Patterns of Sources and Sinks in the Complex Ginzburg-Landau Equation

Jonathan A. Sherratt

Department of Mathematics
& Maxwell Institute for Mathematical Sciences

Heriot-Watt University

University of Oxford, January 21, 2010

This talk can be downloaded from my web site
`www.ma.hw.ac.uk/~jas`

This work is in collaboration with:

Matthew Smith

(Microsoft Research Ltd., Cambridge)

Jens Rademacher

(CWI, Amsterdam)

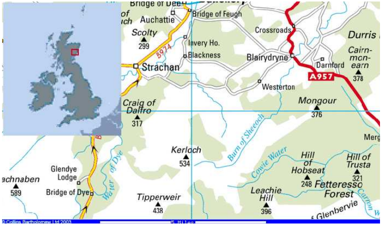
Outline

Outline

Habitat Boundaries in Ecology

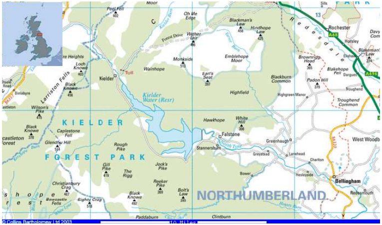
- Ecological habitats are often surrounded by unfavourable environments
- Examples: a wood surrounded by open terrain
  moorland surrounded by farmland
  marsh surrounded by dry ground
- An appropriate boundary condition is "population density=0"

Example: Red Grouse on Kerloch Moor



- Red grouse is a cyclic population (period 4-6 years)
- The study site is moorland, with farmland at its Northern edge
- Farmland is very hostile for red grouse

Second Example: Field Voles in Kielder Forest



Field voles in Kielder Forest are also cyclic (period 4 years)

Boundary Condition at the Reservoir Edge

- Voles are an important prey species for owls and kestrels
- The open expanse of Kielder Water will greatly facilitate hunting at its edge

Short eared owl

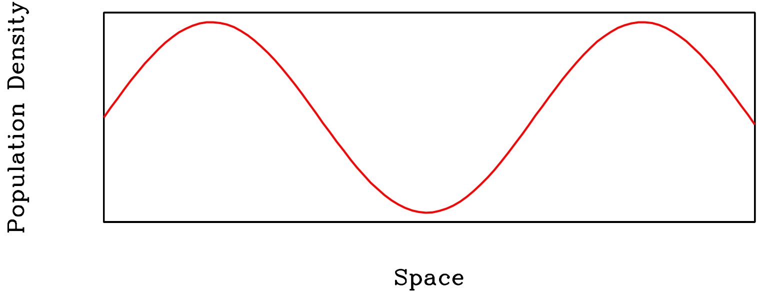
Common kestrel

Boundary Condition at the Reservoir Edge

- Voles are an important prey species for owls and kestrels
- The open expanse of Kielder Water will greatly facilitate hunting at its edge
- Therefore we expect very high vole loss at the reservoir edge, implying that a suitable boundary condition is “vole density=0”

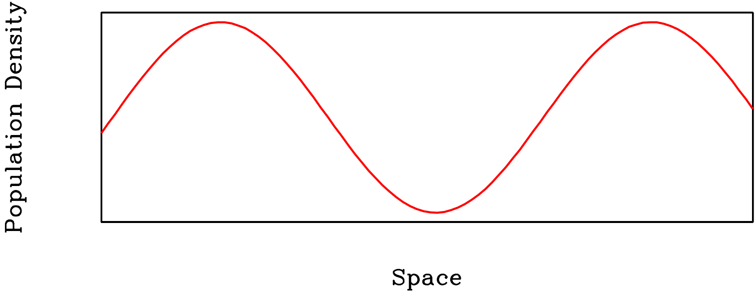
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



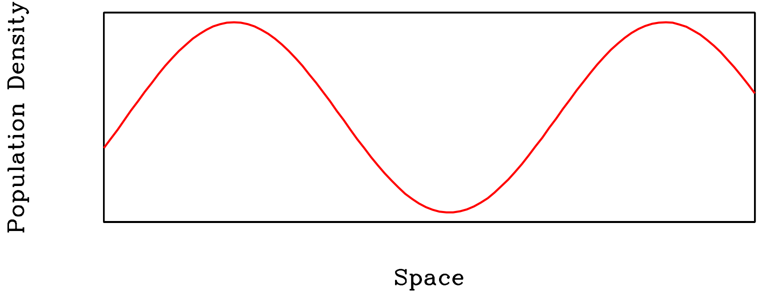
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



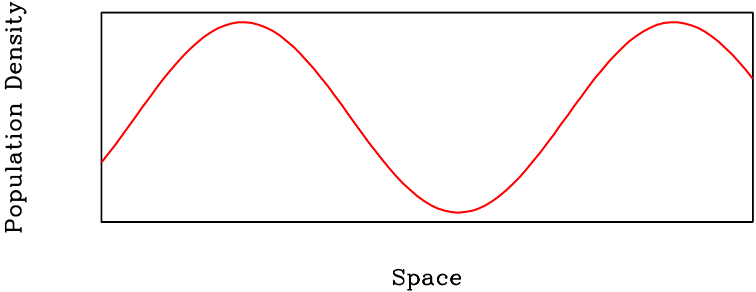
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



