Vegetation Stripes in Semi-Arid Environments

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Mathematical Biology Group Meeting

In collaboration with
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Outline

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Vegetation Pattern Formation

- Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees

Vegetation Pattern Formation (contd)

- On flat ground, irregular mosaics of vegetation are typical

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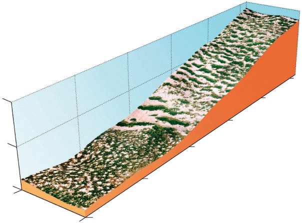

- On slopes, the patterns are stripes, parallel to contours ("Tiger bush")

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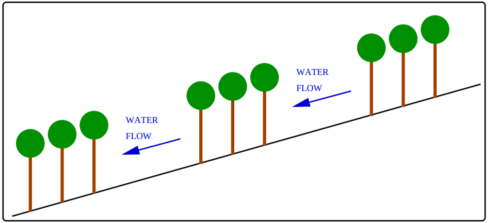

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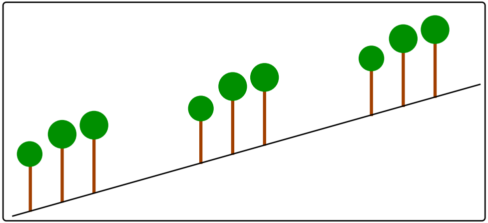

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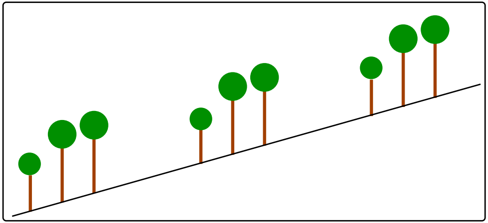

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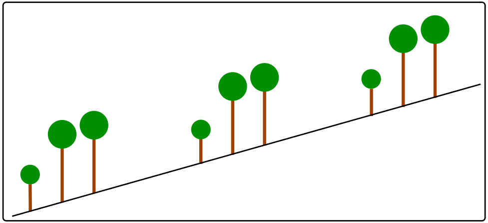

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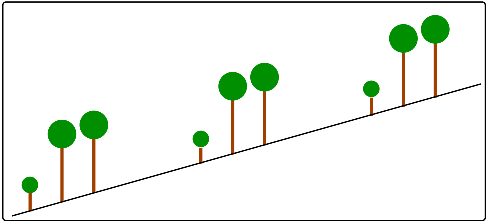

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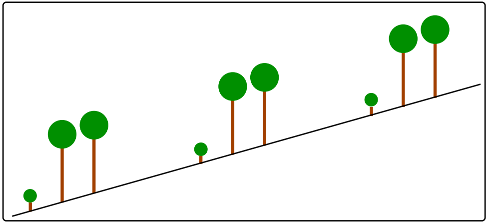

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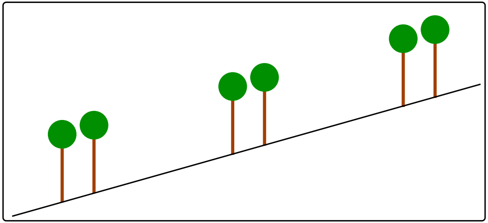

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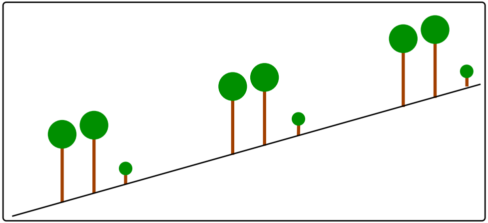

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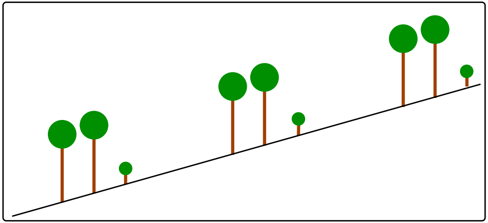

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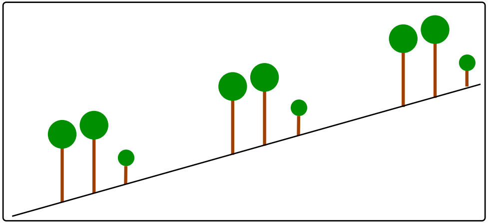

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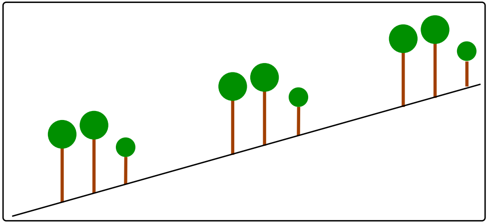

