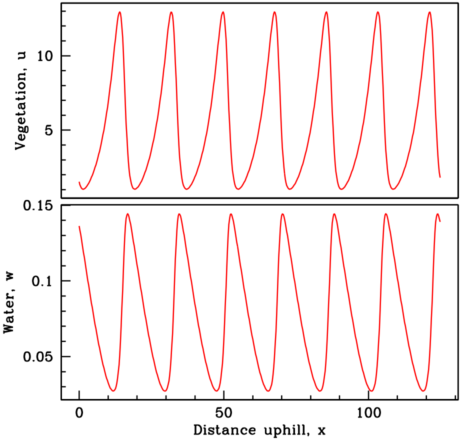

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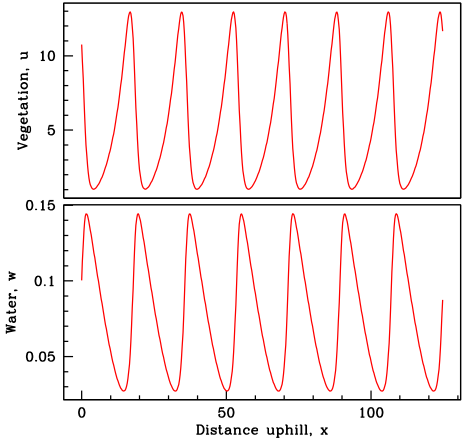

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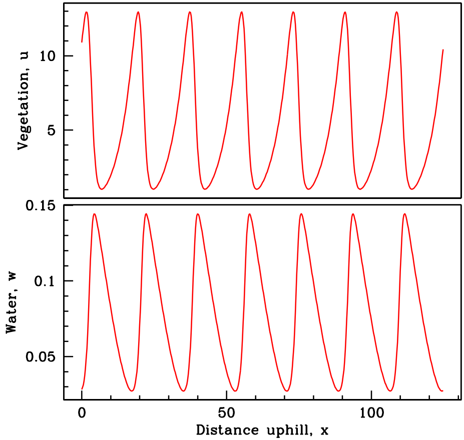

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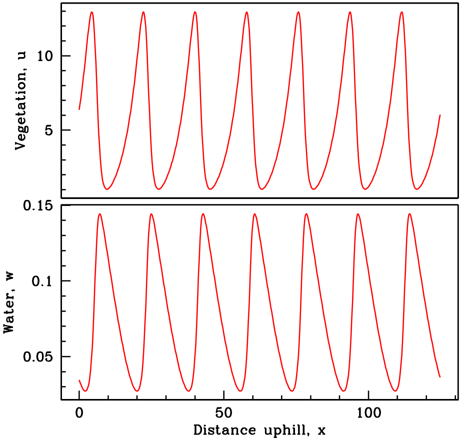

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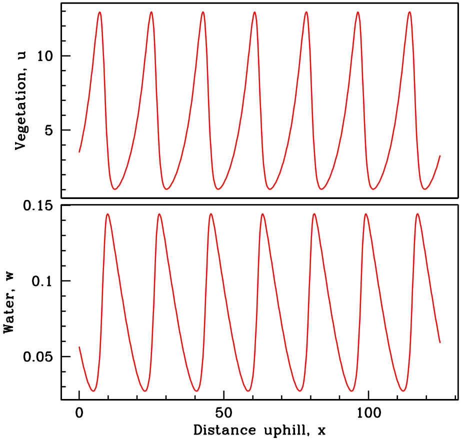

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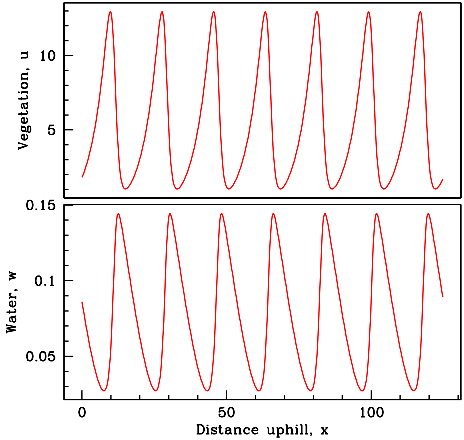

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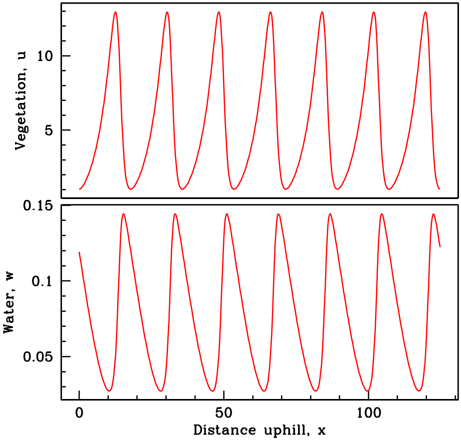

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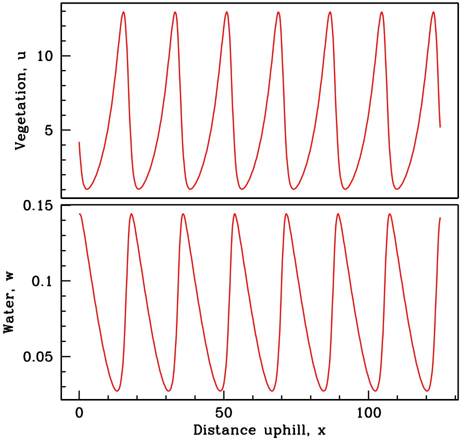