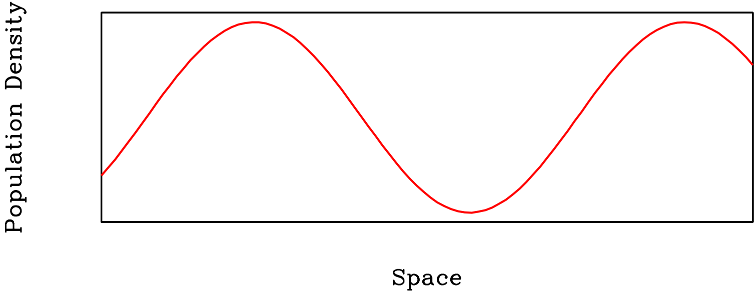
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



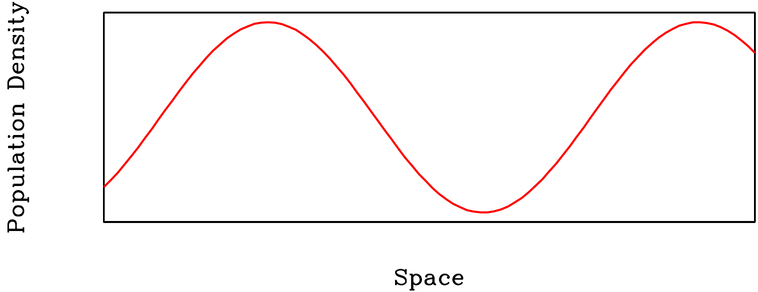
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



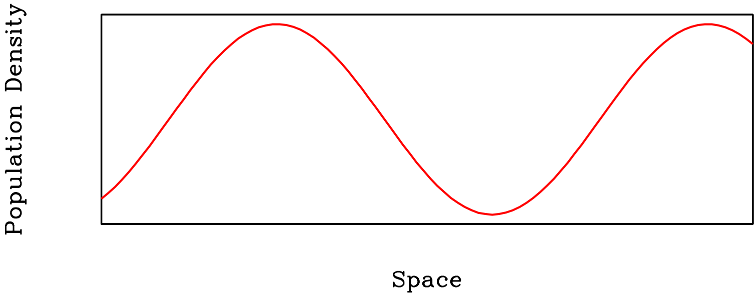
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



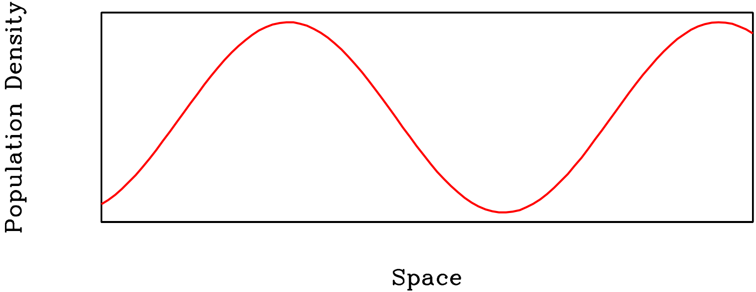
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



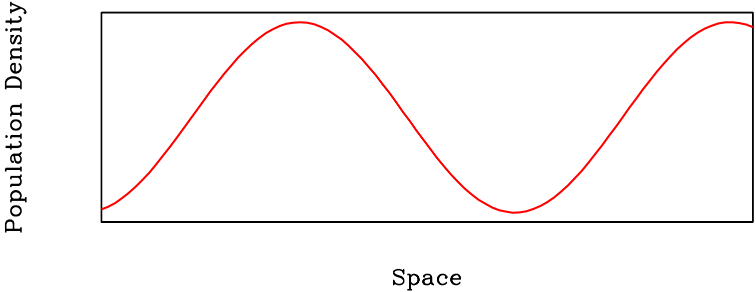
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



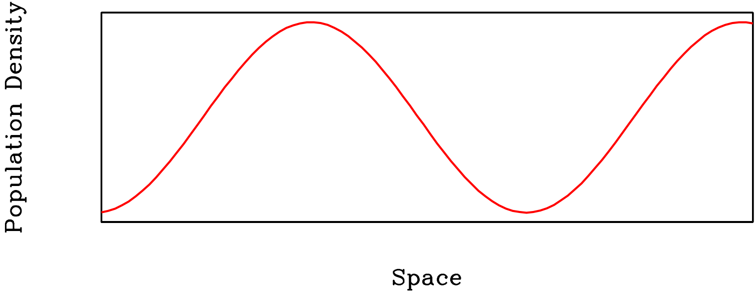
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