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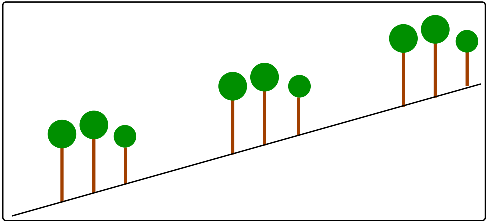

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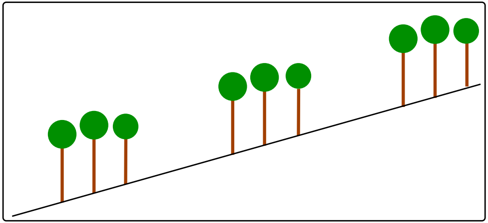

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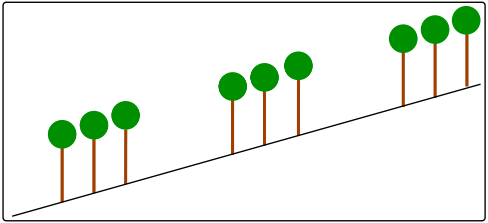

- This mechanism suggests that the stripes would move uphill; this remains controversial.

Outline

1. Ecological Background
2. The Mathematical Model
3. Linear Analysis
4. Travelling Wave Equations
5. Bifurcations in the PDEs
6. Summary

Mathematical Model of Klausmeier

Rate of change of water	=	Rainfall – Evaporation	–	Uptake by plants	+	Flow downhill
Rate of change plant biomass	=	Growth, proportional to water uptake	–	Mortality	+	Random dispersal

$$\partial w/\partial t = A - w - wu^2 + \nu \partial w/\partial x$$

$$\partial u/\partial t = wu^2 - Bu + \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$$

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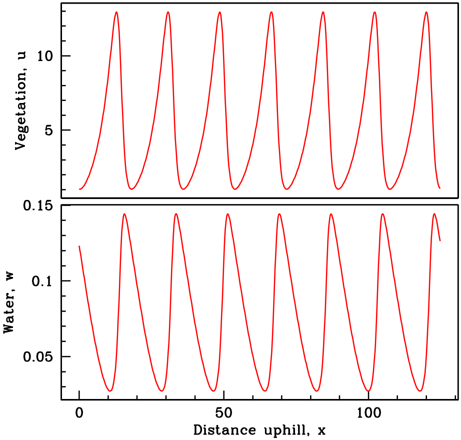
The nonlinearity in wu^2 arises because the presence of roots increases water infiltration into the soil.

Typical Solution of the Model



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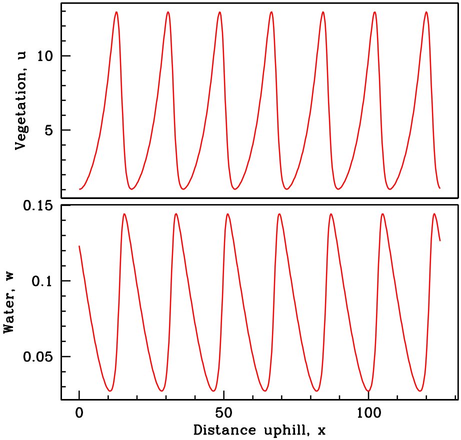

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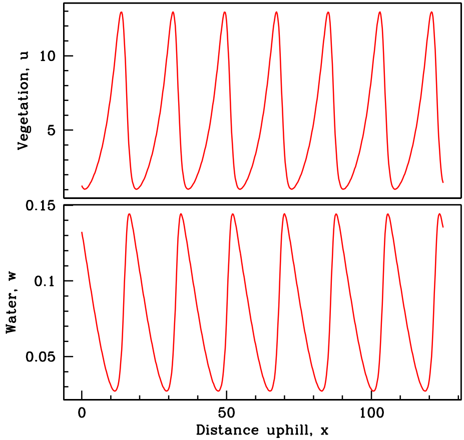

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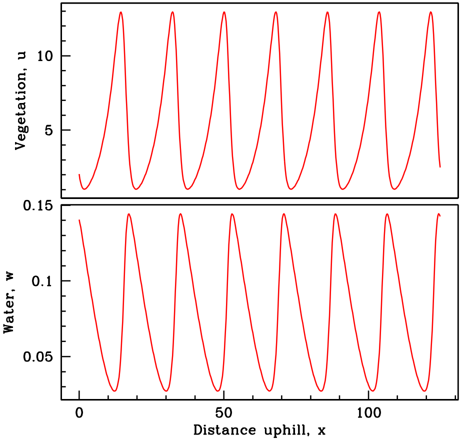