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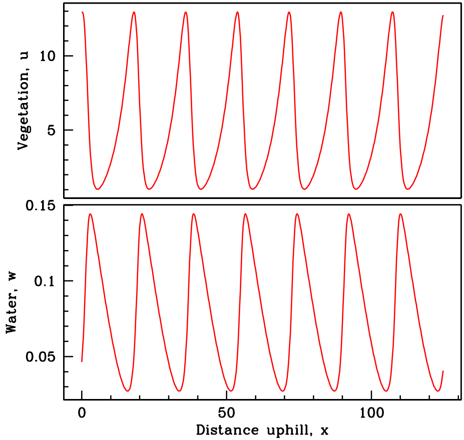

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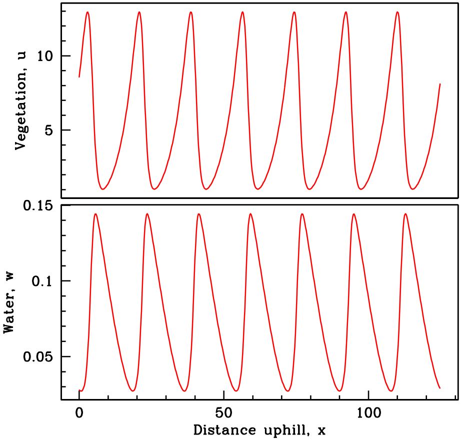

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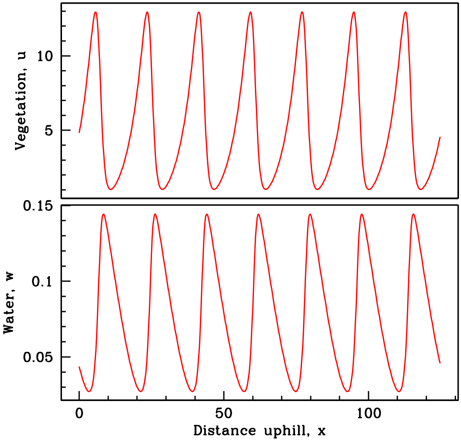

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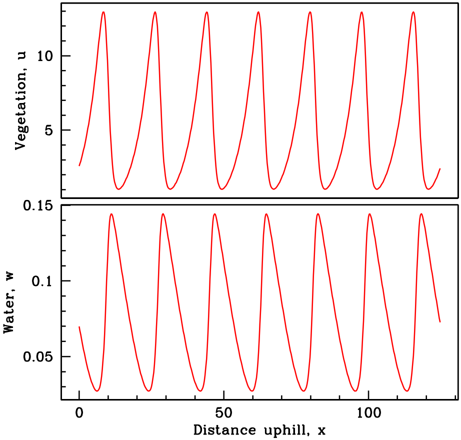

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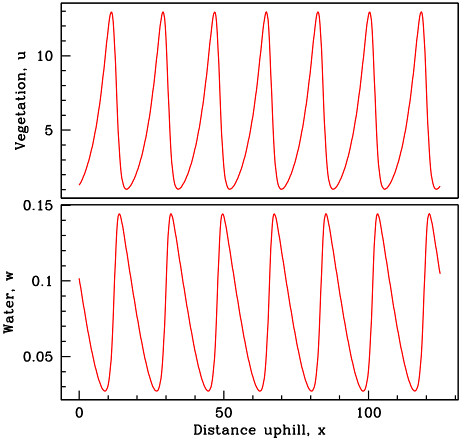

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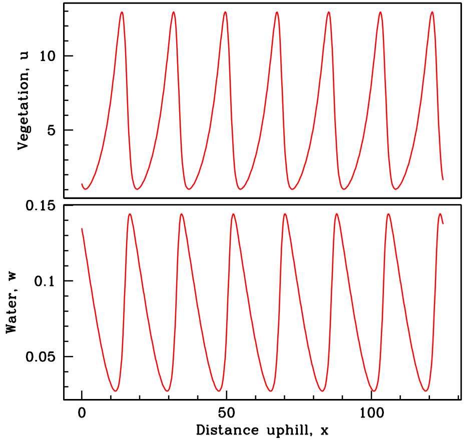

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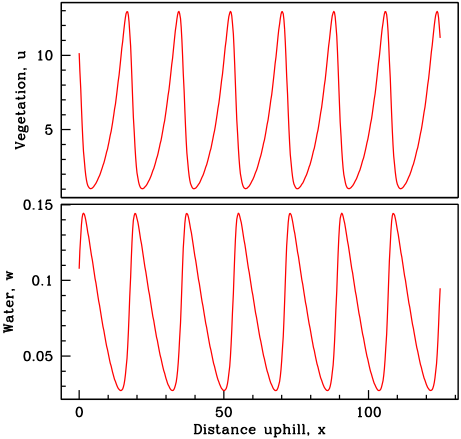

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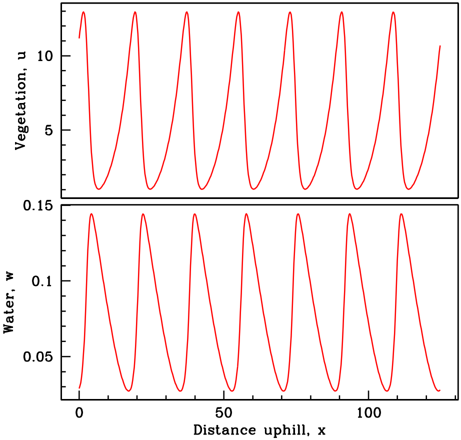