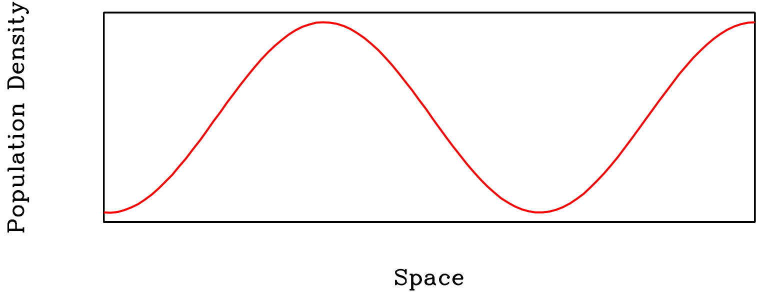
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



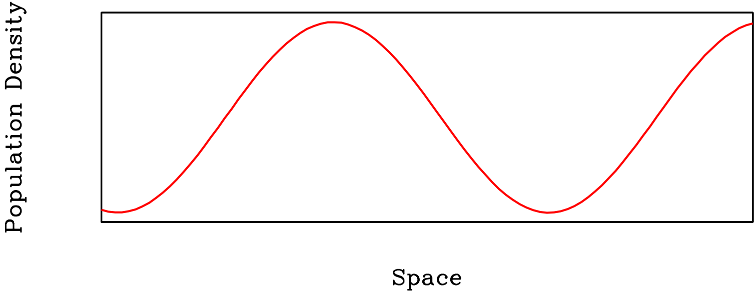
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



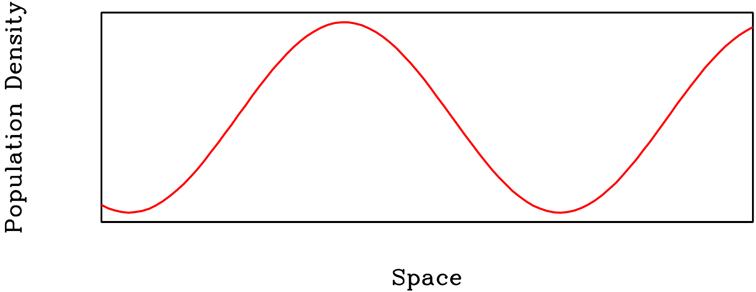
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



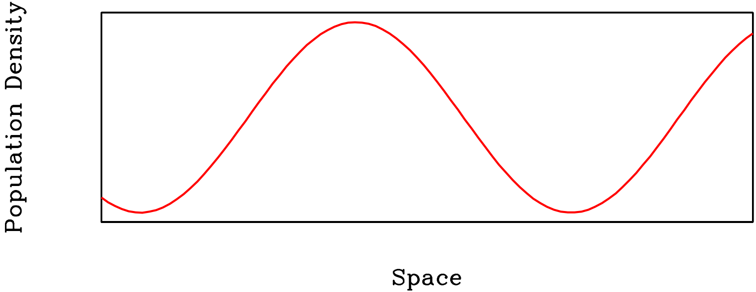
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



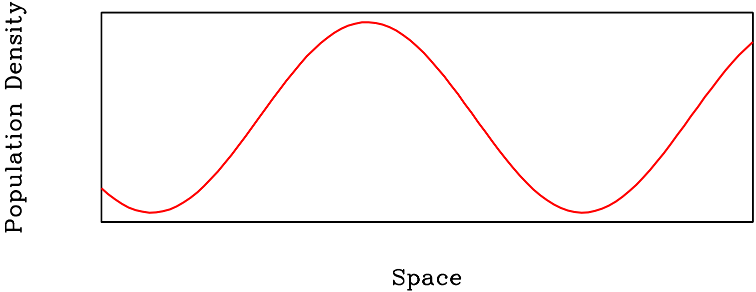
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



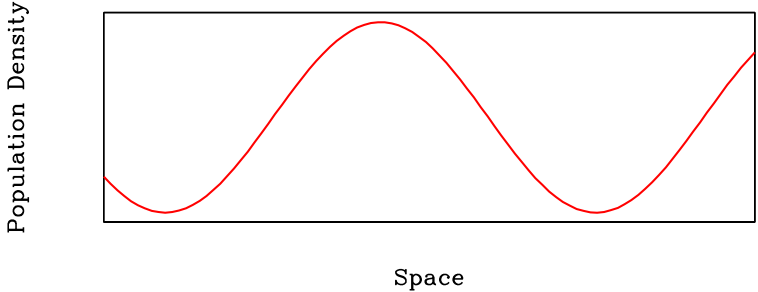
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



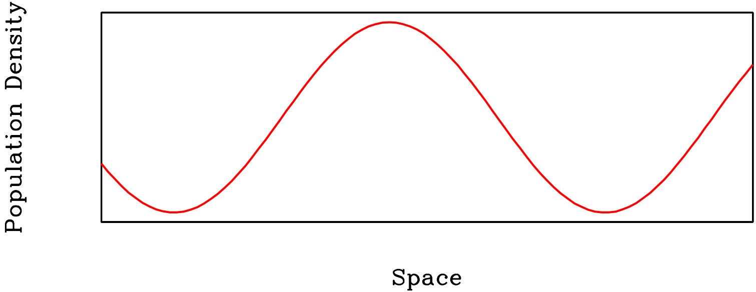
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