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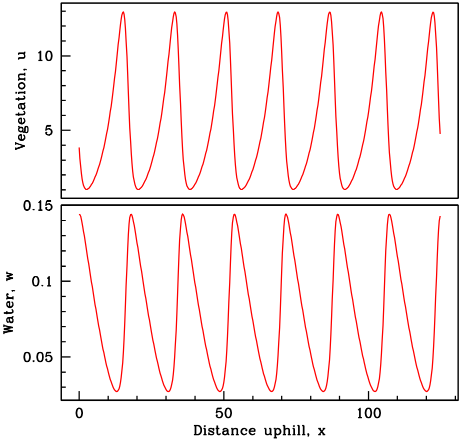

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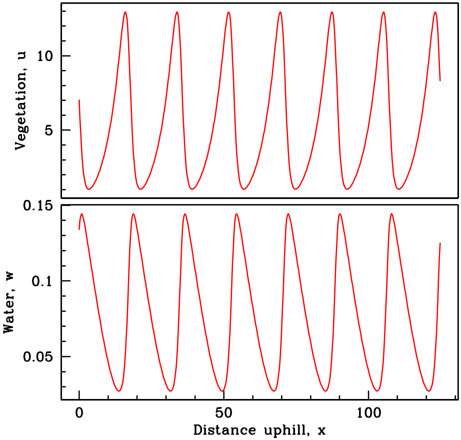

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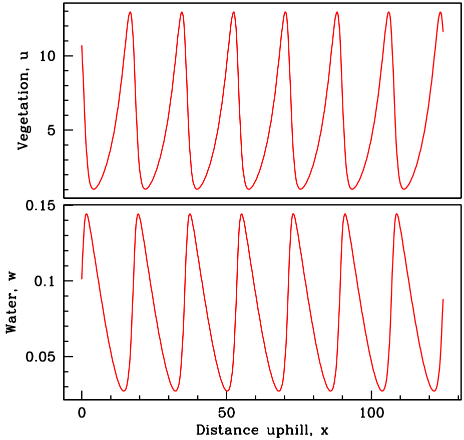

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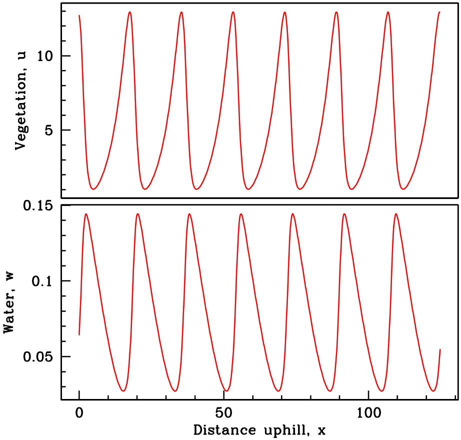

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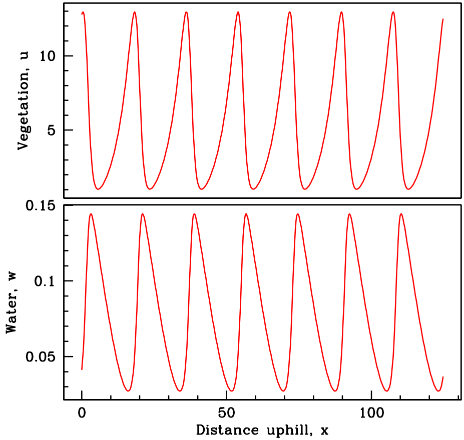

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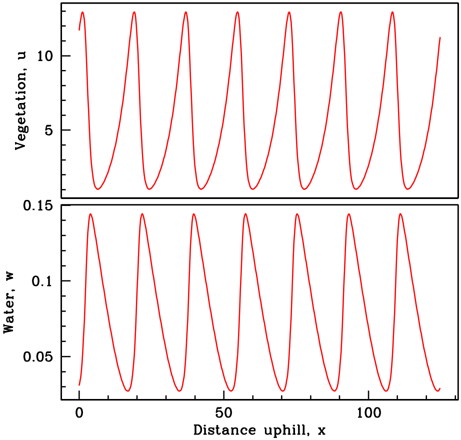

