Spatiotemporal Dynamics in Ecology: Insights from Physics

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This talk can be downloaded from my web site
`www.ma.hw.ac.uk/∼jas`

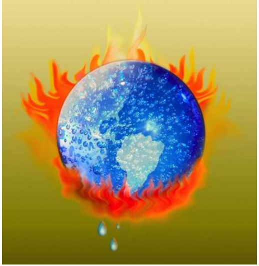
Voles


© - lubomir hlasek
www.hlasek.com
Clethrionomys glareolus 4162

Voles

→

Climate change



Voles

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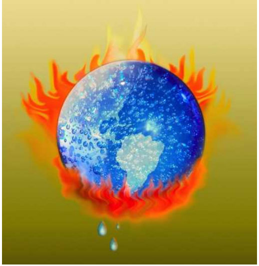


ABSOLUTE STABILITY

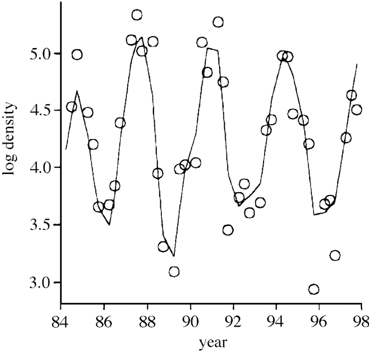
Outline

Outline

Vole Population Cycles

Voles

Clethrionomys glareolus



Vole Population Cycles

Voles

Clethrionomys glareolus

Fennoscandia

Vole Population Cycles

Voles

Clethrionomys glareolus

Weasel

Mustela nivalis

Cyclic Predator-Prey Systems

The interaction between a predator population and its prey can cause population cycles.
This has been modelled extensively using systems of two coupled ODEs.

$$\boxed{\text{predators}} \quad dp/dt = \underbrace{akph/(1+kh)}_{\text{benefit from predation}} - \underbrace{bp}_{\text{death}}$$

$$\boxed{\text{prey}} \quad dh/dt = \underbrace{rh(1-h/h_0)}_{\text{intrinsic birth \& death}} - \underbrace{ckph/(1+kh)}_{\text{predation}}$$

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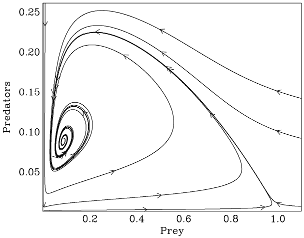
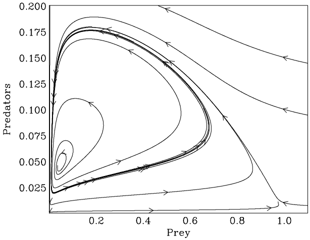


constant coexistence

change $\longrightarrow$ parameters



cycles

Spatiotemporal Data

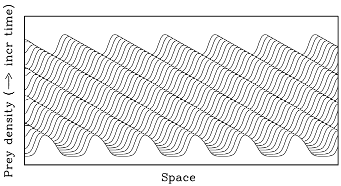
Voles

Clethrionomys glareolus

Field data from different sites
$\Rightarrow$ vole cycles are not spatially homogeneous.

Rather they are organised into a plane wave.

Spatiotemporal Data



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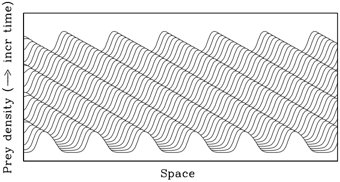
Spatiotemporal Data



Plane wave: periodic as a function of space (1-D) and time.

Mathematically: a solution of the form $f(x \pm ct)$, with the function $f(.)$ periodic.

Terminology: the terms

plane wave,
wavetrain,
periodic travelling wave

are equivalent.