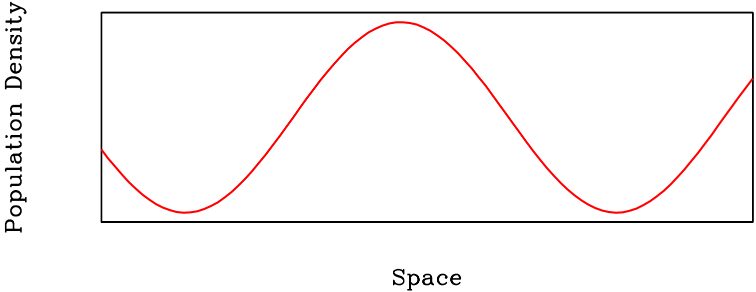
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



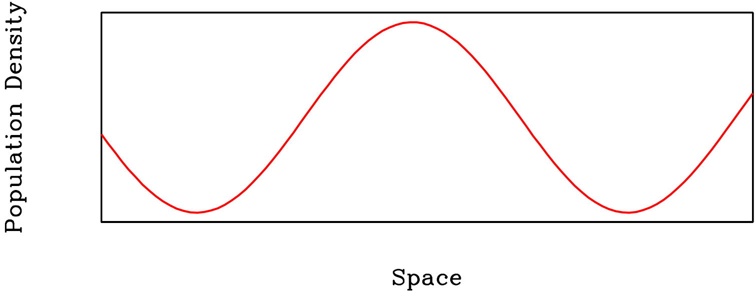
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



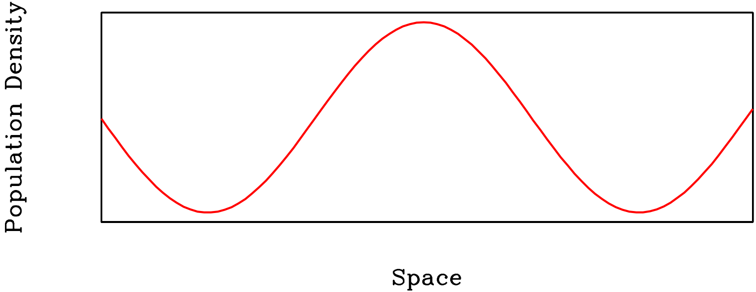
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



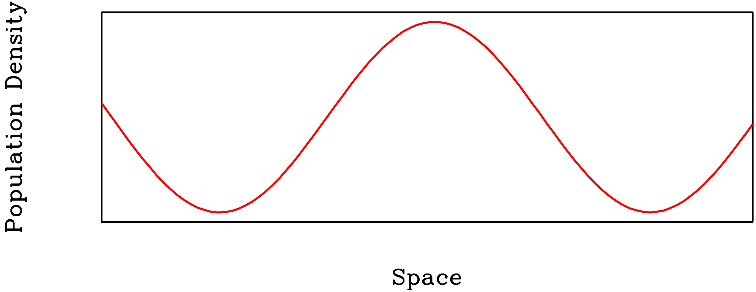
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



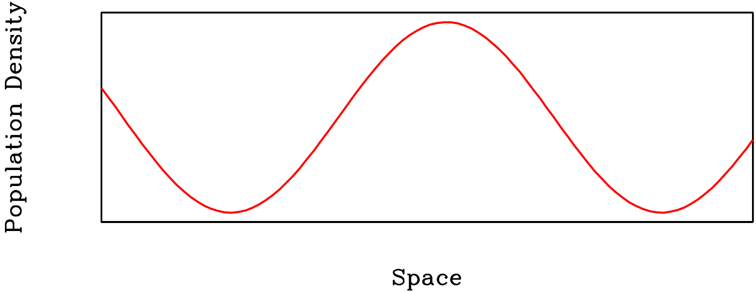
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



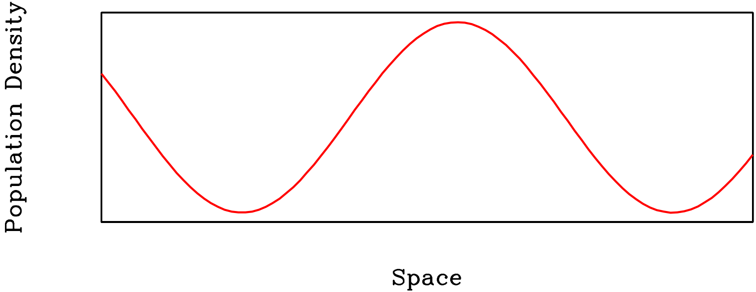
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