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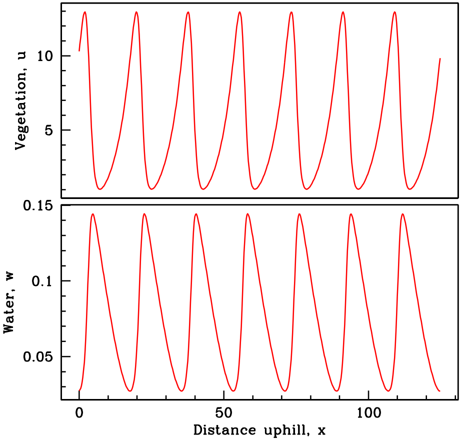

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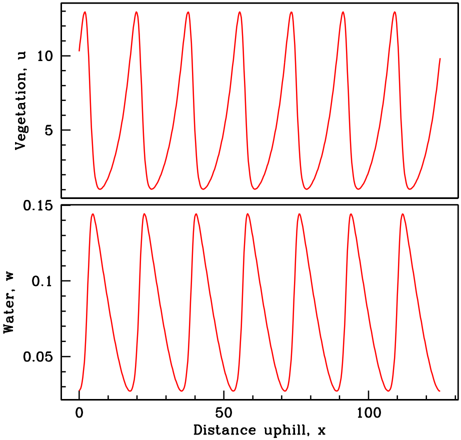

Outline

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$$u_u = \frac{2B}{A - \sqrt{A^2 - 4B^2}} \quad w_u = \frac{A - \sqrt{A^2 - 4B^2}}{2} \quad \text{unstable}$$

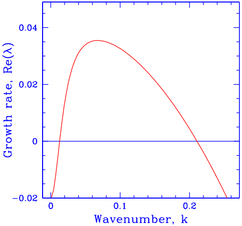
$$u_s = \frac{2B}{A + \sqrt{A^2 - 4B^2}} \quad w_s = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

stable to homog pertns for $B < 2$

- Patterns develop when (u_s, w_s) is unstable to inhomogeneous perturbations

Approximate Conditions for Patterning

Look for solutions $(u, w) = (u_s, w_s) + (u_0, w_0) \exp\{ikx + \lambda t\}$



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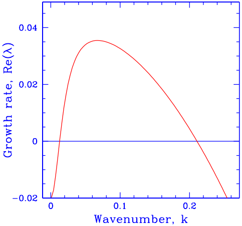

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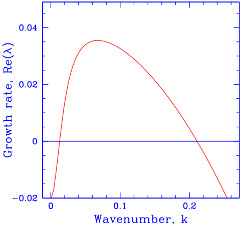

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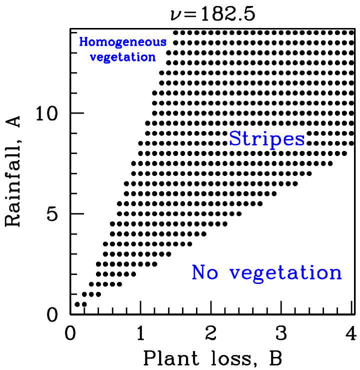
One can niavely assume that existence of (u_s, w_s) gives a second condition

An Illustration of Conditions for Patterning



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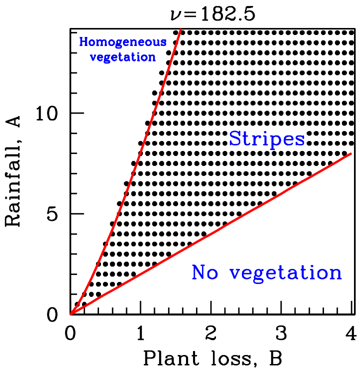

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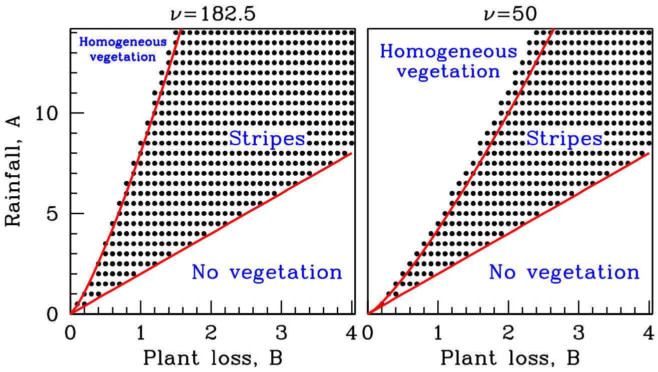

