

Vegetation Stripes in Semi-Arid Environments

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University of Glasgow, 6 December 2005

In collaboration with
Gabriel Lord



Outline

- 1 Ecological Background
- 2 The Mathematical Model
- 3 Linear Analysis
- 4 Travelling Wave Equations
- 5 Bifurcations in the PDEs
- 6 Conclusions

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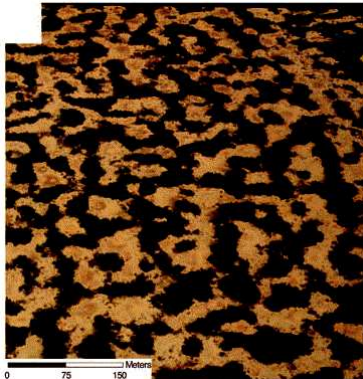
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Vegetation Pattern Formation



- Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico (rainfall 100-700 mm/year)
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees

More Pictures of Vegetation Patterns



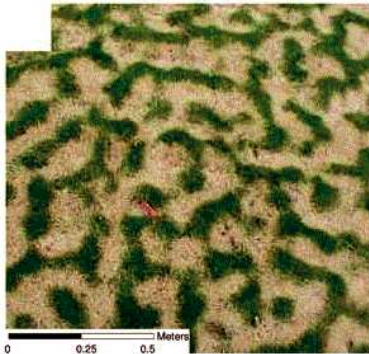
Labyrinth of bushy
vegetation in Niger

More Pictures of Vegetation Patterns



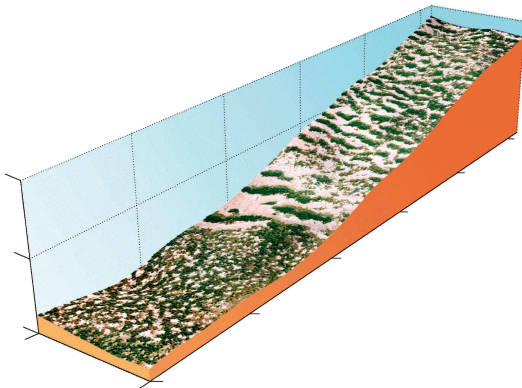
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More Pictures of Vegetation Patterns



Labyrinth of grass
in Israel

Vegetation Pattern Formation (contd)



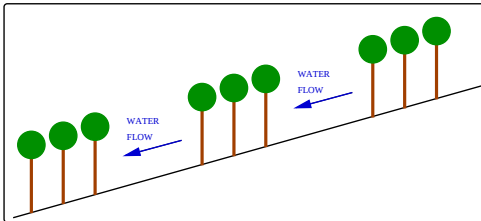
- On flat ground, irregular mosaics of vegetation are typical
- On slopes, the patterns are stripes, parallel to contours (“Tiger bush”)

Mechanisms for Vegetation Patterning

- Basic mechanism: competition for water

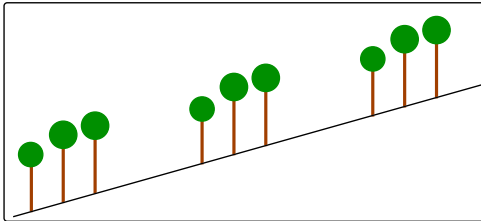
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- Basic mechanism: competition for water
- Possible detailed mechanism: water flow downhill causes stripes



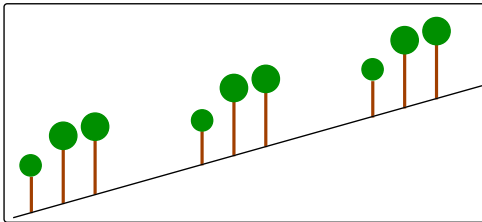
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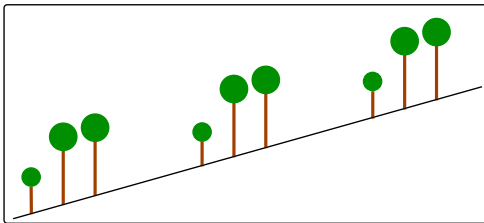
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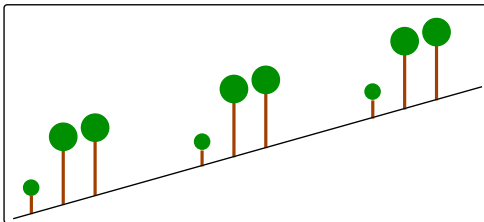
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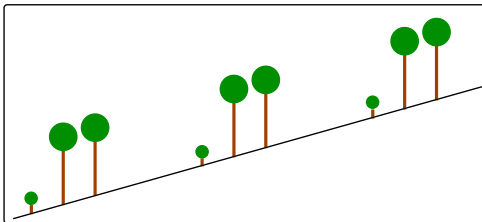
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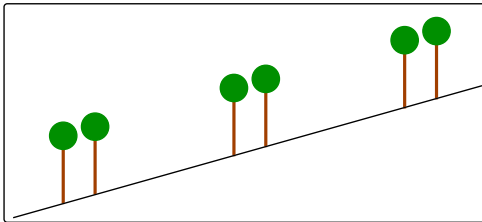
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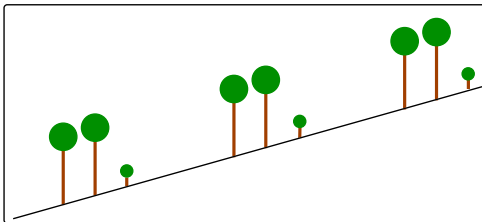
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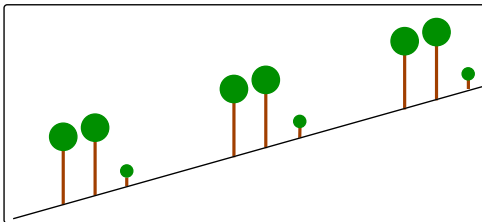
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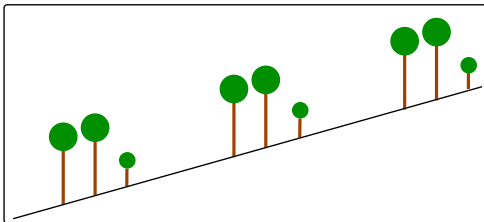
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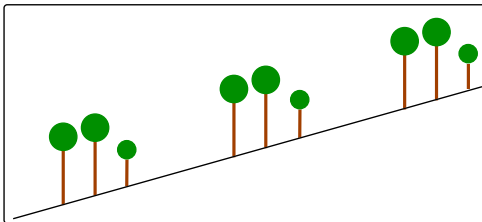
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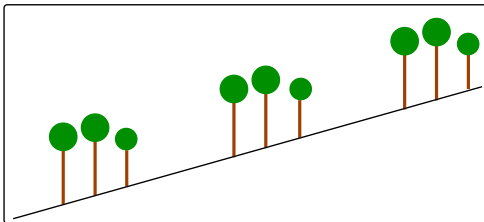
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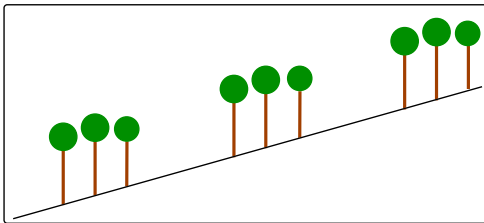
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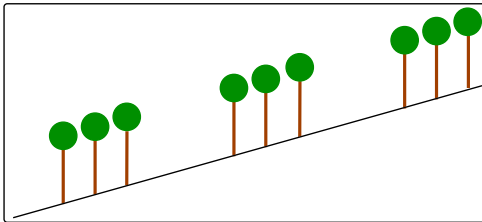
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- This mechanism suggests that the stripes would move uphill; this remains controversial.