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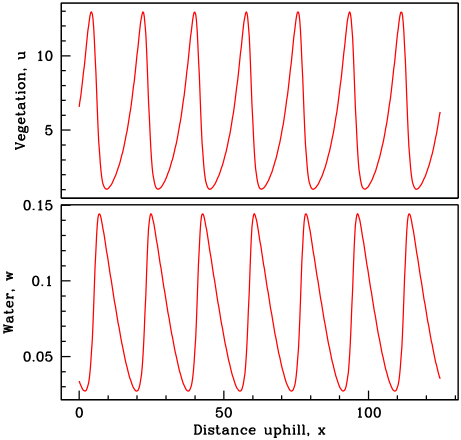

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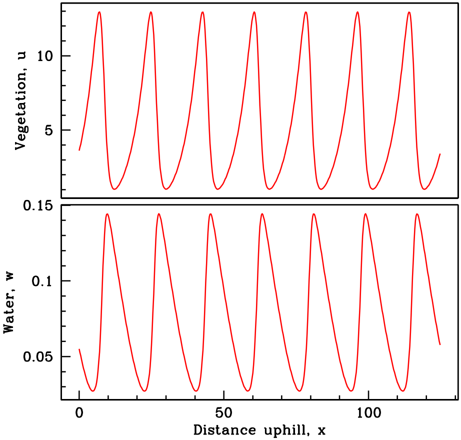

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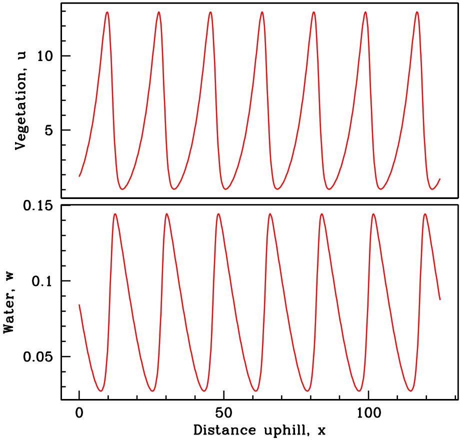

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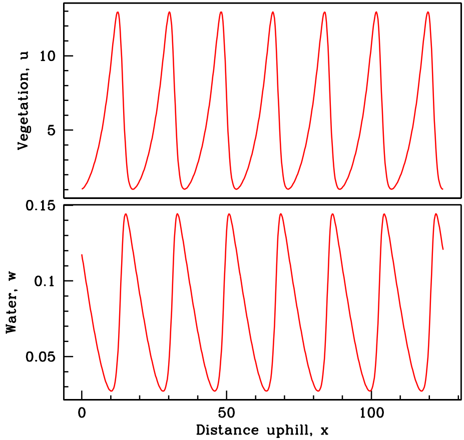

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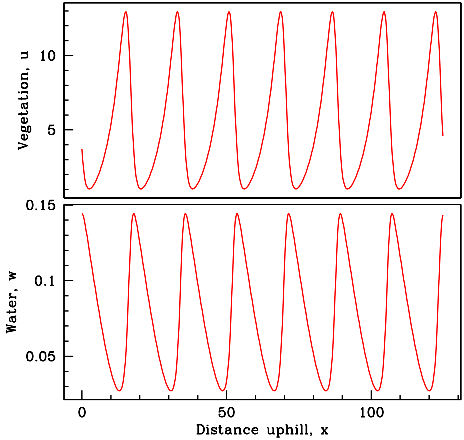

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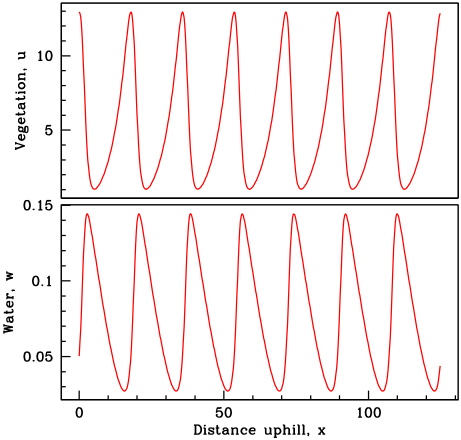

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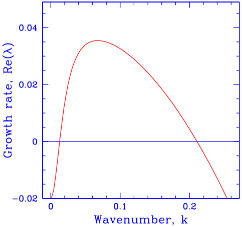
$$u_s = \frac{2B}{A - \sqrt{A^2 - 4B^2}} \quad w_s = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

stable to homog pertns for $B < 2$

- Patterns develop when (u_s, w_s) is unstable to inhomogeneous perturbations

Approximate Conditions for Patterning

Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0)\exp\{ikx + \lambda t\}$



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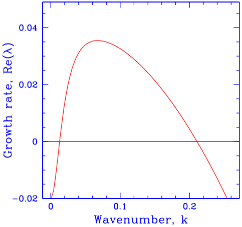