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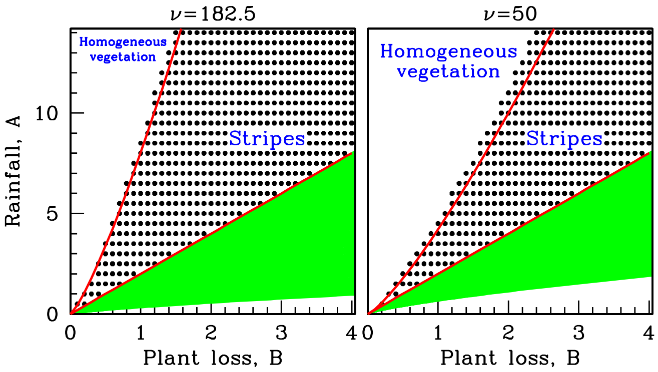


Outline

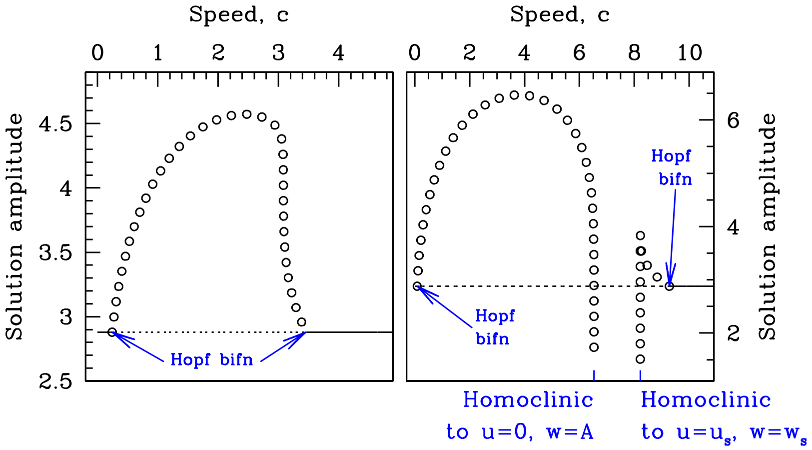
Travelling Wave Equations

The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z)$, $w(x,t) = W(z)$, $z = x - ct$

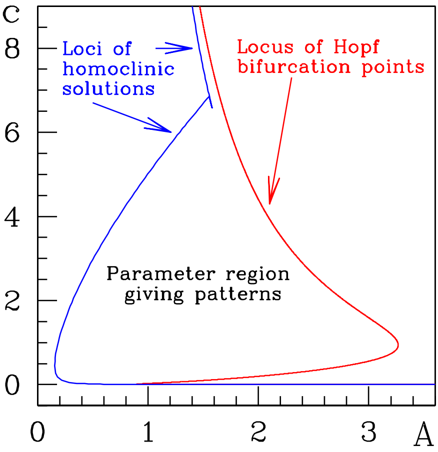
$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these ODEs

Bifurcation Diagram for Travelling Wave ODEs



When do Patterns Form?



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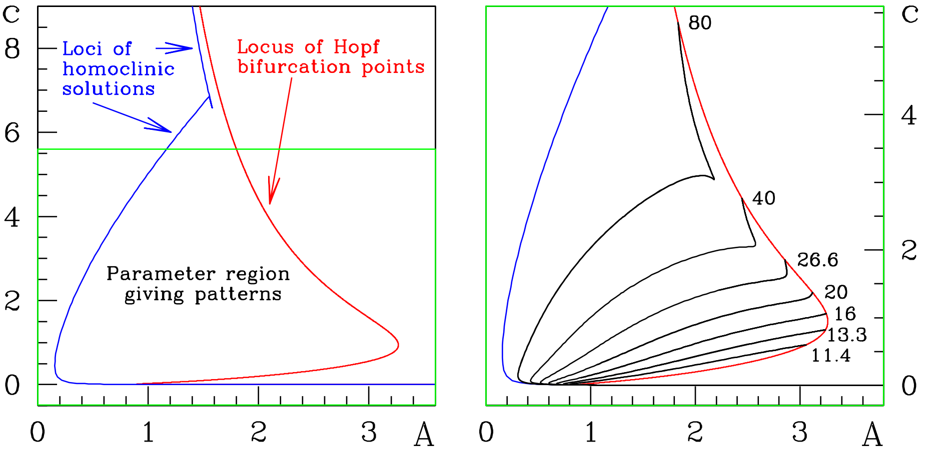
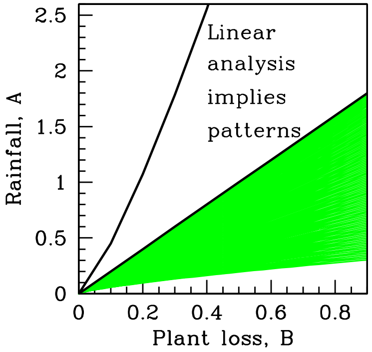