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Mathematical Model of Klausmeier

Rate of change of water = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change of plant biomass = Growth, proportional to water uptake – Mortality + Random dispersal

$$\partial w / \partial t = A - w - wu^2 + \nu \partial w / \partial x$$

$$\partial u / \partial t = wu^2 - Bu + \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2$$

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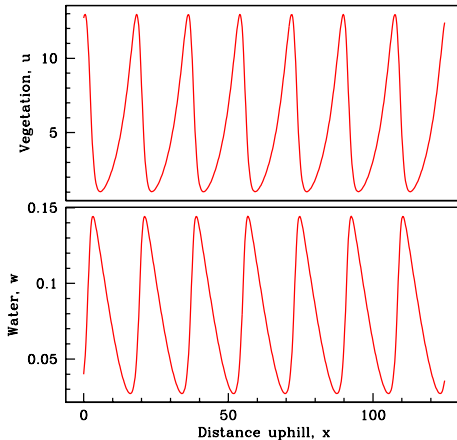
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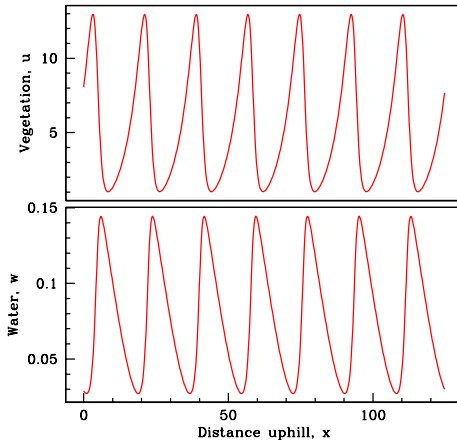
$$\partial u / \partial t = wu^2 - Bu + \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2$$

The nonlinearity in wu^2 arises because the presence of roots increases water infiltration into the soil.