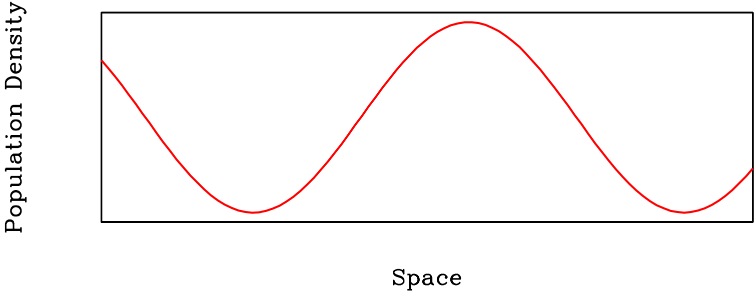
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



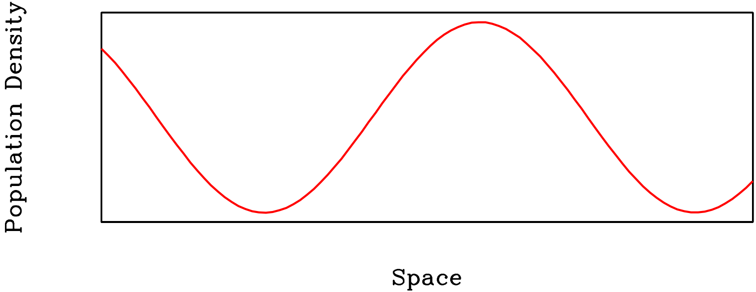
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



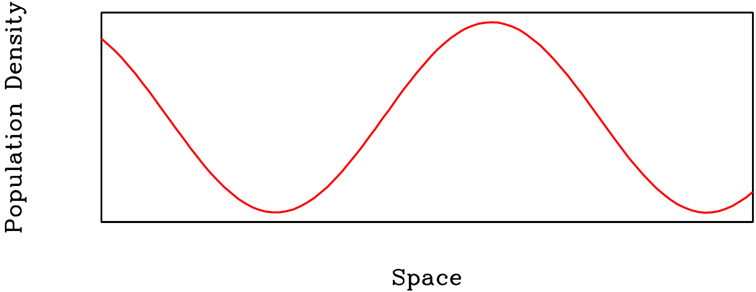
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



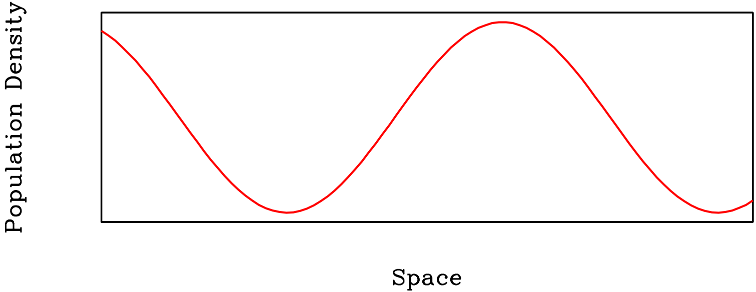
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



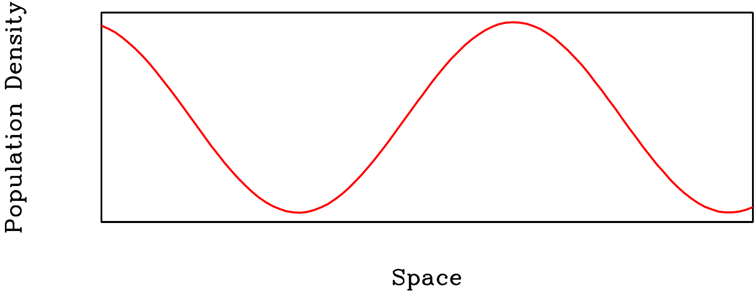
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



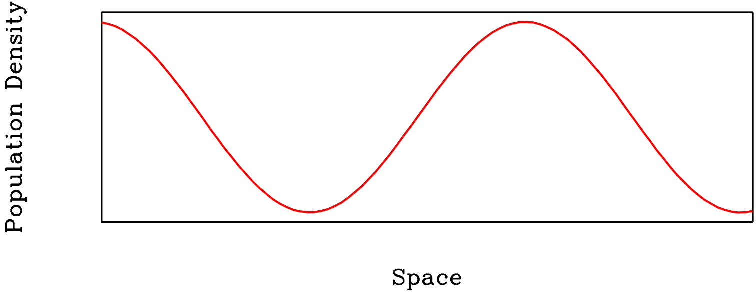
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



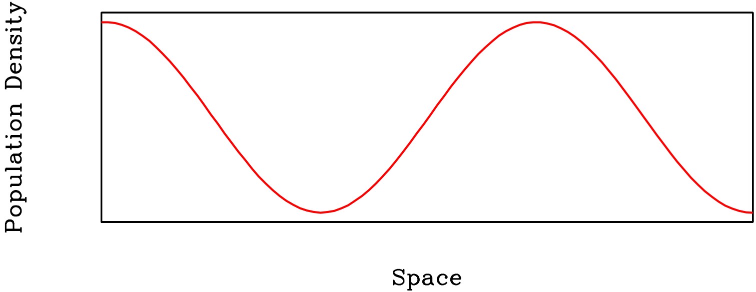
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