Pattern Formation for Low Rainfall



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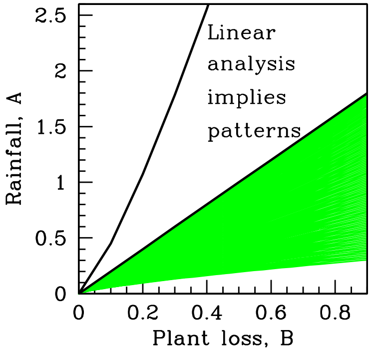

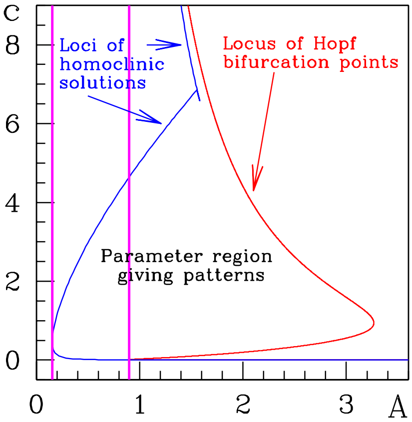

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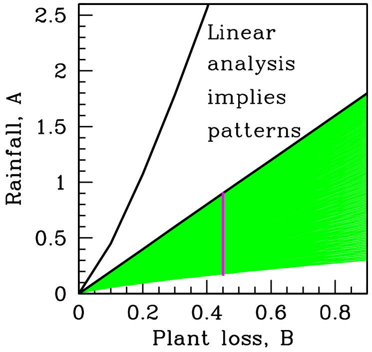

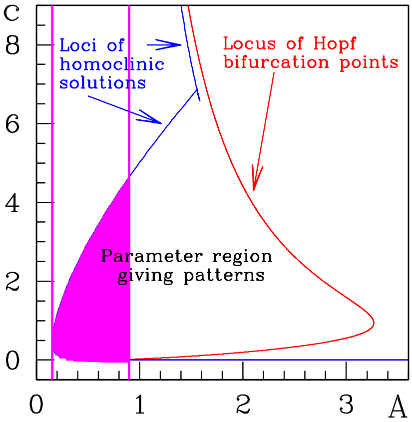

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Discretizing the PDEs

To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

$$\begin{aligned} \partial u_i/\partial t &= w_i u_i^2 - B u_i + (u_{i+1} - 2u_i + 2u_{i\text{-}1})/\Delta x^2 \\ \partial w_i/\partial t &= A - w_i - w_i u_i^2 + \nu(w_{i+1} - w_i)/\Delta x \end{aligned}$$

$(i = 1, \ldots, N)$.

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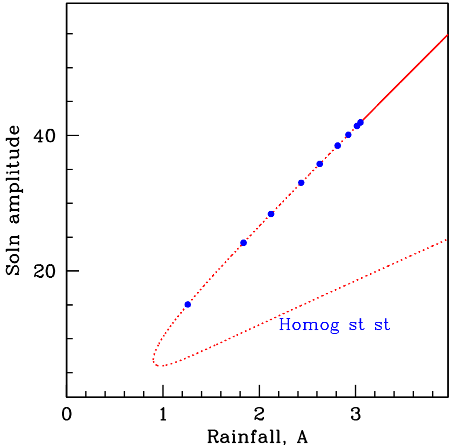
Most of our work has used $N = 40$ and $\Delta x = 2$.
We assume periodic boundary conditions.

Bifurcation Diagram for Discretized PDEs



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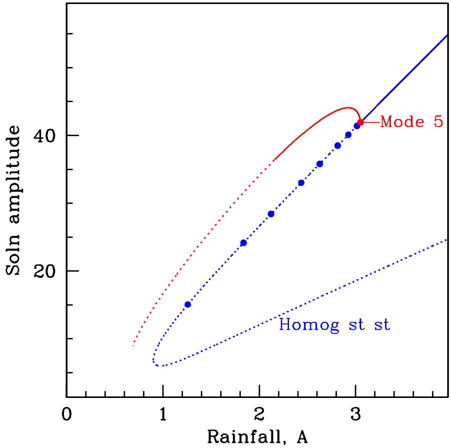