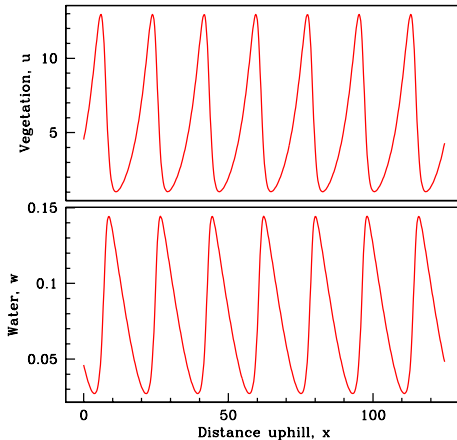
Typical Solution of the Model



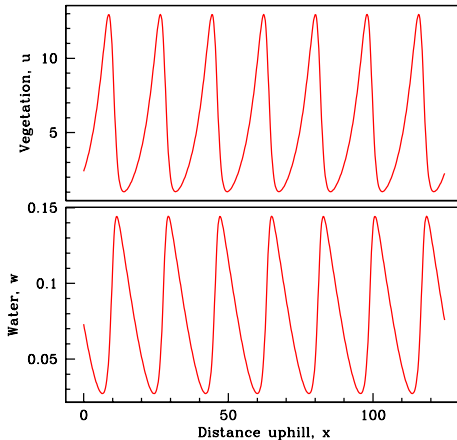
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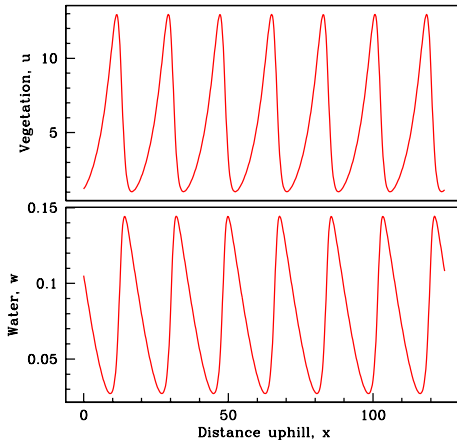
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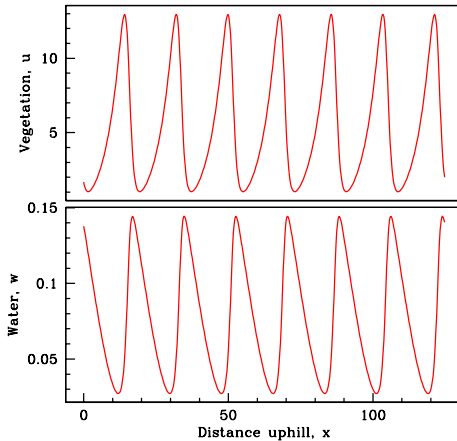
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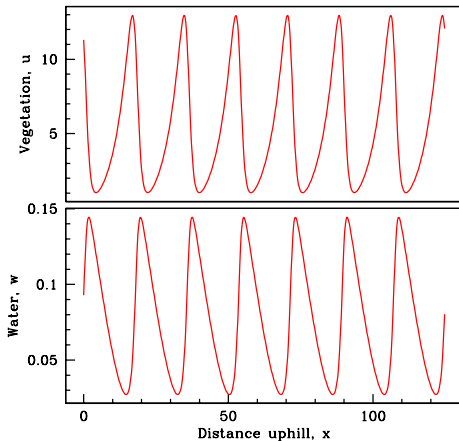
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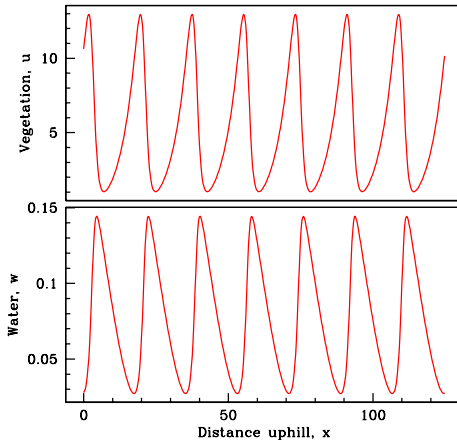
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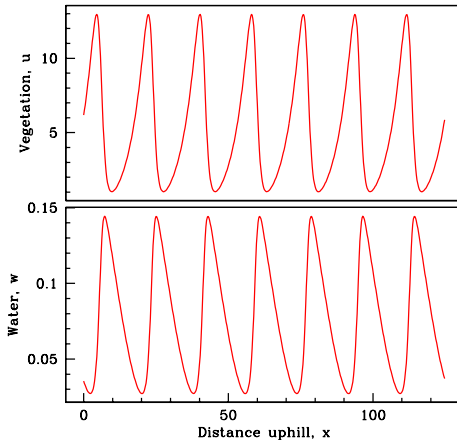
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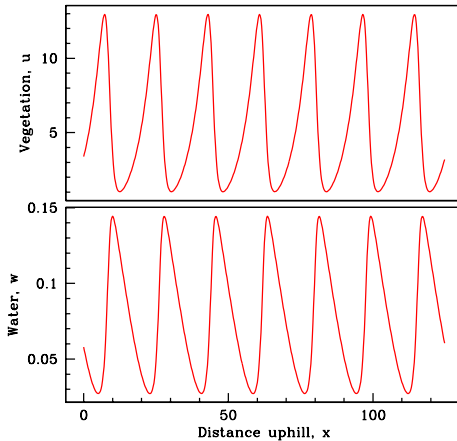
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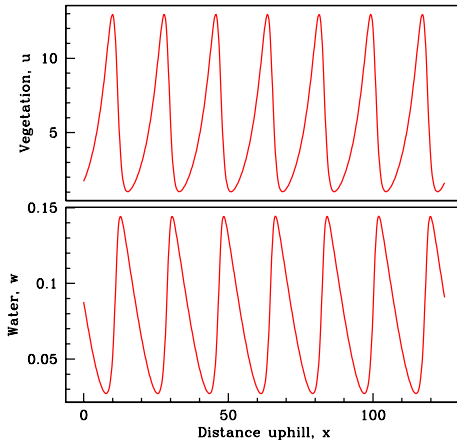
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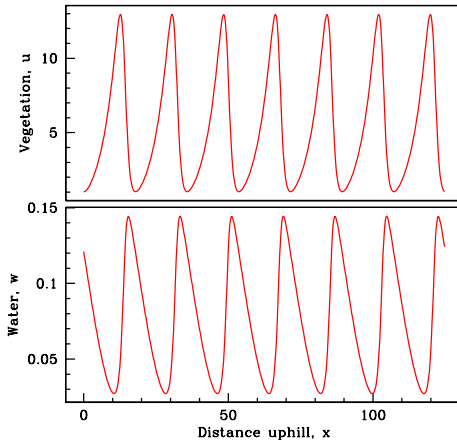
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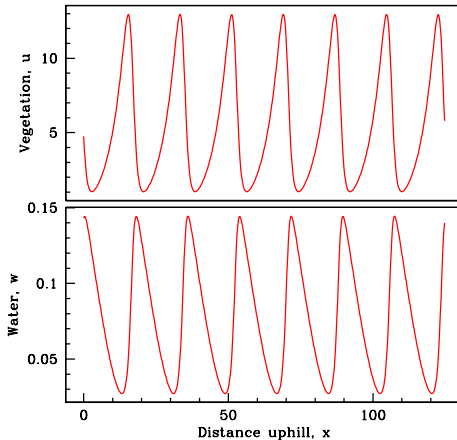
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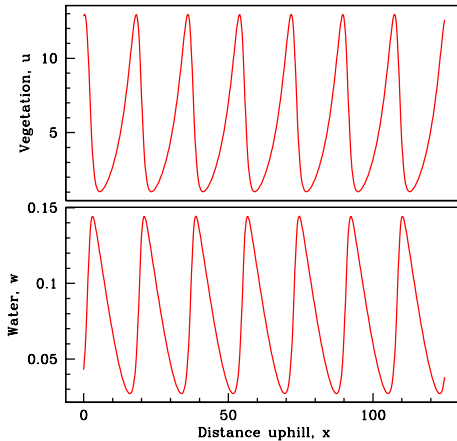
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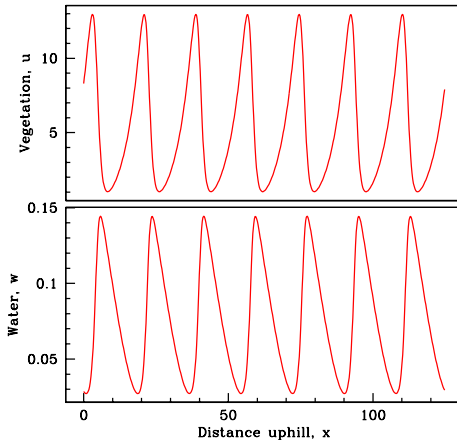
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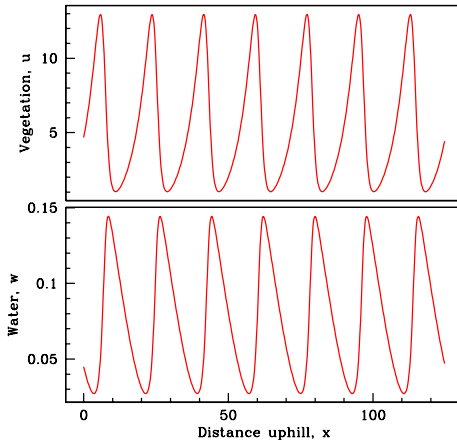
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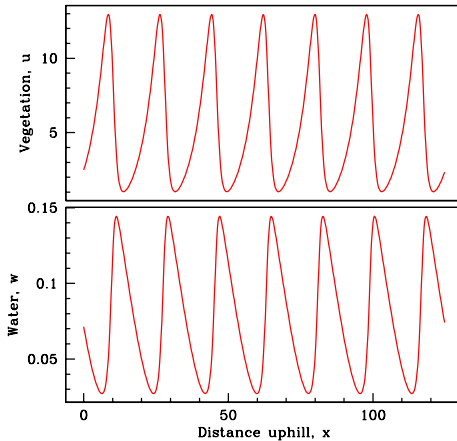
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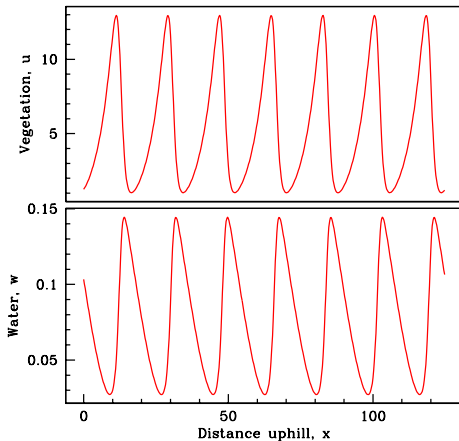
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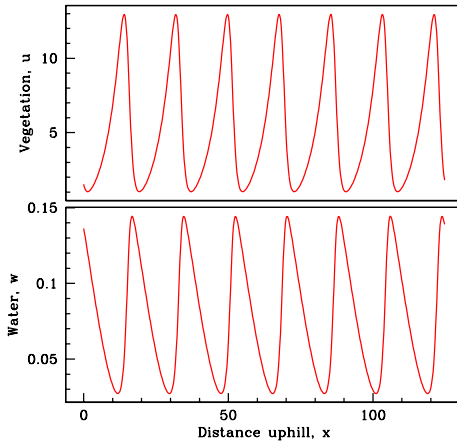
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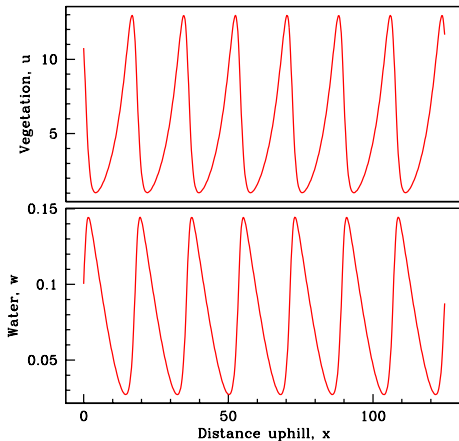
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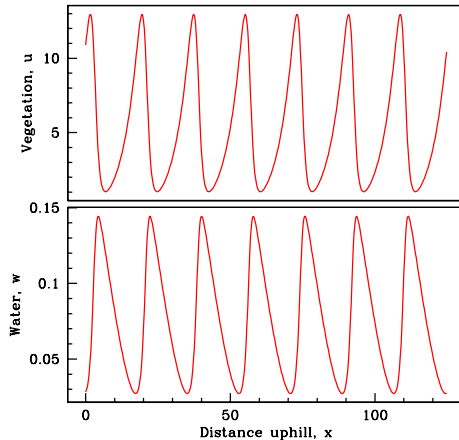
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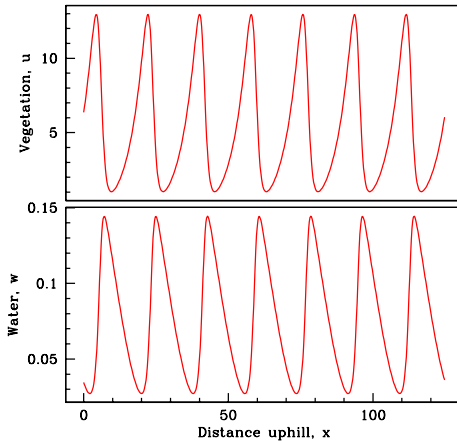