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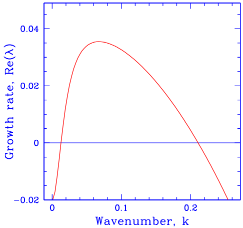
Simplification using $\nu \gg 1$ implies that for pattern formation

$$A < \nu^{1/2} B^{5/4} / 8^{1/4}$$

Approximate Conditions for Patterning

Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0)\exp\{ikx + \lambda t\}$



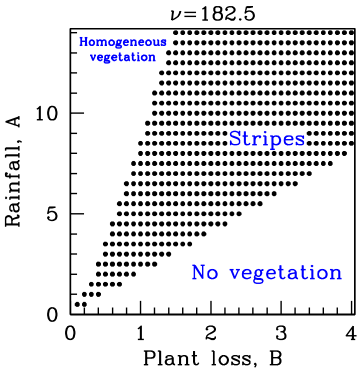
The dispersion relation $\mathrm{Re}[\lambda(k)]$ is algebraically complicated

Simplification using $\nu \gg 1$ implies that for pattern formation

$$2B < A < \nu^{1/2} B^{5/4} / 8^{1/4}$$

One can niavely assume that existence of (u_s, w_s) gives a second condition

An Illustration of Conditions for Patterning



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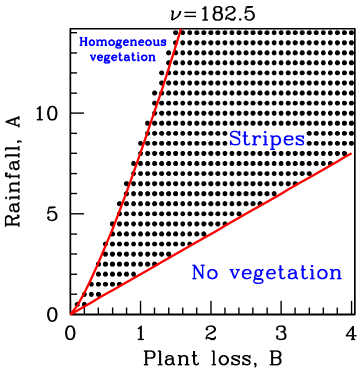

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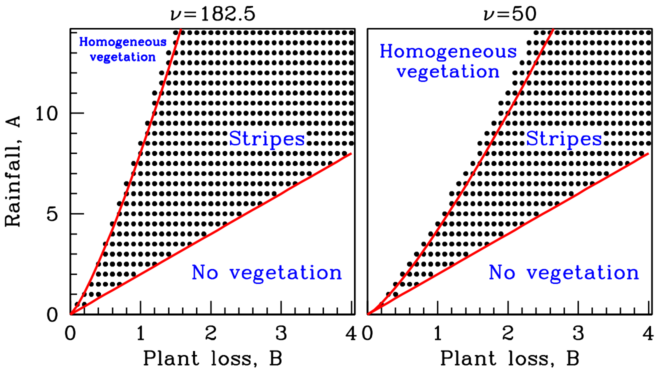

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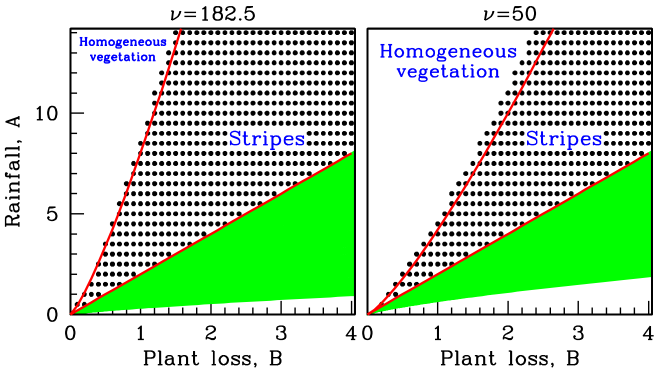
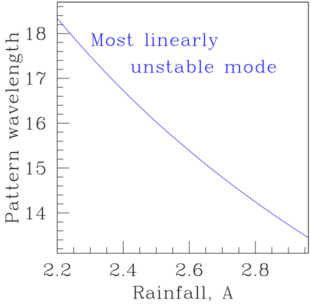


Predicting Pattern Wavelength

Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography.
Wavelength can be predicted from the linear analysis



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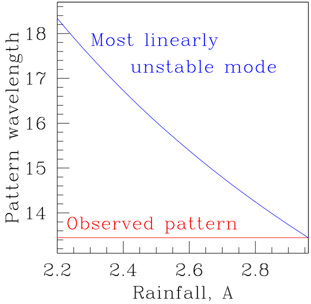


However this prediction doesn't fit the patterns seen in numerical simulations.

Outline

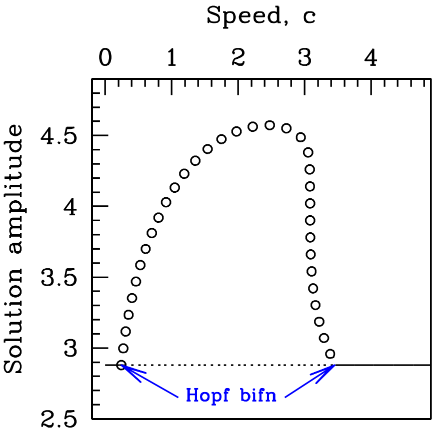
Travelling Wave Equations

The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z)$, $w(x,t) = W(z)$, $z = x - ct$