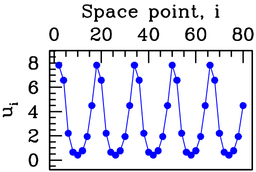

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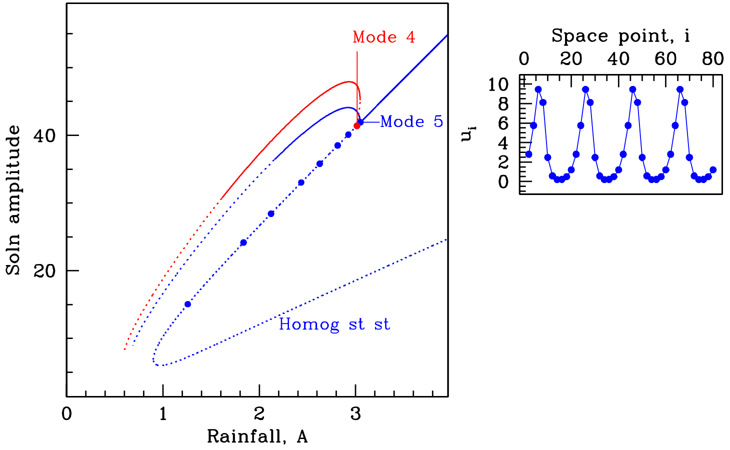

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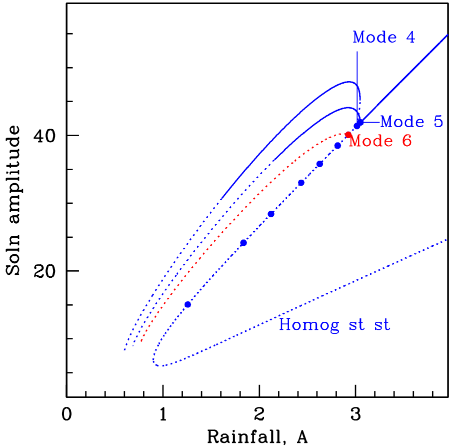

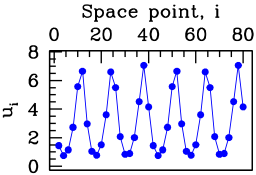

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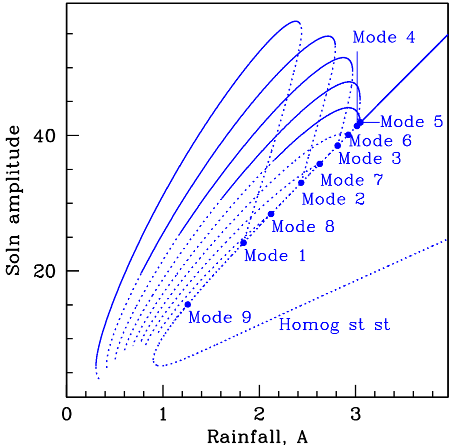
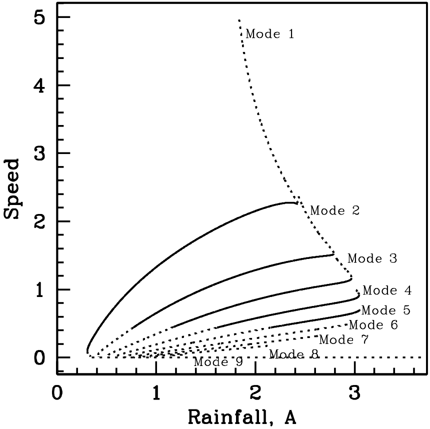


Speed vs Rainfall for Discretized PDEs

c vs A for PDEs



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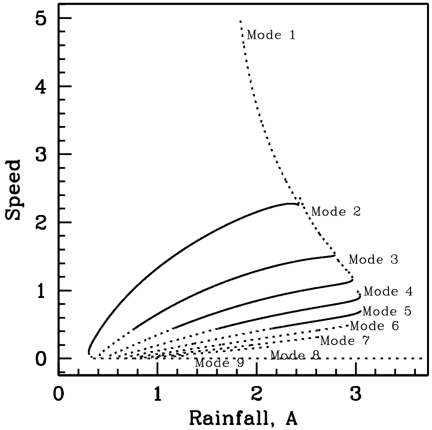
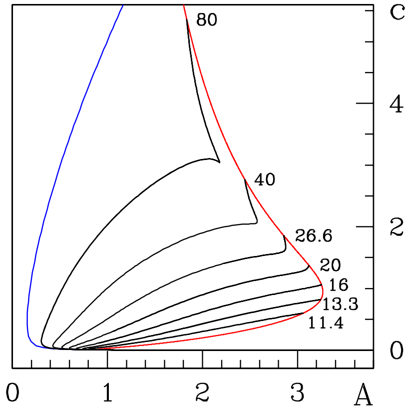


c vs A for travelling wave PDEs



Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation

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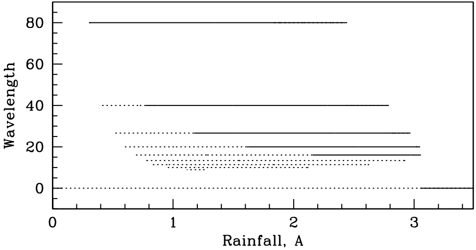
- We consider initial conditions that are small perturbations of the coexistence steady state (u_s, v_s).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation