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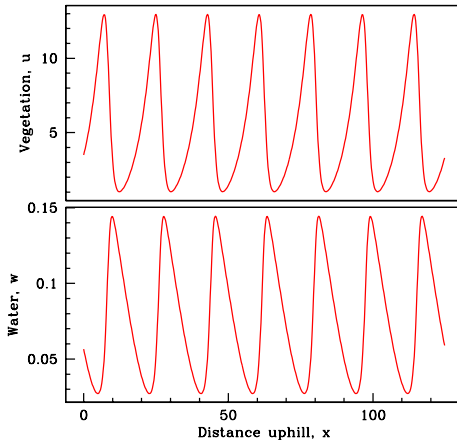
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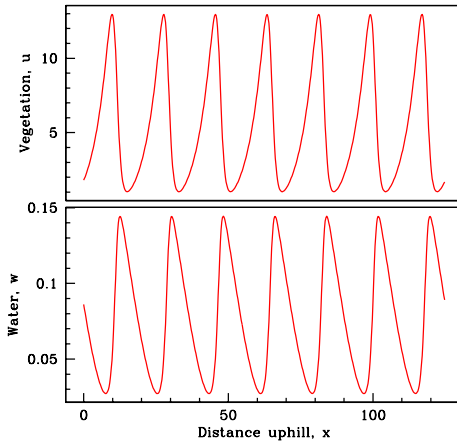
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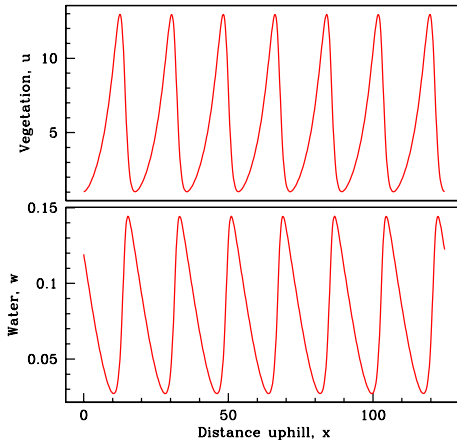
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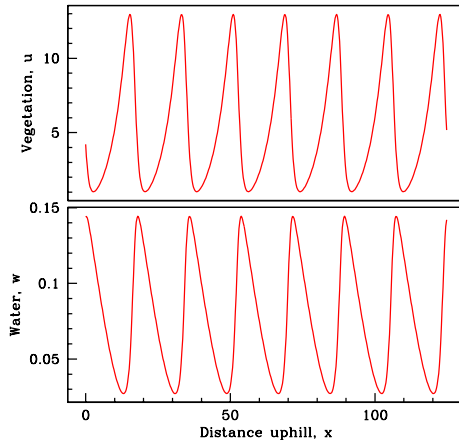
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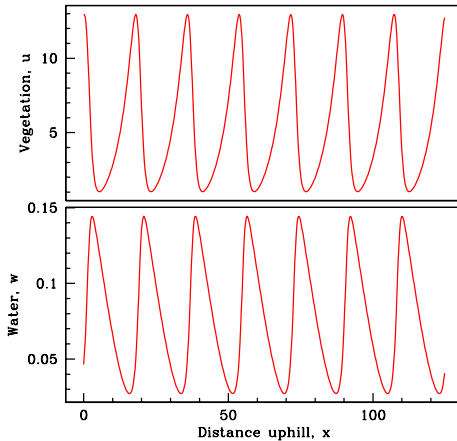
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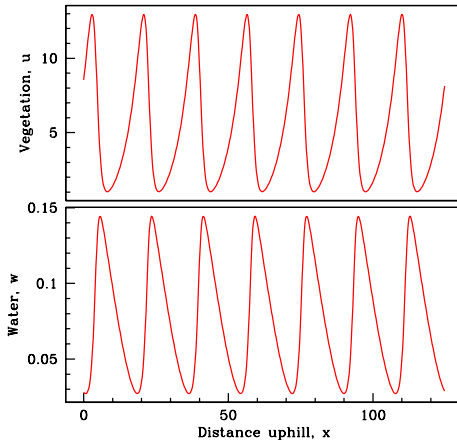
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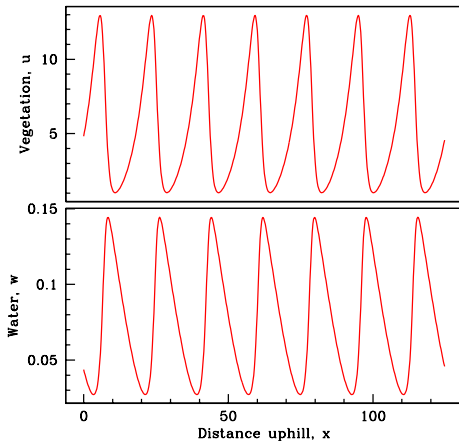
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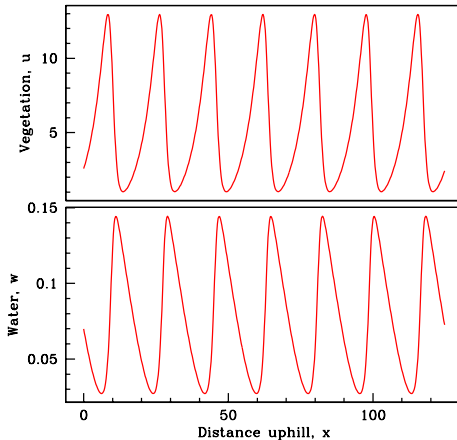
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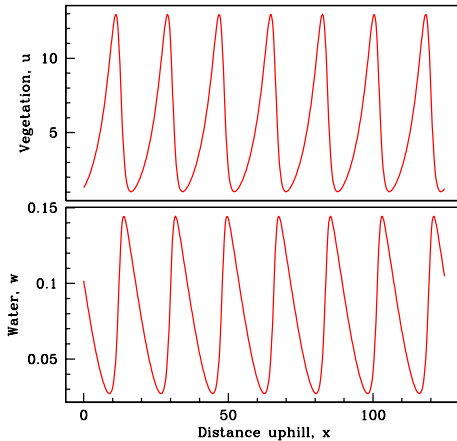
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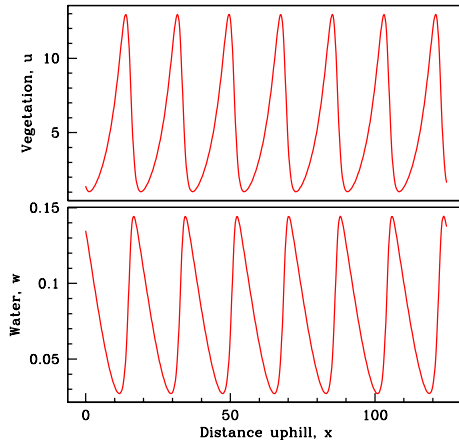
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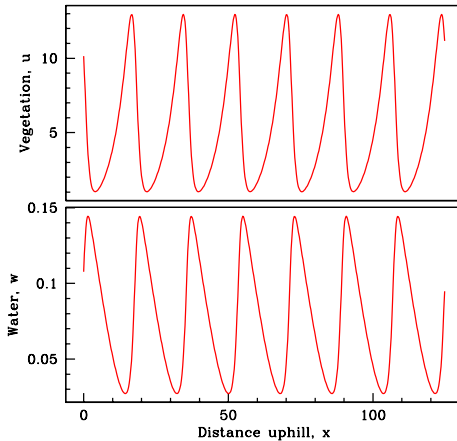
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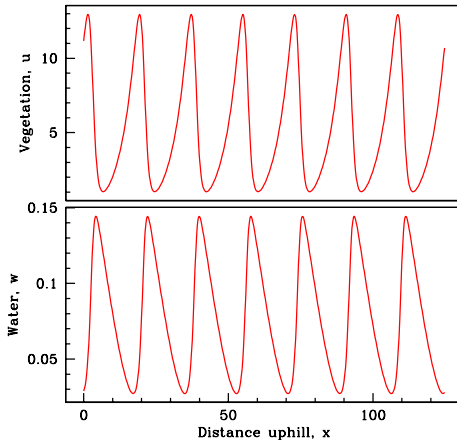
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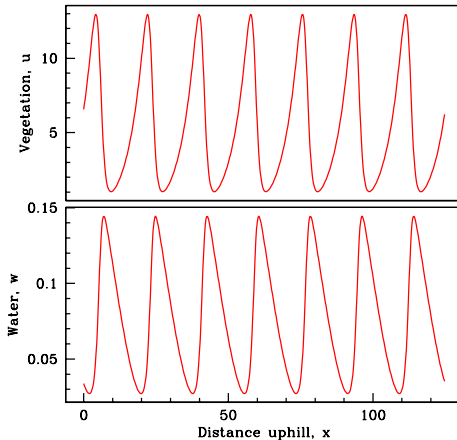
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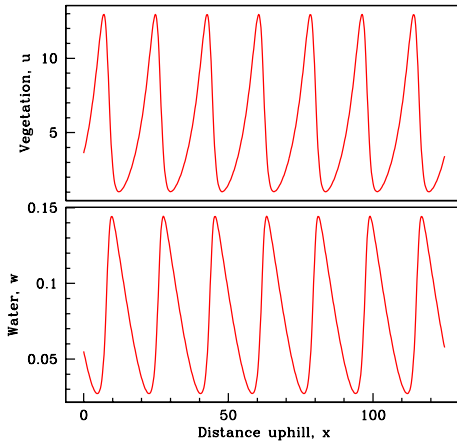
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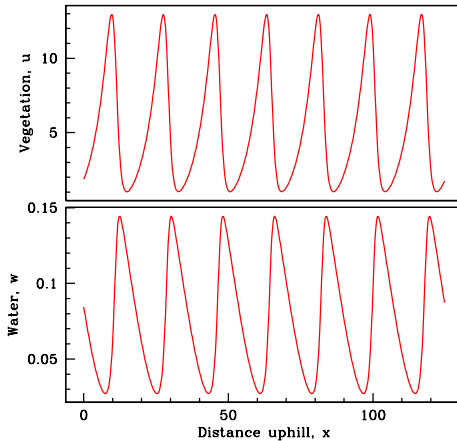
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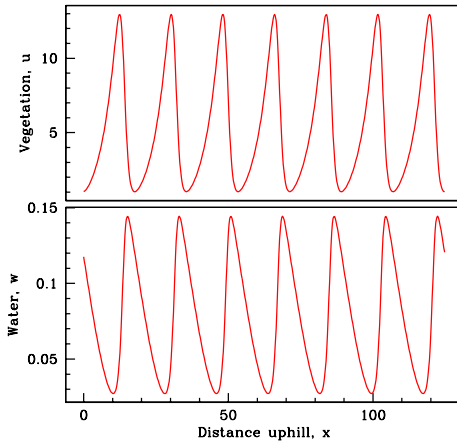
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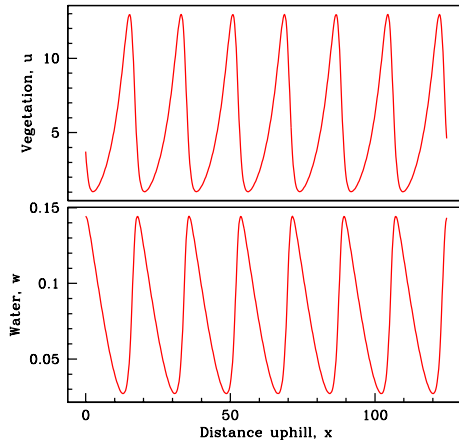
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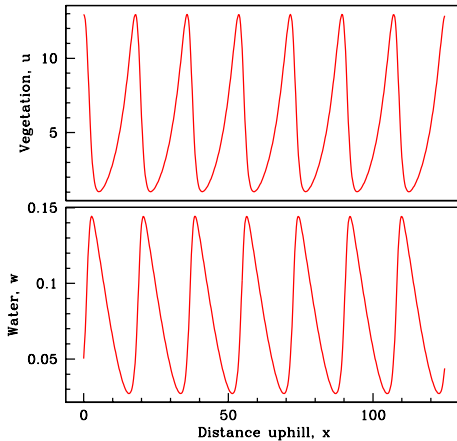
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$$u_u = \frac{2B}{A - \sqrt{A^2 - 4B^2}} \quad w_u = \frac{A - \sqrt{A^2 - 4B^2}}{2} \text{unstable}$$