Modelling Predator-Prey Invasion

To study spatiotemporal dynamics, we add diffusion terms to represent local dispersal.

predators
$$\partial p/\partial t = \underbrace{D_p \partial^2 p/\partial x^2}_{\text{dispersal}} + \underbrace{akph/(1+kh)}_{\text{benefit from predation}} - \underbrace{bp}_{\text{death}}$$

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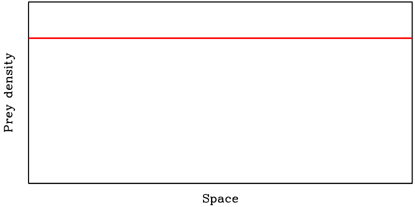
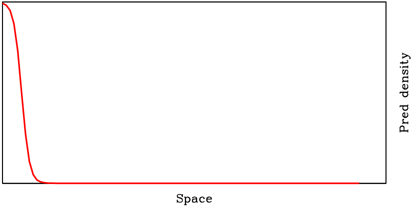
Modelling Predator-Prey Invasion

To study spatiotemporal dynamics, we add diffusion terms to represent local dispersal.
The most natural initial conditions correspond to invasion.



Initially we set the prey to the prey-only equilibrium throughout the domain.



Initially we set the predators to zero except near the left hand boundary.

Modelling Predator-Prey Invasion

To study spatiotemporal dynamics, we add diffusion terms to represent local dispersal.
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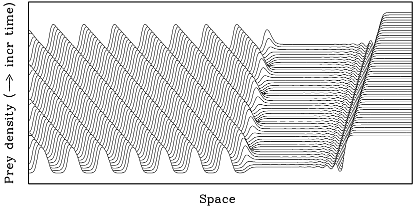

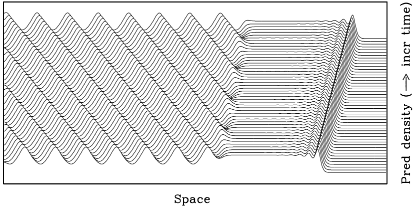

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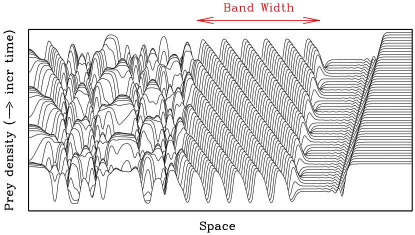
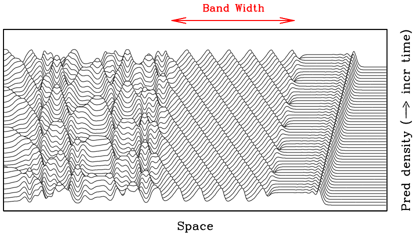




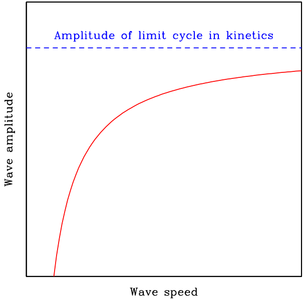
Outline

Plane Wave Stability

An oscillatory reaction-diffusion system has a one-parameter family of plane wave solutions

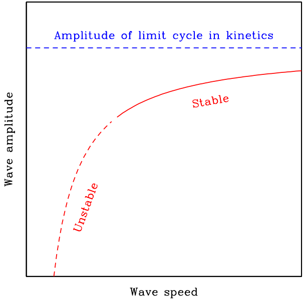
(if the diffusion coefficients are sufficiently close to one another)

(Kopell, Howard (1973) *Stud Appl Math* 52:291)



Plane Wave Stability

Some members of the plane wave family are stable as solutions of the partial differential equations, while others are unstable.



Plane Wave Stability

The invasion process selects a particular member of the plane wave family (Sherratt (1998) *Physica D* 117:145).

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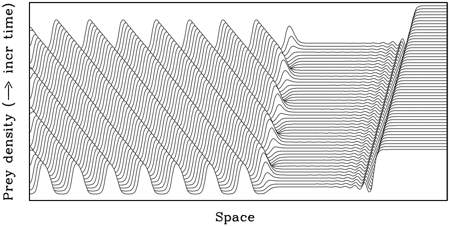
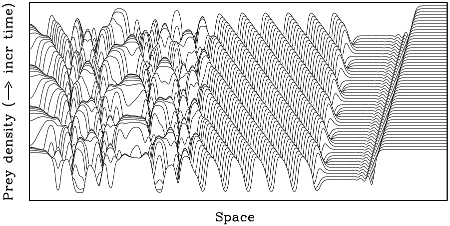
For these parameters, the selected plane wave is stable.