$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these ODEs

Bifurcation Diagram for Travelling Wave ODEs



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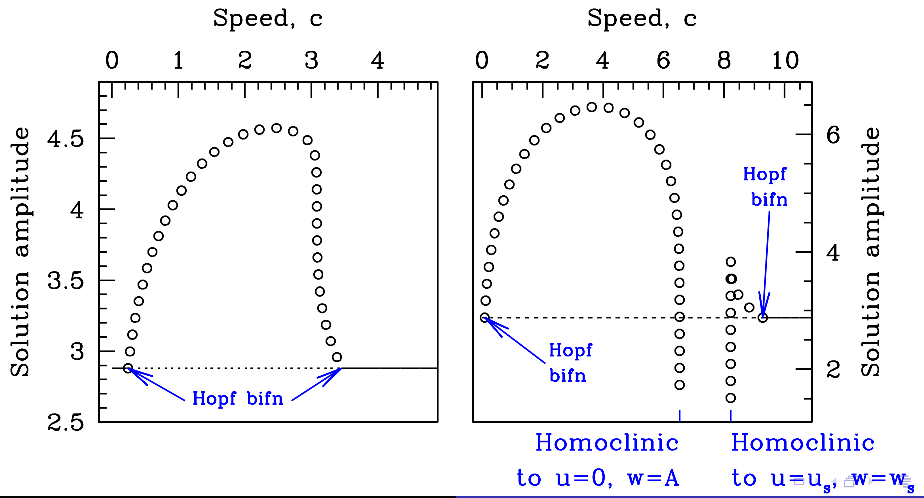
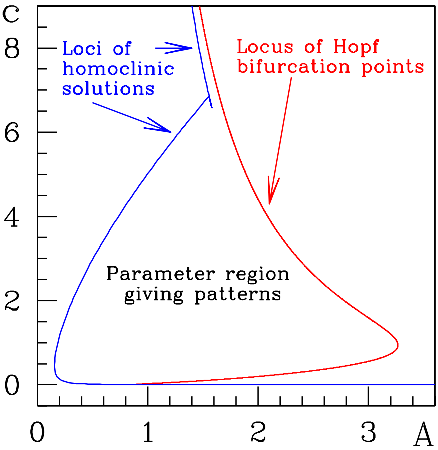


When do Patterns Form?



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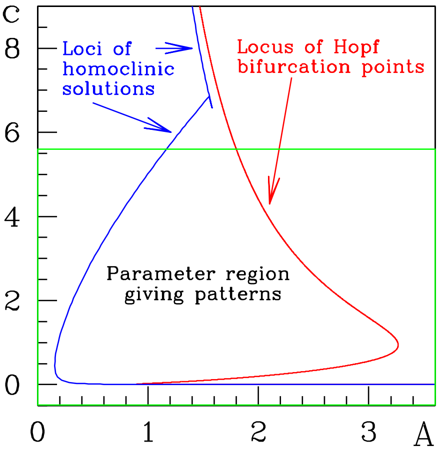
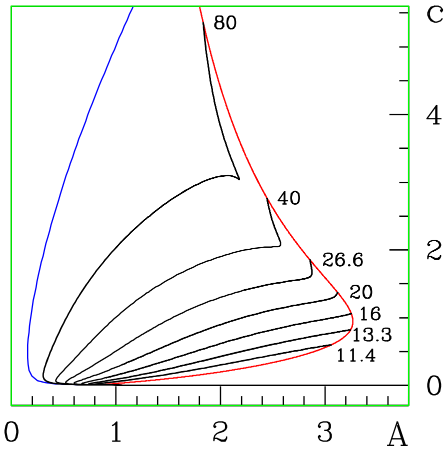
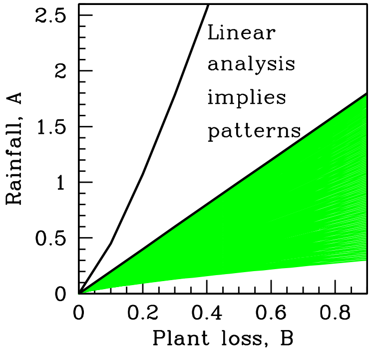




Pattern Formation for Low Rainfall



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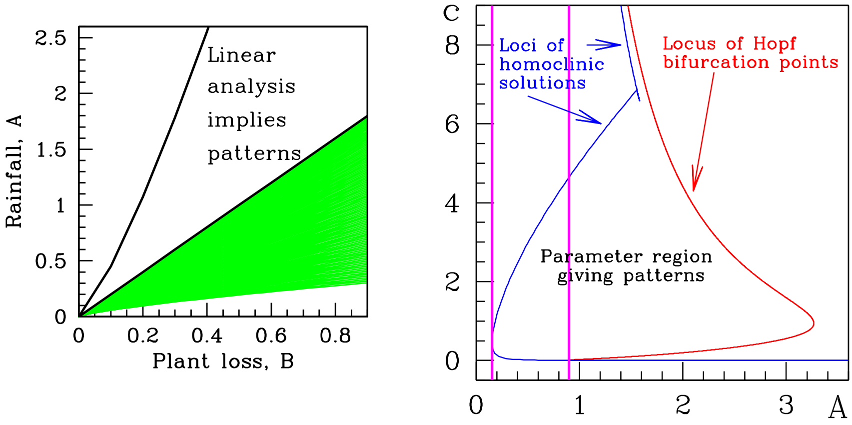

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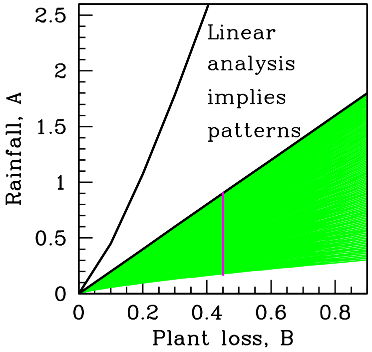