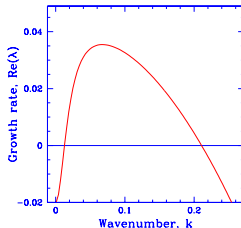
$$u_s = \frac{2B}{A + \sqrt{A^2 - 4B^2}} \quad w_s = \frac{A + \sqrt{A^2 - 4B^2}}{2} \text{stable to homog}$$

pertns for $B < 2$

- Patterns develop when (u_s, w_s) is unstable to inhomogeneous perturbations

Approximate Conditions for Patterning

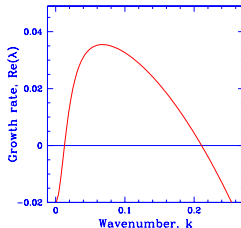
Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0) \exp\{ikx + \lambda t\}$



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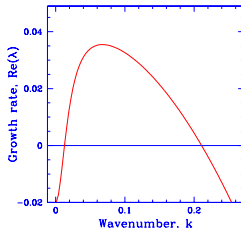
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Simplification using $\nu \gg 1$ implies that for pattern formation

$$A < \nu^{1/2} B^{5/4} / 8^{1/4}$$

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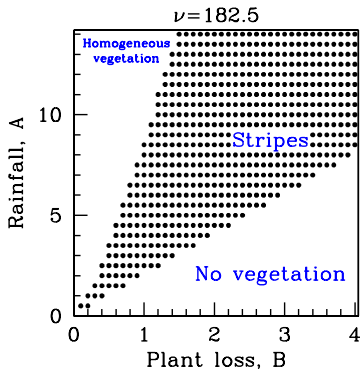
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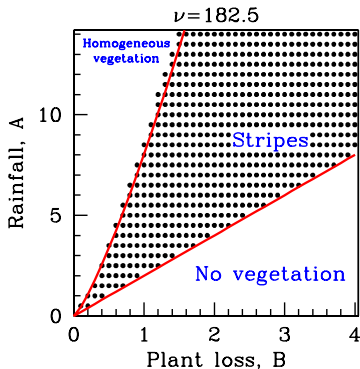
$$2B < A < \nu^{1/2} B^{5/4} / 8^{1/4}$$

One can naively assume that existence of (u_s, w_s) gives a second condition

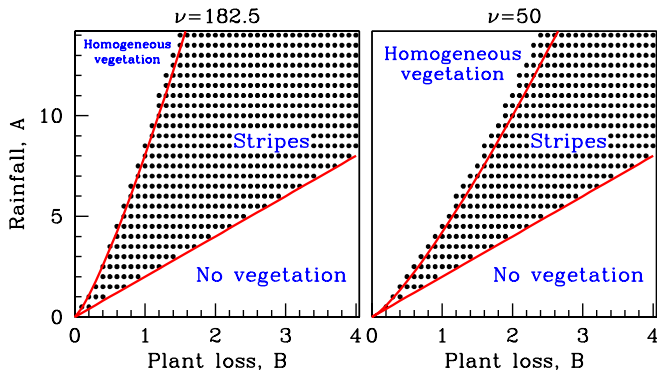
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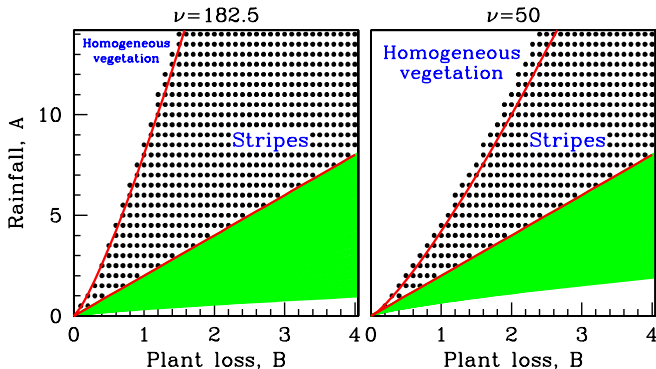
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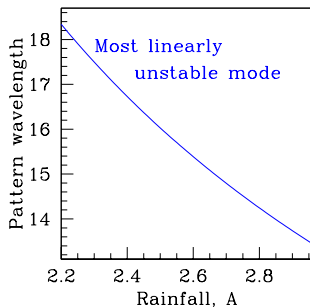


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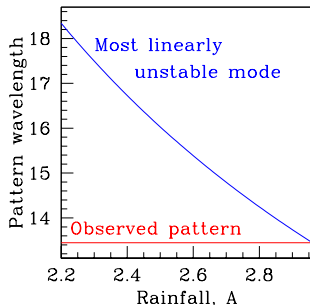
Predicting Pattern Wavelength

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However this prediction doesn't fit the patterns seen in numerical simulations.