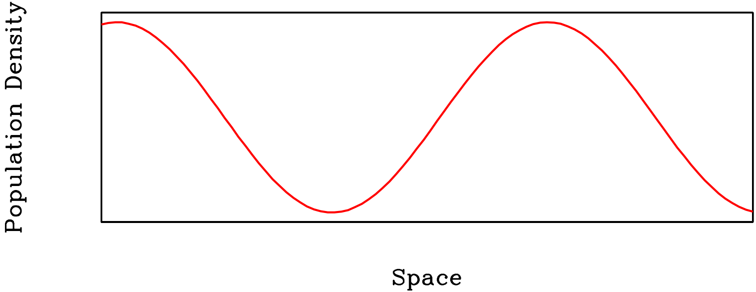
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



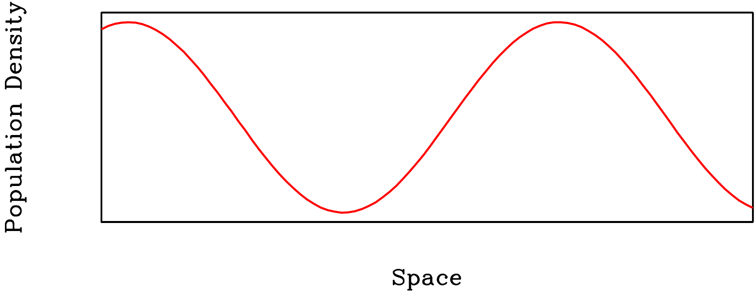
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



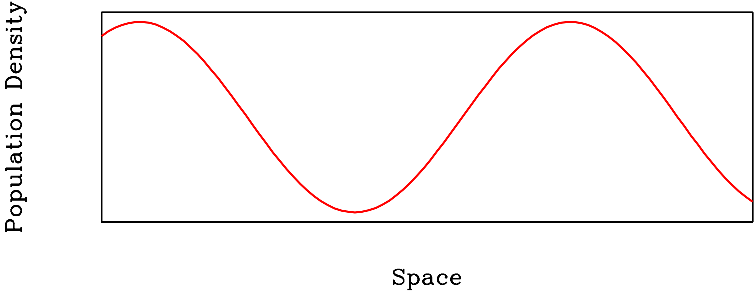
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



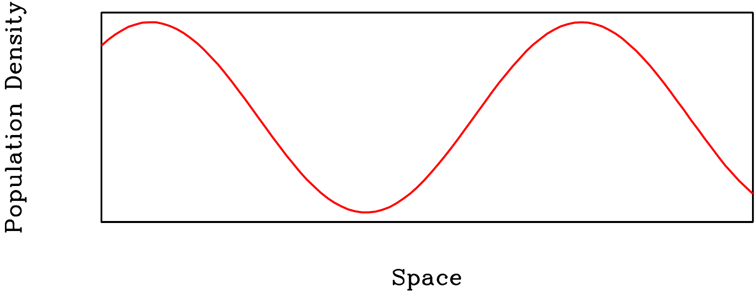
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



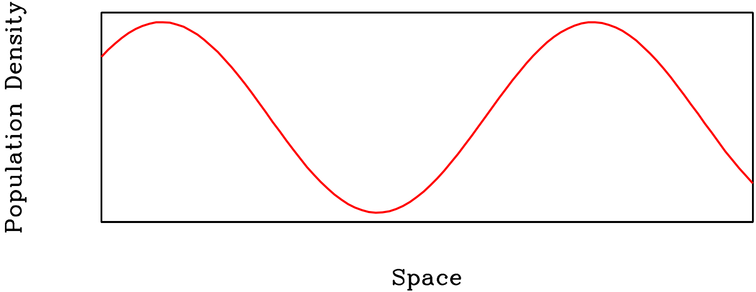
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



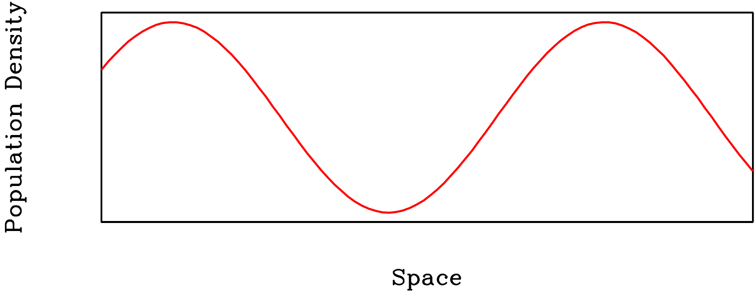
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



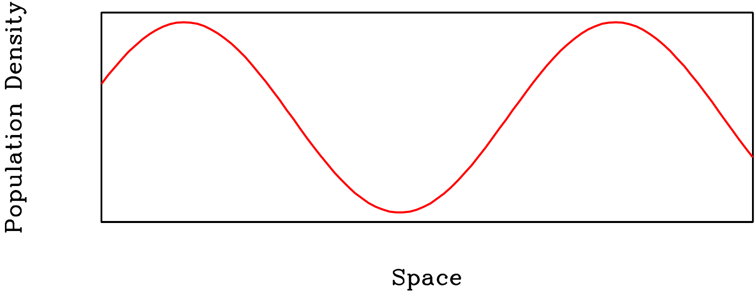
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