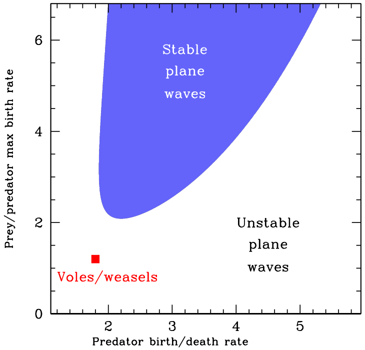
Plane Wave Stability

The invasion process selects a particular member of the plane wave family (Sherratt (1998) *Physica D* 117:145).

A "plane wave band" occurs when the selected plane wave is unstable.

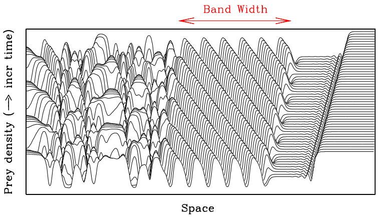


Back to Voles and Weasels



For vole–weasel parameters, the selected plane wave is unstable.

Back to Voles and Weasels



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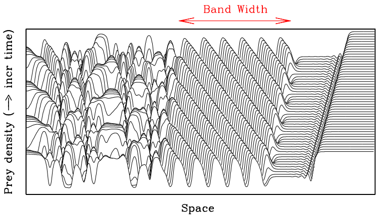
Conclusion: we expect to observe plane waves of field voles only if the band width is large compared to the domain length.

Back to Voles and Weasels



For vole–weasel parameters, the selected plane wave is unstable.

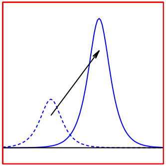
Conclusion: we expect to observe plane waves of field voles only if the band width is large compared to the domain length.

Question: how can we calculate the band width?

Outline

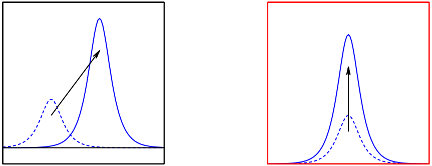
Convective and Absolute Stability

- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving.
  This is "convective instability".



Convective and Absolute Stability

- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".
- Alternatively, a solution can be unstable with perturbations growing without moving. This is "absolute instability".



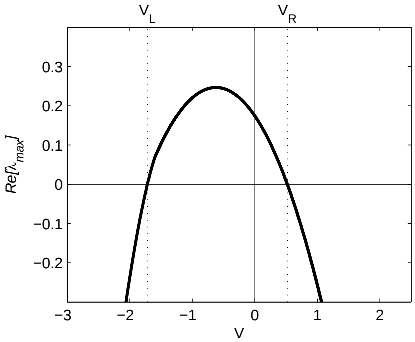
Absolute Stability in a Moving Frame of Reference

Absolute stability refers to the growth/decay of stationary perturbations.

We must consider the growth/decay of perturbations moving with a specified velocity V, i.e. absolute stability in a frame of reference moving with velocity V.

$$\begin{aligned} \text{Define } \lambda_{max}(V) &= \text{temporal eigenvalue of the most unstable linear mode} \\ \nu_{max}(V) &= \text{the corresponding spatial eigenvalue} \end{aligned}$$

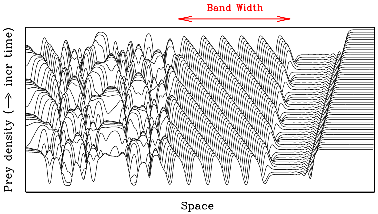
Absolute Stability in a Moving Frame of Reference



A tutorial guide to calculating absolute stability is freely available at

`http://research.microsoft.com/en-us/projects/loptw/tutorial.aspx`

Defining the Width of the Plane Wave Band



- We define the left-hand edge of the plane wave band as where unstable linear modes first become amplified by an arbitrary factor $\mathcal{F}$.