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Travelling Wave Equations

The patterns move at constant shape and speed

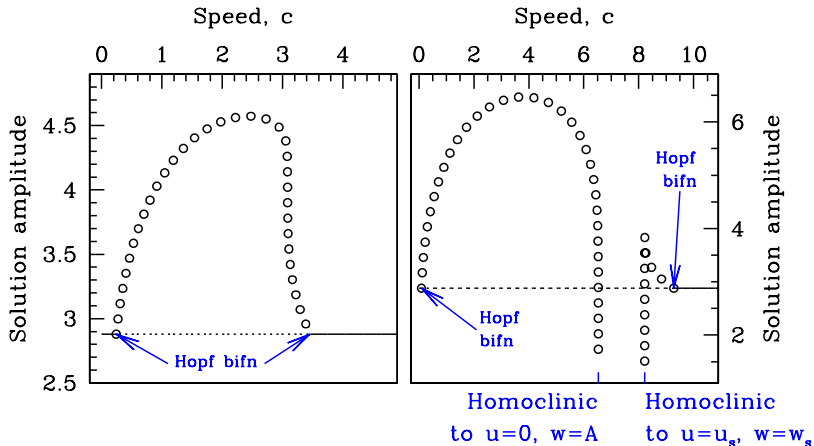
$$\Rightarrow u(x, t) = U(z), w(x, t) = W(z), z = x - ct$$

$$d^2U/dz^2 + c dU/dz + WU^2 - BU = 0$$

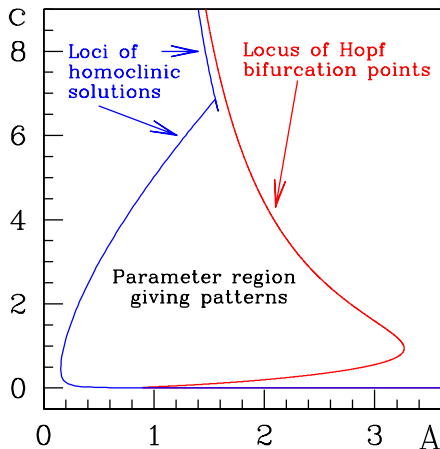
$$(\nu + c)dW/dz + A - W - WU^2 = 0$$

The patterns are periodic (limit cycle) solutions of these ODEs

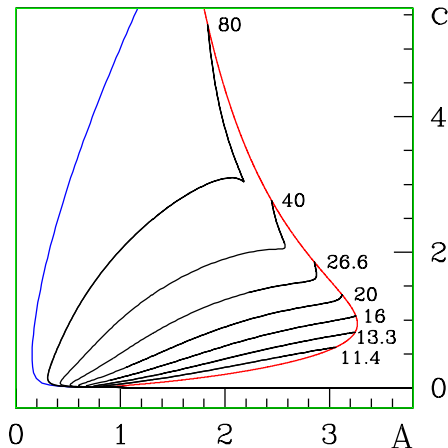
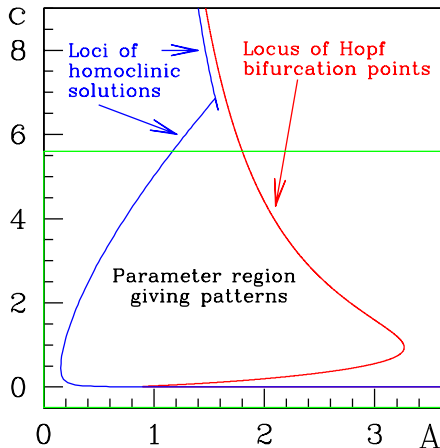
Bifurcation Diagram for Travelling Wave ODEs



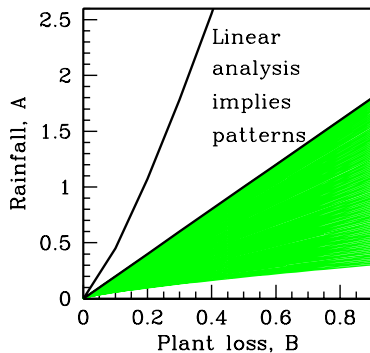
When do Patterns Form?



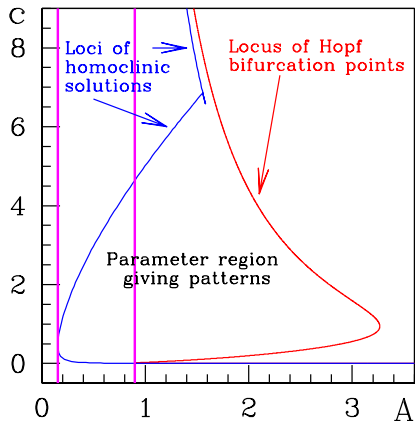
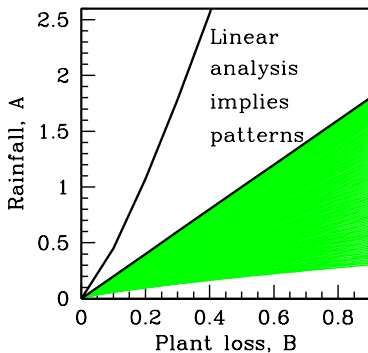
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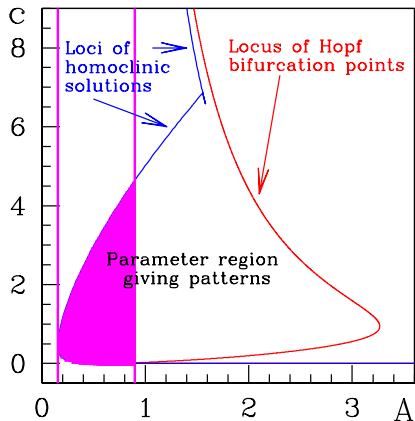
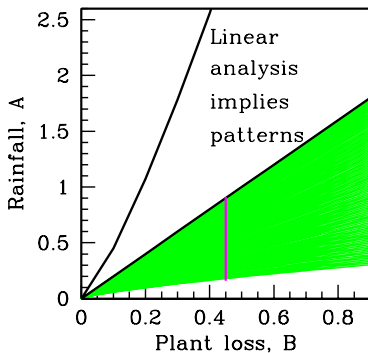
Pattern Formation for Low Rainfall



Pattern Formation for Low Rainfall



Pattern Formation for Low Rainfall



Outline

- 1 Ecological Background
- 2 The Mathematical Model
- 3 Linear Analysis
- 4 Travelling Wave Equations
- 5 Bifurcations in the PDEs**
- 6 Conclusions

Discretizing the PDEs

To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

$$\begin{aligned}\partial u_i / \partial t &= w_i u_i^2 - B u_i + (u_{i+1} - 2u_i + 2u_{i-1}) / \Delta x^2 \\ \partial w_i / \partial t &= A - w_i - w_i u_i^2 + \nu (w_{i+1} - w_i) / \Delta x\end{aligned}$$

($i = 1, \dots, N$).

