Modelling Vegetation Patterns in Semi-Arid Environments

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This talk can be downloaded from my web site
www.ma.hw.ac.uk/~jas
In collaboration with
Gabriel Lord
Outline

1. Ecological Background
2. The Mathematical Model
3. Model Predictions: When Do Patterns Occur?
4. Model Predictions: Which Pattern Forms?
5. Conclusions
1. Ecological Background
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Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico (rainfall 100-700 mm/year)
First identified by aerial photos in 1950s
Plants vary from grasses to shrubs and trees
Labyrinth of bushy vegetation in Niger
Striped pattern of bushy vegetation in Niger
Vegetation Pattern Formation (contd)

- On flat ground, irregular mosaics of vegetation are typical.
- On slopes, the patterns are stripes, parallel to contours ("Tiger bush").
Mechanisms for Vegetation Patterning

- Basic mechanism: competition for water
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- Basic mechanism: competition for water
- Possible detailed mechanism: water flow downhill causes stripes
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This mechanism suggests that the stripes would move uphill; this remains controversial.
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Mathematical Model of Klausmeier

\[ \frac{\partial w}{\partial t} = A - w - wu^2 + \nu \frac{\partial w}{\partial x} \]

\[ \frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2} \]


**Mathematical Model of Klausmeier**

Rate of change = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change = Growth, proportional to water uptake – Mortality + Random dispersal

\[
\frac{\partial w}{\partial t} = A - w - wu^2 + \nu \frac{\partial w}{\partial x}
\]

\[
\frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2}
\]

The nonlinearity in \(wu^2\) arises because the presence of roots increases water infiltration into the soil.
Typical Solution of the Model

![Graph showing the typical solution of the model with two plots: one for vegetation density (u) and one for water content (w) versus distance uphill (x).]
Typical Solution of the Model
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model
Typical Solution of the Model

![Typical Solution of the Model](image-url)
Typical Solution of the Model

- Vegetation, $u$
  - Axis: 0 to 10
- Water, $w$
  - Axis: 0 to 0.15

Distance uphill, $x$
- 0 to 100
Typical Solution of the Model

Vegetation, u

Water, w

Distance uphill, x
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing typical solution of the model with vegetation and water profiles over distance uphill.]
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, \( u \)

Water, \( w \)

Distance uphill, \( x \)
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![Graph showing vegetation and water profiles](image-url)
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Typical Solution of the Model

![Graph showing a typical solution of the model with two plots: one for vegetation (u) and another for water (w) as a function of distance uphill (x).]
Typical Solution of the Model

- Vegetation, \( u \):
  - Y-axis: 0 to 10
  - X-axis: Distance uphill, \( x \)

- Water, \( w \):
  - Y-axis: 0.05 to 0.15
  - X-axis: Distance uphill, \( x \)

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Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Diagram showing vegetation and water distribution over distance](image-url)
Typical Solution of the Model

![Graph showing vegetation and water dynamics over distance uphill](image-url)
Typical Solution of the Model

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Distance uphill, $x$
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Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water levels over distance uphill]
Typical Solution of the Model

![Graph showing vegetation and water distribution](image-url)
Typical Solution of the Model

- Vegetation, $u$
  - Maximum: 10
  - Minimum: 0.15

- Water, $w$
  - Maximum: 0.1
  - Minimum: 0.05

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
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![Graph showing vegetation and water distribution over distance](image-url)
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For all parameter values, there is a stable “desert” steady state $u = 0, w = A$. 
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When $A \geq 2B$, there are also non-trivial steady states. If $A$ is relatively small, this steady state destabilises, giving patterns.
An Illustration of Conditions for Patterning
Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography. Mathematical prediction of wavelength as a function of parameters (rainfall, plant loss, slope) is difficult because there are multiple pattern solutions.
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We determine **pattern existence** via numerical bifurcation analysis of the pattern ordinary differential equations, and **pattern stability** via numerical bifurcation analysis of the discretized model partial differential equations.
Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of a spatially uniform state.
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.
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Perturbations to \((u_s, v_s)\)
For a range of rainfall levels, there is more than one stable pattern. Which will be selected?

We consider initial conditions that are small perturbations of a spatially uniform state.

All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.
Key Result

For a wide range of rainfall levels, there are multiple stable patterns.
The existence of multiple stable patterns raises the possibility of hysteresis. We consider slow variations in the rainfall parameter $A$. Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year.
Hysteresis

Rainfall

Time

Space

<< Mode 5 >> <<<<<< Mode 1 >>>>> < Mode 3 >

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Hysteresis

Rainfall vs. Time

Space

<< Mode 5 >> <<<< Mode 1 >>>>> < Mode 3 >
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In general, pattern wavelength depends on initial conditions. When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at a constant value.

\[
\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}
\]
Other Potential Mechanisms for Vegetation Patterns

- **Rietkirk** Klausmeier model with diffusion of water in the soil
- **van de Koppel** Klausmeier model with grazing
- **Maron** two variable model (plant density and water in the soil) with water transport based on porous media theory
- **Lejeune** short range activation (shading) and long range inhibition (competition for water)

All of these models predict patterns. To discriminate between them requires a detailed understanding of each model.
Ecological Background

The Mathematical Model

Model Predictions: When Do Patterns Occur?

Model Predictions: Which Pattern Forms?

Conclusions

Predictions of Pattern Wavelength

Other Potential Mechanisms for Vegetation Patterns

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