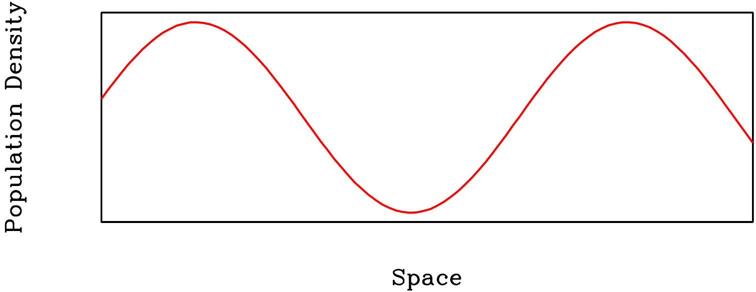
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



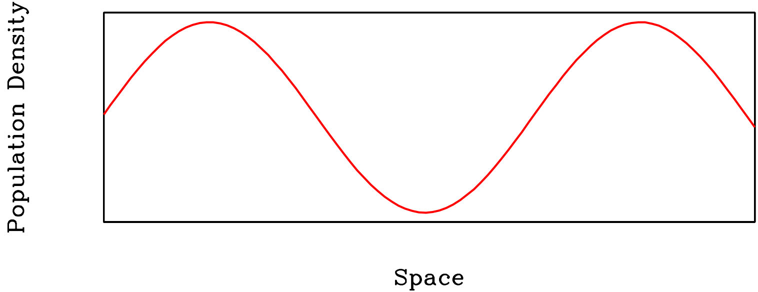
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



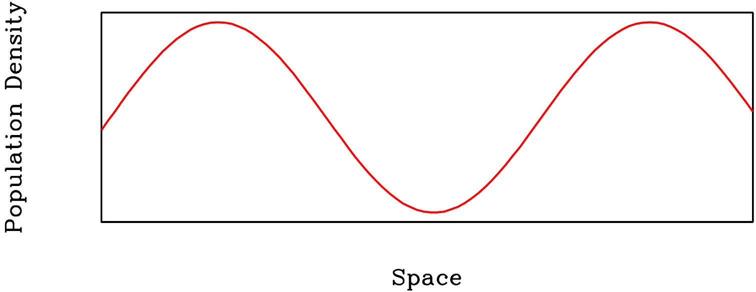
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



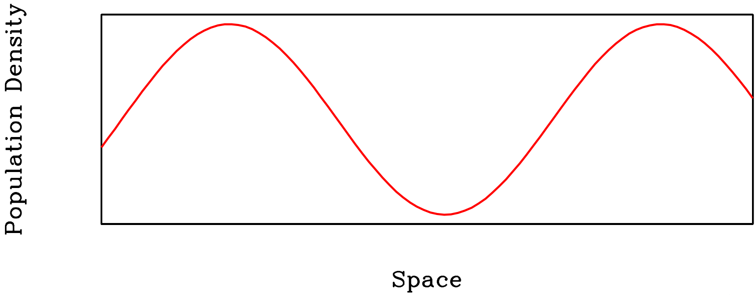
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



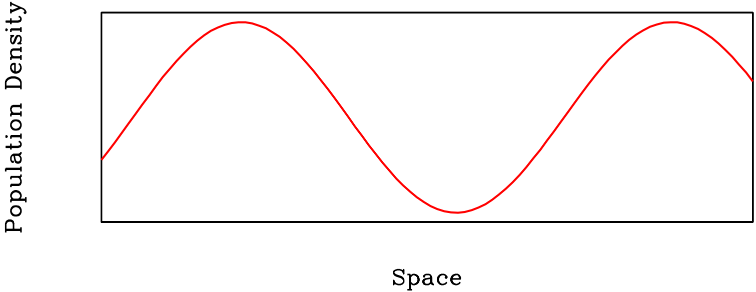
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



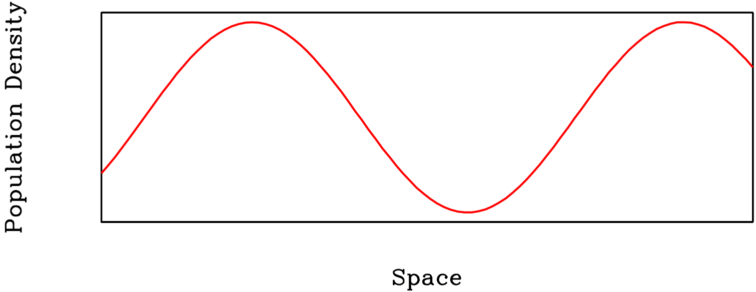
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