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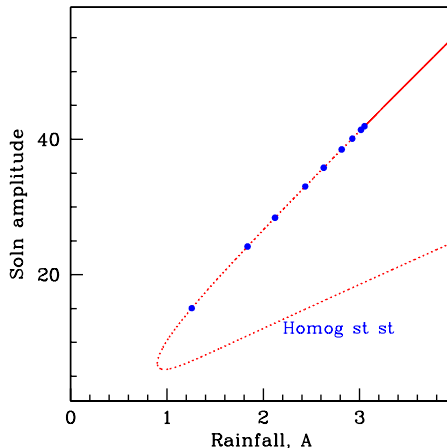
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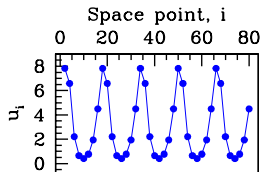
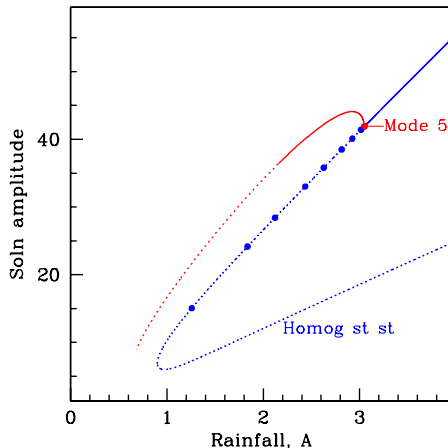
Most of our work has used $N = 40$ and $\Delta x = 2$.

We assume periodic boundary conditions.

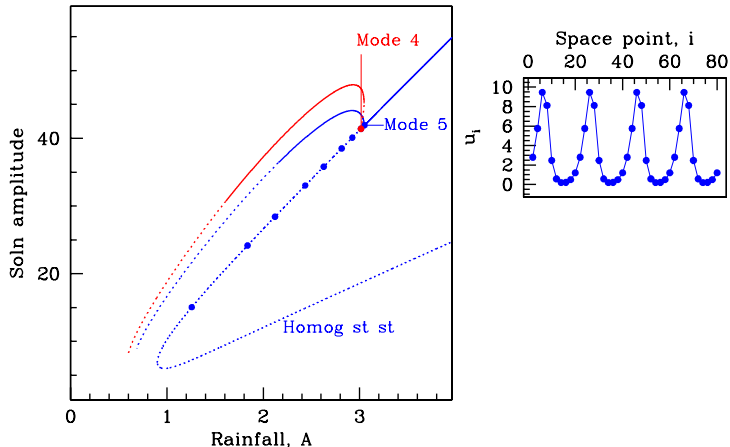
Bifurcation Diagram for Discretized PDEs



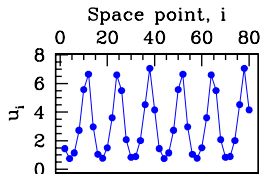
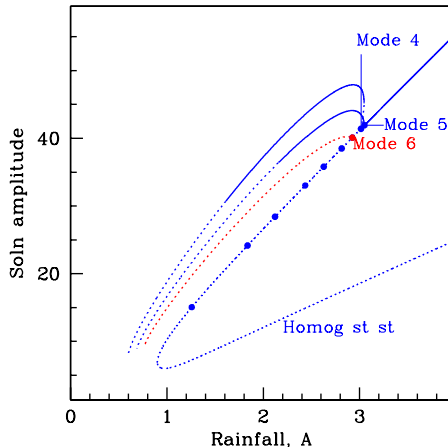
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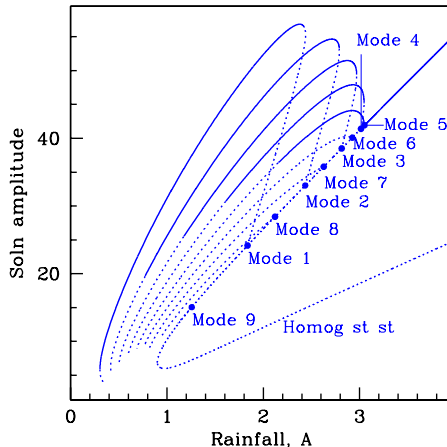
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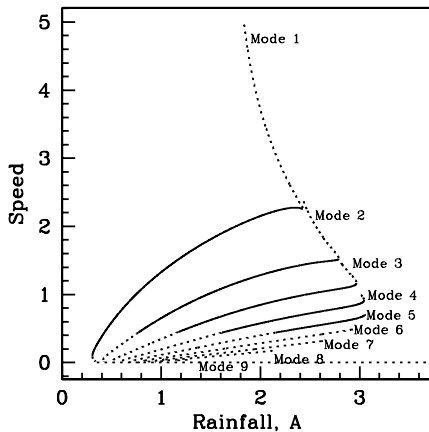


Bifurcation Diagram for Discretized PDEs



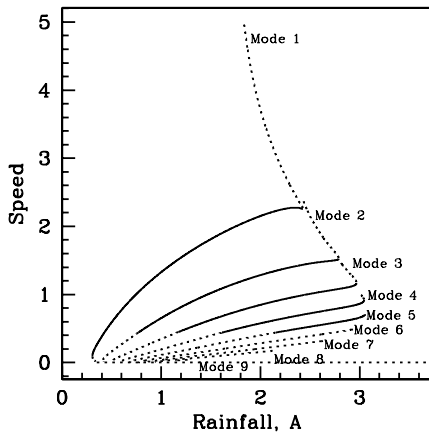
Speed vs Rainfall for Discretized PDEs

c vs A for PDEs

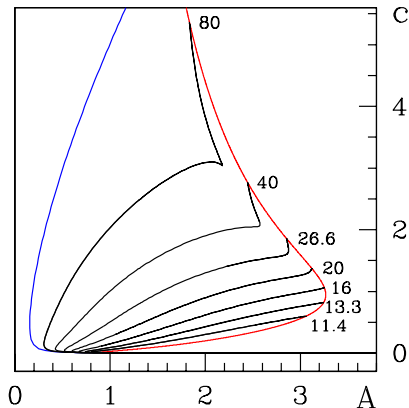


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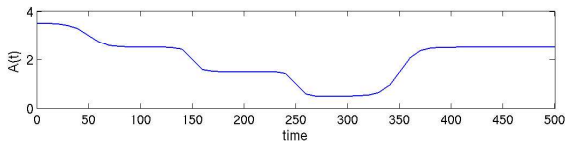
c vs A for travelling wave PDEs



Key Result

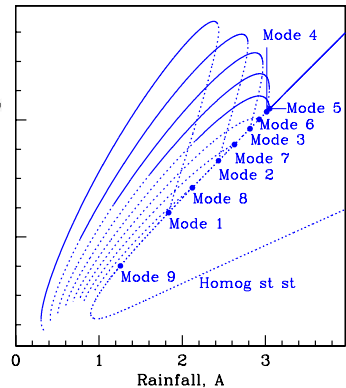
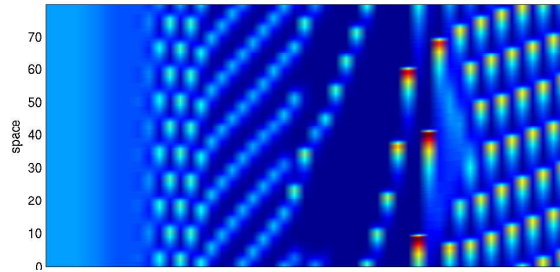
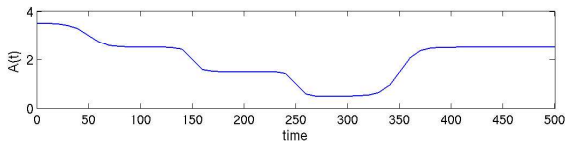
For a wide range of rainfall levels,
there are multiple stable patterns.