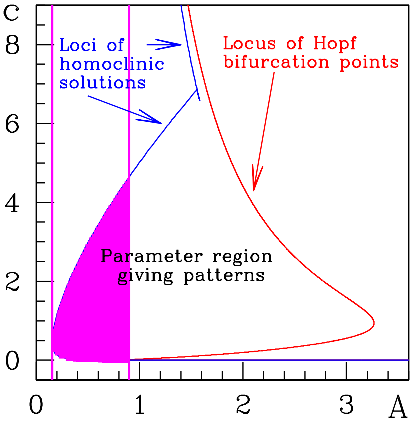

Outline

Discretizing the PDEs

To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

$$
\begin{aligned}
\partial u_i/\partial t &= w_i u_i^2 - B u_i + (u_{i+1} - 2u_i + 2u_{i\text{-}1})/\Delta x^2 \\
\partial w_i/\partial t &= A - w_i - w_i u_i^2 + \nu (w_{i+1} - w_i)/\Delta x
\end{aligned}
$$

$(i = 1, \ldots, N)$.

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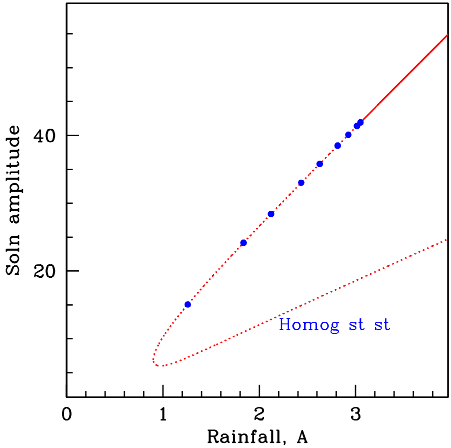
$(i = 1, \ldots, N)$.
We use upwinding for the convective term.
Most of our work has used $N = 40$ and $\Delta x = 2$.
We assume periodic boundary conditions.

Bifurcation Diagram for Discretized PDEs



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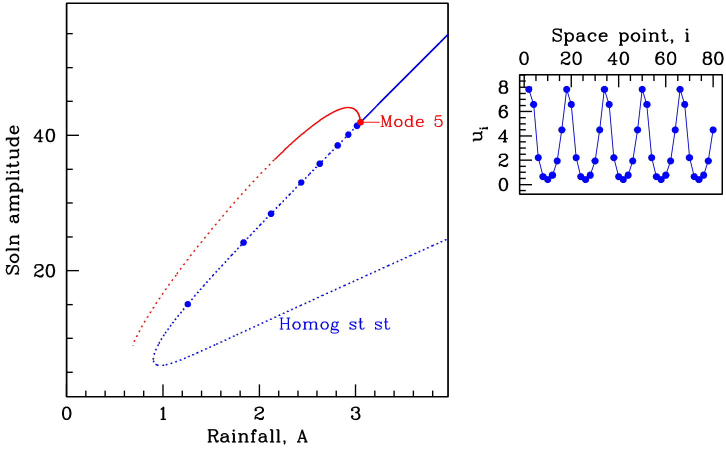

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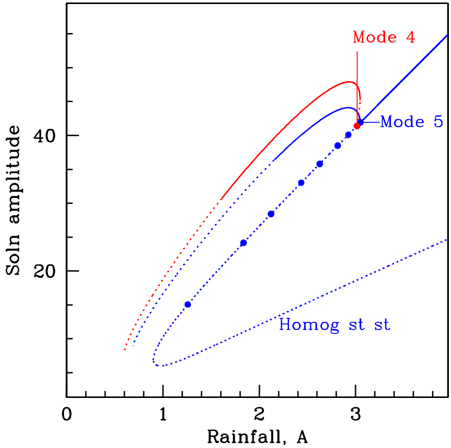

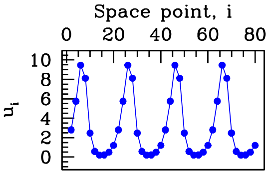

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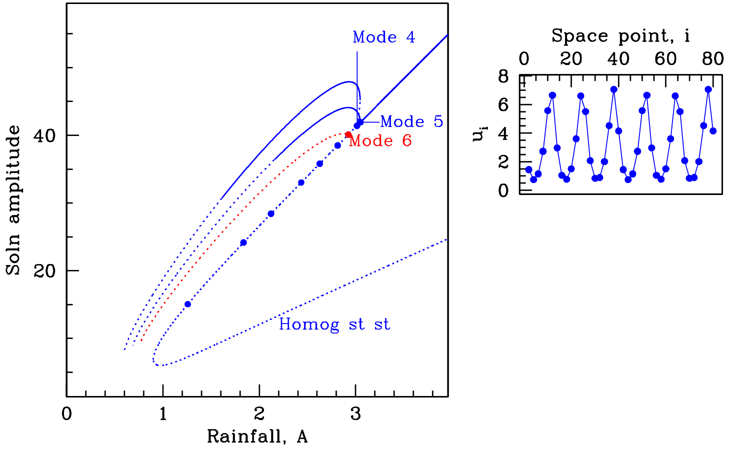

