Vegetation Stripes in Semi-Arid Environments

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Mathematical Biology Group Meeting
In collaboration with Gabriel Lord
Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico.

- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees
On flat ground, irregular mosaics of vegetation are typical.
Vegetation Pattern Formation (contd)

- On flat ground, irregular mosaics of vegetation are typical.

- On slopes, the patterns are stripes, parallel to contours (“Tiger bush”).
Mechanisms for Vegetation Patterning

- Basic mechanism: competition for water
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- Possible detailed mechanism: water flow downhill causes stripes
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![Vegetation Pattern Diagram]
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This mechanism suggests that the stripes would move uphill; this remains controversial.
Ecological Background

The Mathematical Model

Linear Analysis

Travelling Wave Equations

Bifurcations in the PDEs

Summary

Outline

1. Ecological Background
2. The Mathematical Model
3. Linear Analysis
4. Travelling Wave Equations
5. Bifurcations in the PDEs
6. Summary
Mathematical Model of Klausmeier

Rate of change = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change = Growth, proportional to water uptake – Mortality + Random dispersal

\[
\frac{\partial w}{\partial t} = A - w - wu^2 + \nu \frac{\partial w}{\partial x}
\]

\[
\frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
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The nonlinearity in $wu^2$ arises because the presence of roots increases water infiltration into the soil.
Typical Solution of the Model

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Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

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Typical Solution of the Model

- **Vegetation, $u$**
- **Water, $w$**

Distance uphill, $x$
Typical Solution of the Model

Mathematical Model of Klausmeier

Graph showing typical solutions of the model with axes labeled as:
- Vegetation, $u$ on the vertical axis
- Water, $w$ on the vertical axis
- Distance uphill, $x$ on the horizontal axis.
Typical Solution of the Model
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

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Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0, w = A$. 
Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$.
- When $A \geq 2B$, there are also two non-trivial steady states

$$
\begin{align*}
    u_u &= \frac{2B}{A - \sqrt{A^2 - 4B^2}} \\
    w_u &= \frac{A - \sqrt{A^2 - 4B^2}}{2} \\
    u_s &= \frac{2B}{A + \sqrt{A^2 - 4B^2}} \\
    w_s &= \frac{A + \sqrt{A^2 - 4B^2}}{2}
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\end{align*}
\]

- $u_s$ and $w_s$ are unstable for $B < 2$
- Patterns develop when $(u_s, w_s)$ is unstable to inhomogeneous perturbations
Approximate Conditions for Patterning

Look for solutions \((u, w) = (u_s, w_s) + (u_0, w_0) \exp\{ikx + \lambda t\}\)

The dispersion relation \(\text{Re}[\lambda(k)]\) is algebraically complicated
Approximate Conditions for Patterning

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Simplification using \(\nu \gg 1\) implies that for pattern formation

\[ A < \nu^{1/2} B^{5/4} / 8^{1/4} \]
Approximate Conditions for Patterning

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Simplification using \(\nu \gg 1\) implies that for pattern formation

\[2B < A < \nu^{1/2} B^{5/4} / 8^{1/4}\]

One can naively assume that existence of \((u_s, w_s)\) gives a second condition.
An Illustration of Conditions for Patterning

\( \nu = 182.5 \)

- Homogeneous vegetation
- Stripes
- No vegetation

Rainfall, \( A \)

Plant loss, \( B \)
An Illustration of Conditions for Patterning

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- Homogeneous vegetation
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Rainfall, A

Plant loss, B
An Illustration of Conditions for Patterning

Rainfall, A

Plant loss, B

Homogeneous vegetation

Stripes

No vegetation

ν = 182.5

ν = 50

Homogeneous vegetation

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An Illustration of Conditions for Patterning

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- Homogeneous vegetation
- Stripes
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The patterns move at constant shape and speed
\[ u(x, t) = U(z), \quad w(x, t) = W(z), \quad z = x - ct \]

\[
\begin{align*}
\frac{d^2 U}{dz^2} + c \frac{dU}{dz} + WU^2 - BU &= 0 \\
(\nu + c)\frac{dW}{dz} + A - W - WU^2 &= 0
\end{align*}
\]

The patterns are periodic (limit cycle) solutions of these ODEs
Bifurcation Diagram for Travelling Wave ODEs

- Homoclinic to $u=0, w=A$
- Homoclinic to $u=u_s, w=w_s$

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When do Patterns Form?