Hysteresis

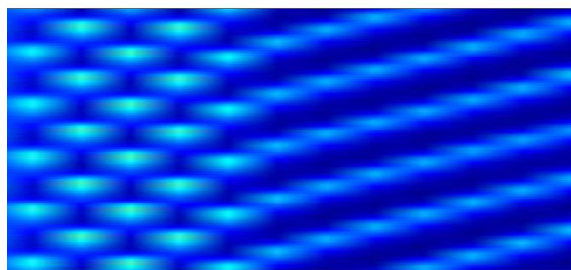
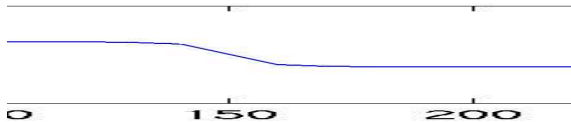


- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A
- Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year

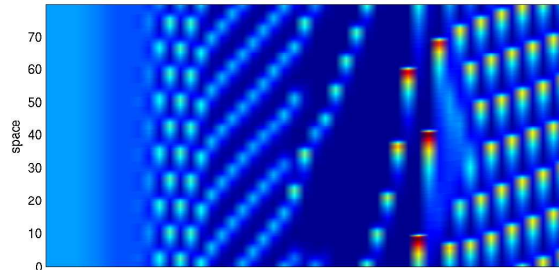
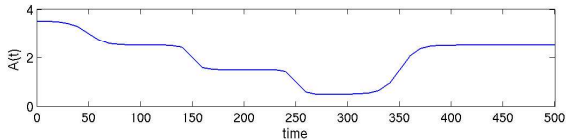
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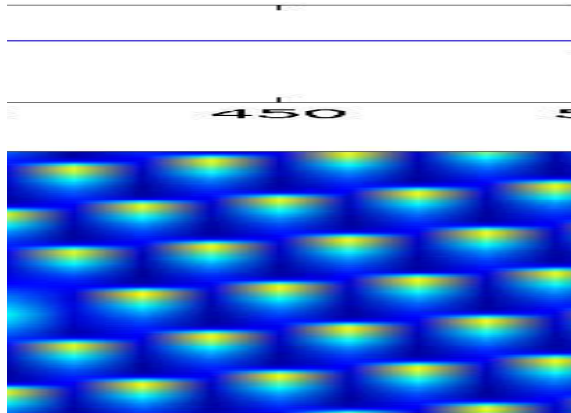
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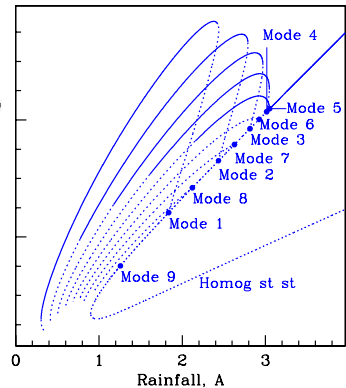
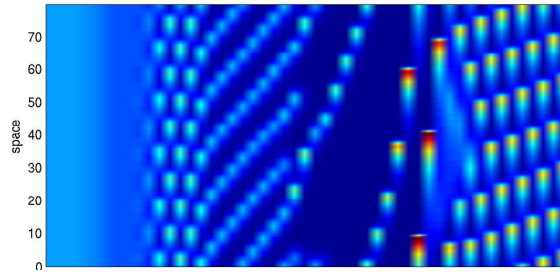
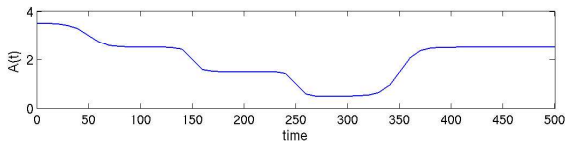
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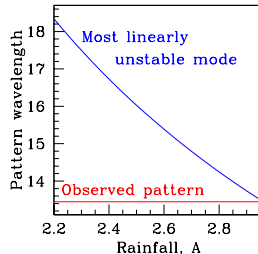
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Predictions of Pattern Wavelength

- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.

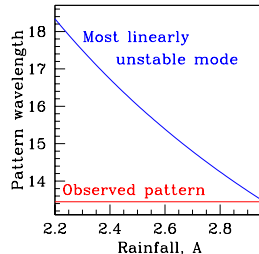
$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



Predictions of Pattern Wavelength

- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.

$$\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}$$



- When vegetation stripes arise from desert via an increase in rainfall, pattern wavelength will be the whole domain.

Other Potential Mechanisms for Vegetation Patterns

Rietkirk Klausmeier model with diffusion of water in the soil

van de Koppel Klausmeier model with grazing

Maron two variable model (plant density and water in the soil) with water transport based on porous media theory

Lejeune short range activation (shading) and long range inhibition (competition for water)

All of these models predict patterns. To discriminate between them requires a detailed understanding of each model.

Mathematical Moral

Predictions based only on
linear stability analysis are
misleading for this model

List of Frames

- 1 **Ecological Background**
 - Vegetation Pattern Formation
 - More Pictures of Vegetation Patterns
 - Vegetation Pattern Formation (contd)
 - Mechanisms for Vegetation Patterning
- 2 **The Mathematical Model**
 - Mathematical Model of Klausmeier
 - Typical Solution of the Model
- 3 **Linear Analysis**
 - Homogeneous Steady States
 - Approximate Conditions for Patterning
 - An Illustration of Conditions for Patterning
 - Predicting Pattern Wavelength
- 4 **Travelling Wave Equations**
 - Travelling Wave Equations
 - Bifurcation Diagram for Travelling Wave ODEs
 - When do Patterns Form?
 - Pattern Formation for Low Rainfall

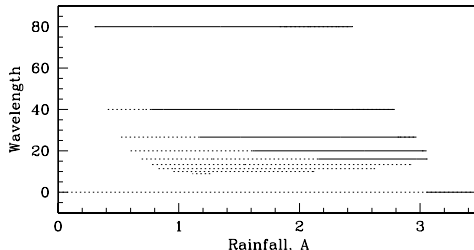
- 5 **Bifurcations in the PDEs**
 - Discretizing the PDEs
 - Bifurcation Diagram for Discretized PDEs
 - Speed vs Rainfall for Discretized PDEs
 - Key Result
 - Hysteresis
- 6 **Conclusions**
 - Predictions of Pattern Wavelength
 - Other Potential Mechanisms for Vegetation Patterns
 - Mathematical Moral

Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation

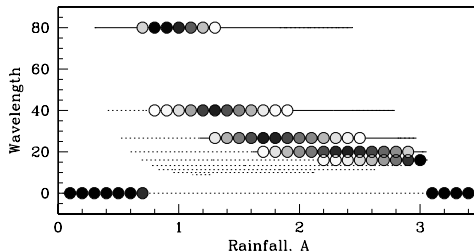
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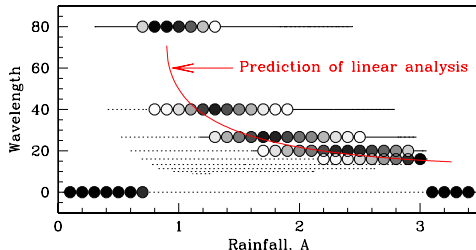
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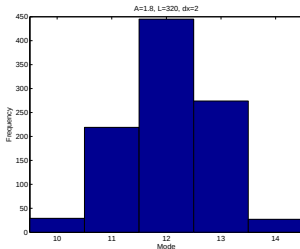
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The wavelength is close to that predicted by linear stability analysis

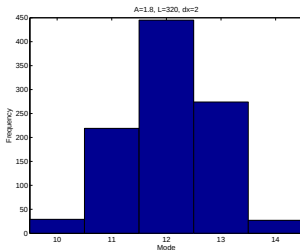
Pattern Selection on Larger Domains

The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged



Pattern Selection on Larger Domains

The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged



But it does not apply for other initial conditions, such as perturbations about (u_u, w_u)