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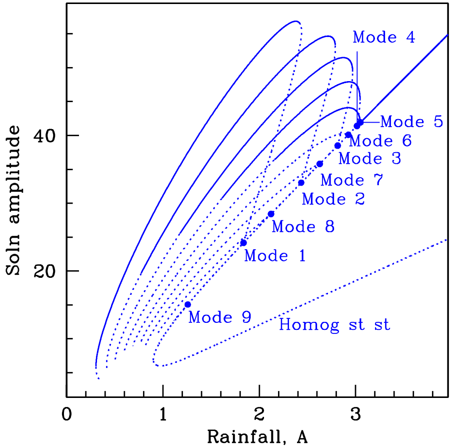
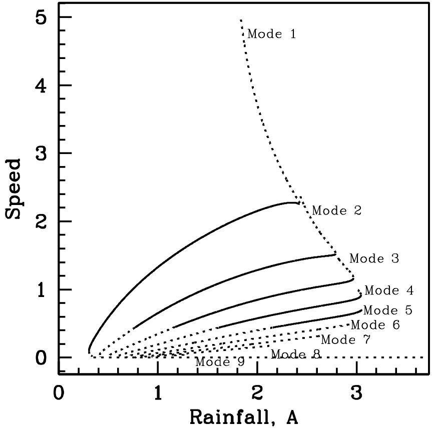


Speed vs Rainfall for Discretized PDEs

c vs A for PDEs



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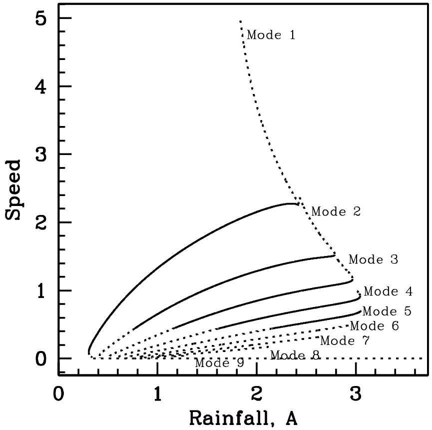
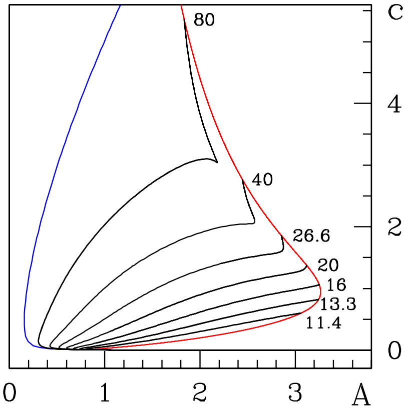


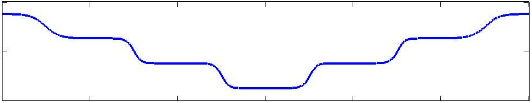
c vs A for travelling wave PDEs



Key Result

For a wide range of rainfall levels, there are multiple stable patterns.

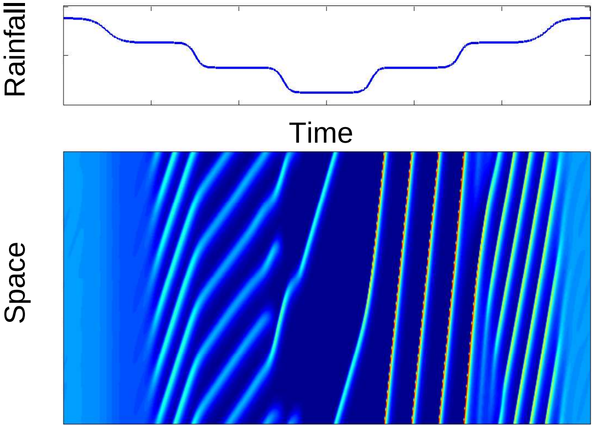
Hysteresis

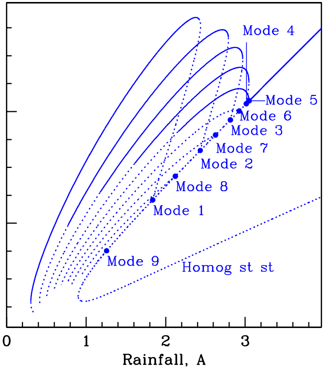


Time

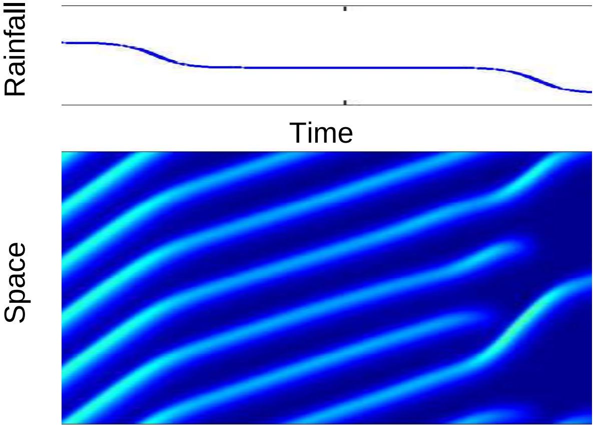
- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A
- Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year

Hysteresis


Rainfall
Time
Space


Mode 4
Mode 5
Mode 6
Mode 3
Mode 7
Mode 2
Mode 8
Mode 1
Mode 9
Homog st st
0
1
2
3
Rainfall, A

Hysteresis



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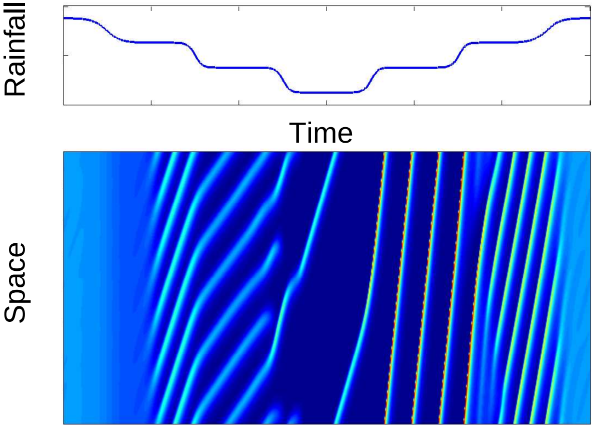