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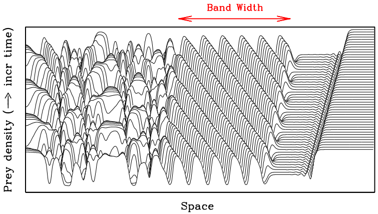

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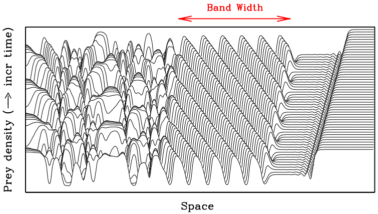

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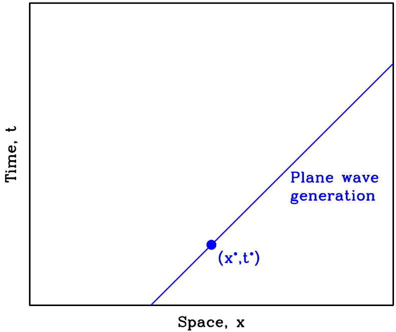
- The dependence on ecological parameters is via $\mathcal{W}$.

The Band Width Formula



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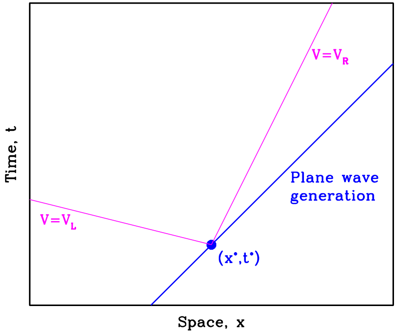

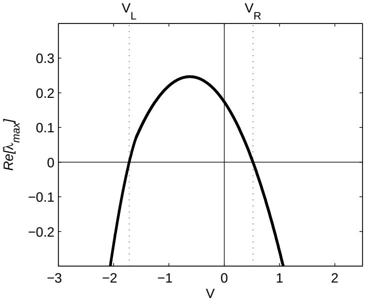

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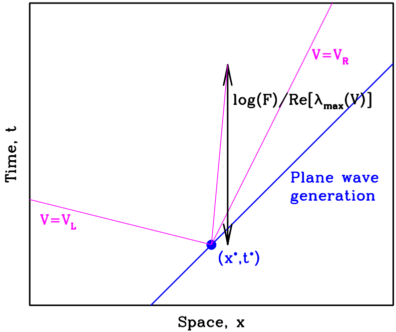
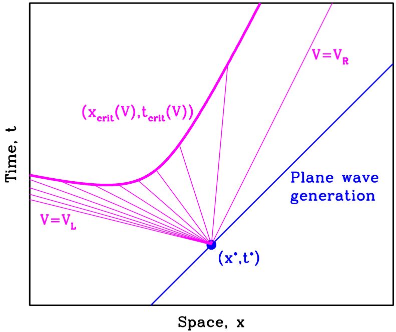


Perturbations moving with velocity V grow as $\exp[\mathrm{Re}\,(\lambda_{max}(V))\cdot t]$

$\Rightarrow$ amplified by the factor $\mathcal{F}$ after time $=\log(\mathcal{F})/\mathrm{Re}\,(\lambda_{max}(V))$

The Band Width Formula



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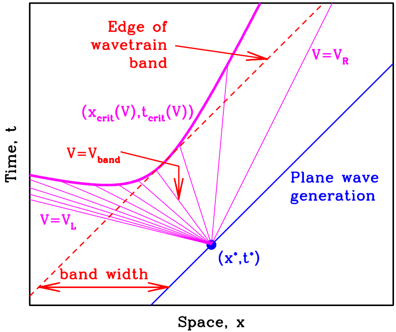