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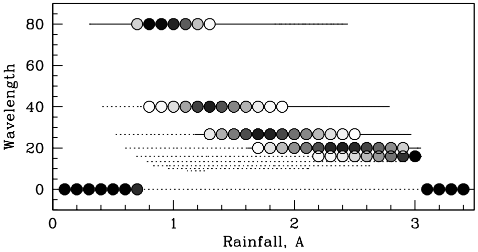

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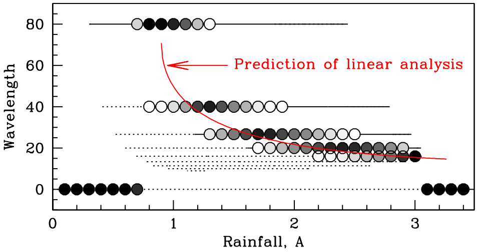
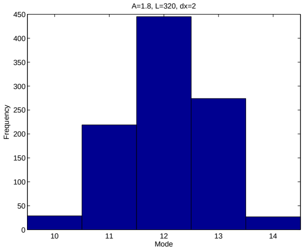


The wavelength is close to that predicted by linear stability analysis

Pattern Selection on Larger Domains

The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged



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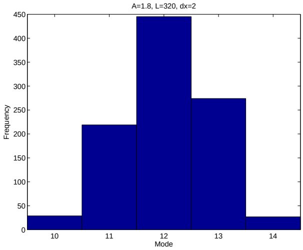
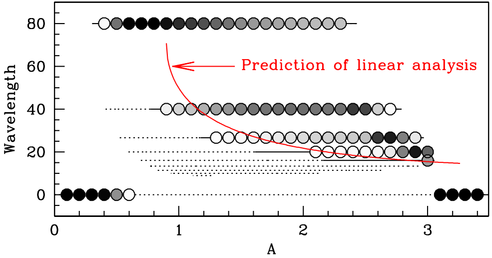
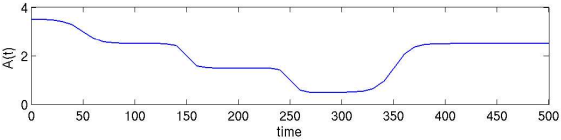


But it does not apply for other initial conditions, such as perturbations about (u_u, w_u)

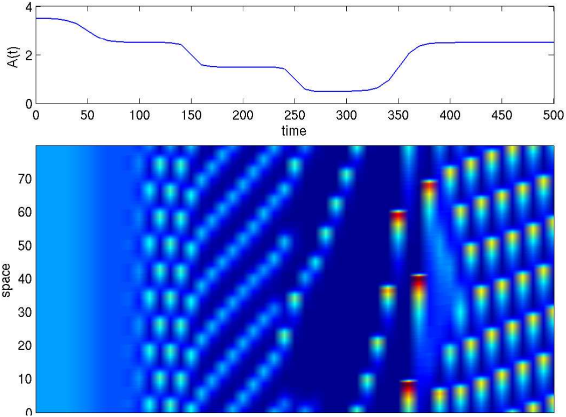


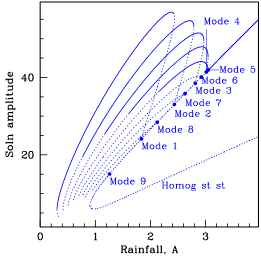
Hysteresis



- The existence of multiple stable patterns raises the possibility of hysteresis
- We consider slow variations in the rainfall parameter A

Hysteresis

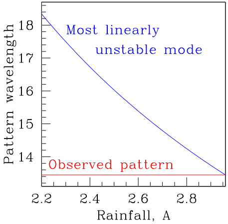

A(t)
time
space


Mode 4
Mode 5
Mode 6
Mode 3
Mode 7
Mode 2
Mode 8
Mode 1
Mode 9
Homog st st
Soln amplitude
Rainfall, A

Outline

Conclusions

- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.



Conclusions

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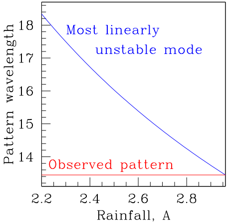

- When vegetation stripes arise from desert via an increase in rainfall, pattern wavelength will be the whole domain.

Moral

Predictions based only on linear stability analysis are misleading for this model

List of Frames