Loci of homoclinic solutions

Locus of Hopf bifurcation points

Parameter region giving patterns
When do Patterns Form?

- Loci of homoclinic solutions
- Locus of Hopf bifurcation points
- Parameter region giving patterns

Graphs showing parameter regions and bifurcation diagrams.
Pattern Formation for Low Rainfall

Rainfall, $A$

Plant loss, $B$

Linear analysis implies patterns

When do Patterns Form?

Pattern Formation for Low Rainfall
Pattern Formation for Low Rainfall

- Linear analysis implies patterns
- Loci of homoclinic solutions
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- Parameter region giving patterns
Pattern Formation for Low Rainfall

- Linear analysis implies patterns.
- Loci of homoclinic solutions.
- Locus of Hopf bifurcation points.
- Parameter region giving patterns.
To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

\[
\begin{align*}
\partial u_i / \partial t & = w_i u_i^2 - Bu_i + (u_{i+1} - 2u_i + 2u_{i-1}) / \Delta x^2 \\
\partial w_i / \partial t & = A - w_i - w_i u_i^2 + \nu (w_{i+1} - w_i) / \Delta x
\end{align*}
\]

\((i = 1, \ldots, N)\).
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We use upwinding for the convective term.
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Most of our work has used \(N = 40\) and \(\Delta x = 2\).

We assume periodic boundary conditions.
Bifurcation Diagram for Discretized PDEs

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Bifurcation Diagram for Discretized PDEs

- Discretizing the PDEs
- Bifurcation Diagram for Discretized PDEs
- Speed vs Rainfall for Discretized PDEs
- Pattern Selection
- Pattern Selection on Larger Domains
- Hysteresis
Bifurcation Diagram for Discretized PDEs
Bifurcation Diagram for Discretized PDEs

![Diagram showing bifurcation points and solution amplitudes as a function of rainfall. Modes 4, 5, and 6 are marked, with homogenous steady state indicated.](diagram.png)
Bifurcation Diagram for Discretized PDEs

![Bifurcation Diagram](image)
Speed vs Rainfall for Discretized PDEs

c vs A for PDEs

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Vegetation Stripes in Semi-Arid Environments
Speed vs Rainfall for Discretized PDEs

$c$ vs $A$ for PDEs

$c$ vs $A$ for travelling wave PDEs
Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state \((u_s, v_s)\).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.
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All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.

The wavelength is close to that predicted by linear stability analysis.
The proximity of the wavelength to the most linearly unstable mode continues as the domain is enlarged.
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But it does not apply for other initial conditions, such as perturbations about \((u_u, w_u)\).
The existence of multiple stable patterns raises the possibility of hysteresis.

We consider slow variations in the rainfall parameter $A$. 

The graph shows the function $A(t)$ over time, indicating a possible phase transition or hysteresis effect due to the changes in rainfall parameter $A$. 

The graph displays a series of peaks and troughs that might correspond to different stable states of the system under varying conditions.
Hysteresis

Discretizing the PDEs
Bifurcation Diagram for Discretized PDEs
Speed vs Rainfall for Discretized PDEs
Pattern Selection
Pattern Selection on Larger Domains
Hysteresis
Conclusions

When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.
Conclusions

- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at its bifurcating value.

- When vegetation stripes arise from desert via an increase in rainfall, pattern wavelength will be the whole domain.
Predictions based only on linear stability analysis are misleading for this model.
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Moral

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