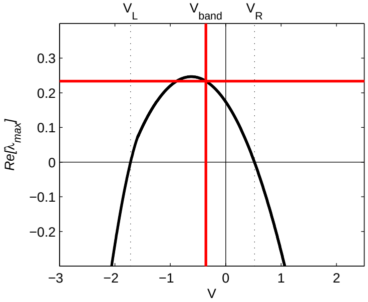

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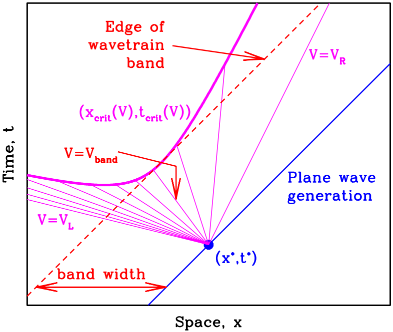
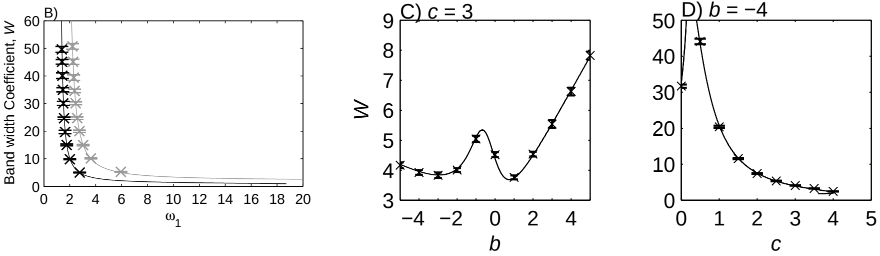


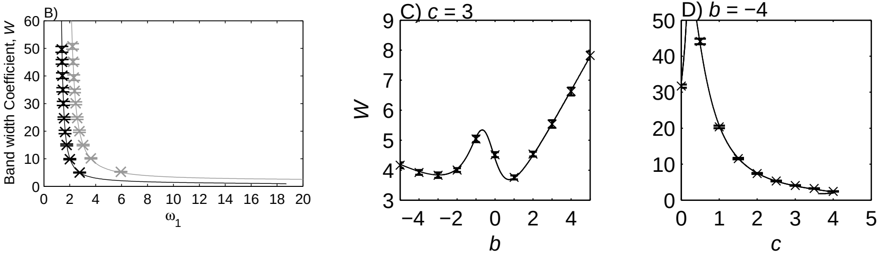
$$\mathcal{W} = 1/\mathrm{Re}\,[\nu_{max}(V_{band})]$$

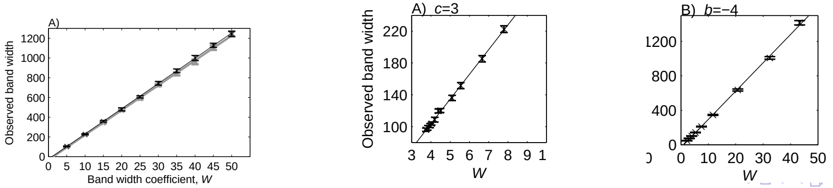
where $(V_{band} - c_{inv})\mathrm{Re}\,[\nu_{max}(V_{band})] = \mathrm{Re}\,[\lambda_{max}(V_{band})]$

The Form of W


B)
Band width Coefficient, W
0
10
20
30
40
50
60
0 2 4 6 8 10 12 14 16 18 20
ω1
C) c = 3
W
3
4
5
6
7
8
9
−4 −2 0 2 4
b
D) b = −4
0
10
20
30
40
50
0 1 2 3 4 5
c

The Form of W

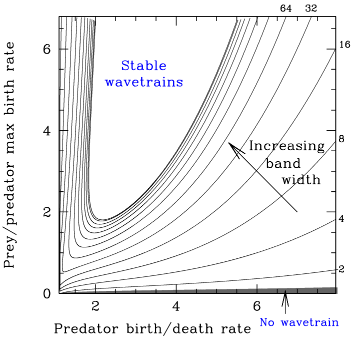

B)
Band width Coefficient, W
ω_1
C) c = 3
W
b
D) b = −4
c


A)
Observed band width
Band width coefficient, W
A) c=3
Observed band width
W
B) b=−4
W

Outline

Band Width Sensitivity

Our formula gives band width vs ecological parameters.



Band Width Sensitivity

Our formula gives band width vs ecological parameters.