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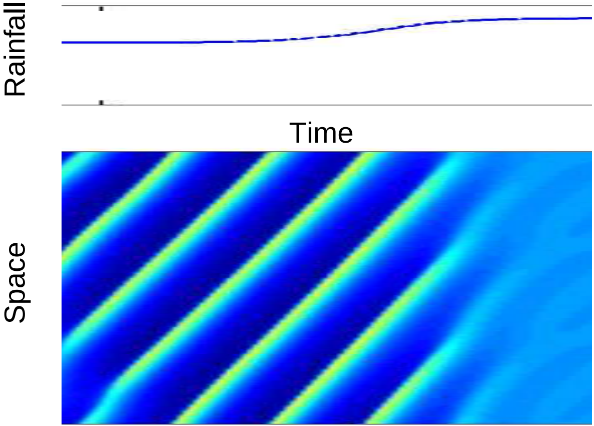

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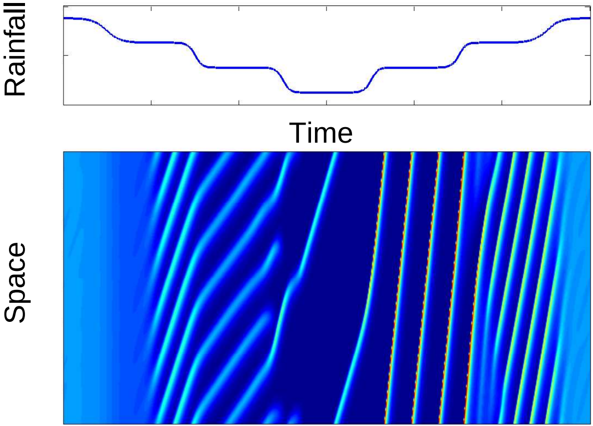
Rainfall
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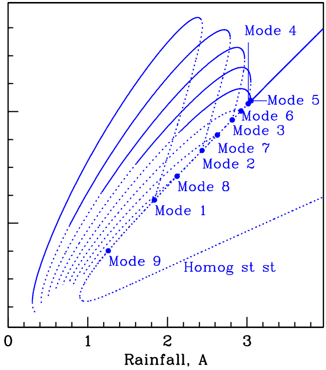
Mode 4
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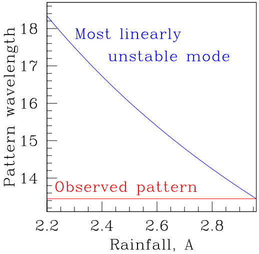
Rainfall, A

Outline

Predictions of Pattern Wavelength

- In general, pattern wavelength depends on initial conditions
- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.

$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



Other Potential Mechanisms for Vegetation Patterns

Rietkirk Klausmeier model with diffusion of water in the soil

van de Koppel Klausmeier model with grazing

Maron two variable model (plant density and water in the soil) with water transport based on porous media theory

Lejeune short range activation (shading) and long range inhibition (competition for water)

All of these models predict patterns. To discriminate between them requires a detailed understanding of each model.

Mathematical Moral

Predictions based only on linear stability analysis are misleading for this model

List of Frames

Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation

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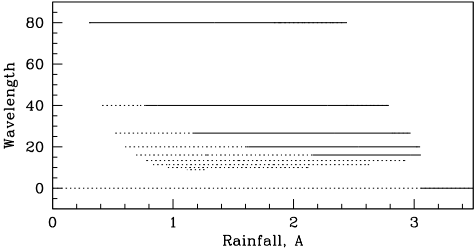

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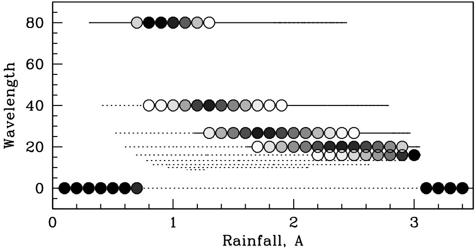

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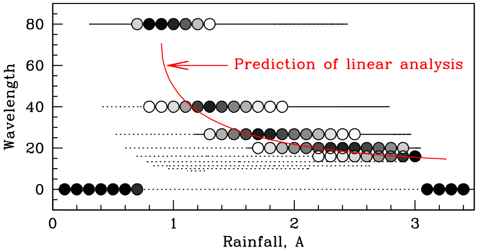
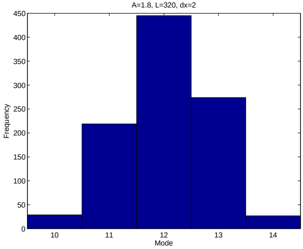


The wavelength is close to that predicted by linear stability analysis

Pattern Selection on Larger Domains

The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged



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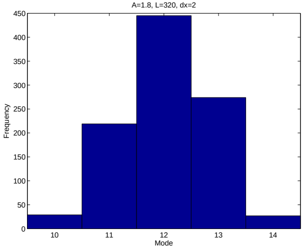

But it does not apply for other initial conditions, such as perturbations about (u_u, w_u)