What does this imply for the vole – weasel interaction?

voles

weasel

Band Width Sensitivity

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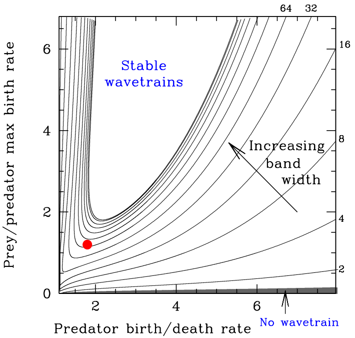

● = weasel–vole parameters.

The plane wave band is very wide: about 300 wavelengths.

Band Width Sensitivity

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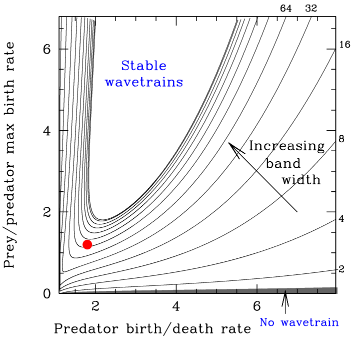


● = weasel–vole parameters.

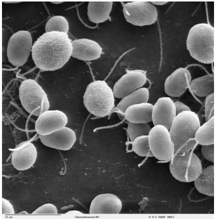
5%↑ in vole birth rate
⇒ 22%↑ in band width.

Band Width Sensitivity

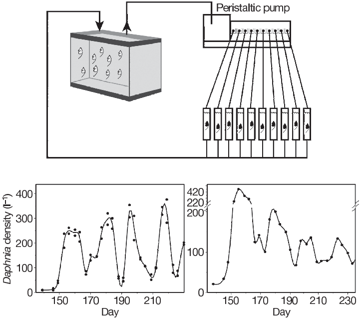
Our formula gives band width vs ecological parameters.

Example: *Daphnia pulex–Chlamydomonas reinhardii* interaction

Daphnia pulex



Chlamydomonas reinhardii



(from McCauley *et al* (2008), *Nature* 455:1240, 2008)

Band Width Sensitivity

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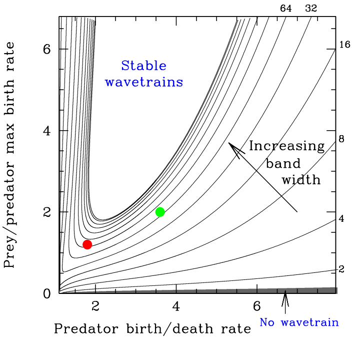

● = plankton parameters (*Daphnia pulex–Chlamydomonas reinhardii*).

5.2%↓ in Daphnia birth rate

⇒ doubling of band width.

Ecological Implications

- It is known that climate change is significantly affecting the parameters of oscillatory ecological systems (e.g. Ims *et al* (2008) *TREE* 23:79).
- Also, climate change $\Rightarrow$ more frequent invasions.
- The width of the plane wave band determines whether one sees spatiotemporal chaos or periodic homogeneous oscillations behind the invasion.
- We have shown that band width depends sensitively on ecological parameters.

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- Also, climate change $\Rightarrow$ more frequent invasions.
- The width of the plane wave band determines whether one sees spatiotemporal chaos or periodic homogeneous oscillations behind the invasion.
- We have shown that band width depends sensitively on ecological parameters.
- This suggests that the implications of climate change for *spatio* temporal dynamics may be even more dramatic than for purely temporal behaviour.

This work is in collaboration with:

Matthew Smith

(Microsoft Research Cambridge)

Jens Rademacher

(CWI, Amsterdam)

Xavier Lambin

(University of Aberdeen)

References

J.A. Sherratt, M.J. Smith: Periodic travelling waves in cyclic populations: field studies and reaction-diffusion models.
J. R. Soc. Interface **5**, 483-505 (2008).

J.A. Sherratt, M.J. Smith, J.D.M. Rademacher: Locating the transition from periodic oscillations to spatiotemporal chaos in the wake of invasion.
Proc. Natl. Acad. Sci. USA, published online (open access).

List of Frames

The Form of V_{band}