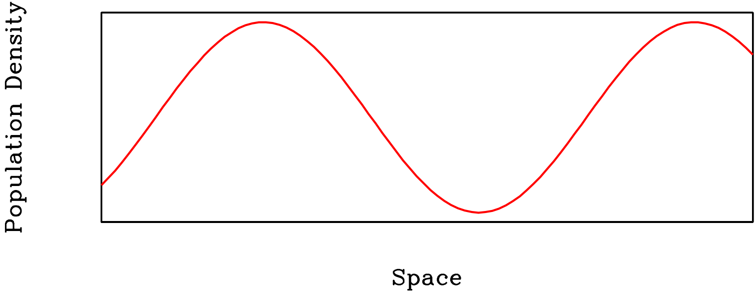
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



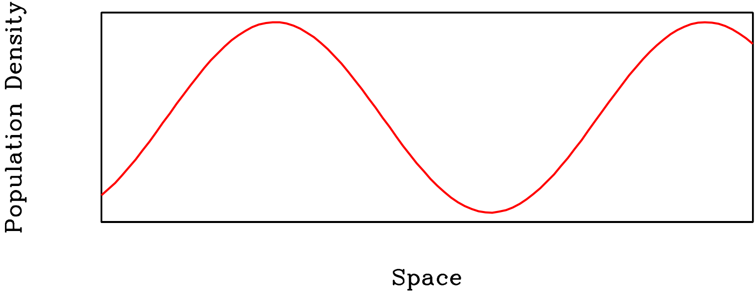
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



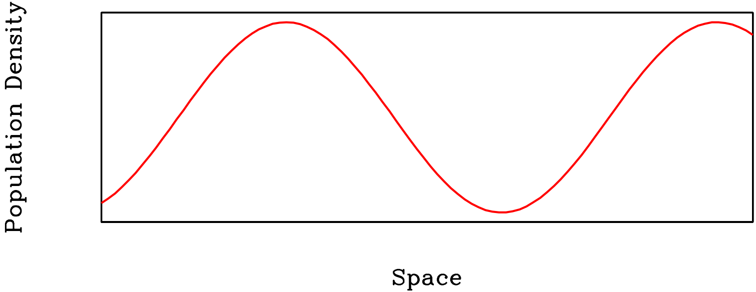
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



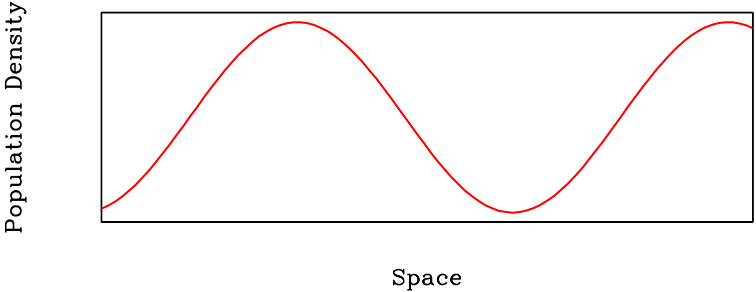
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



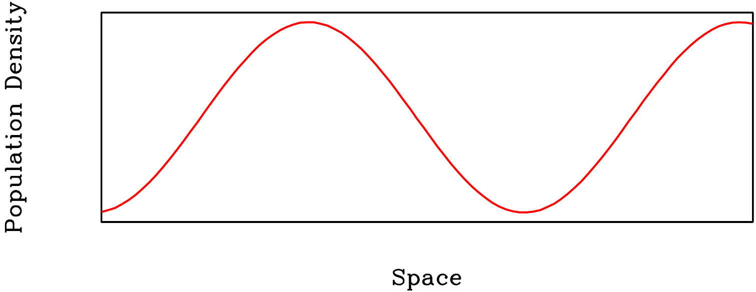
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



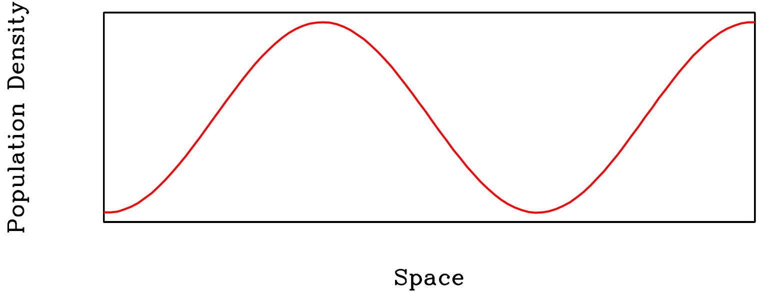
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



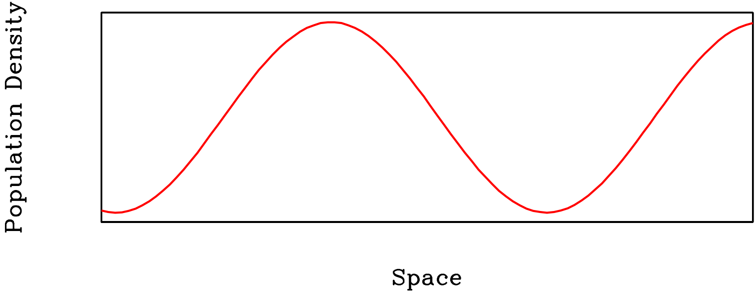
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



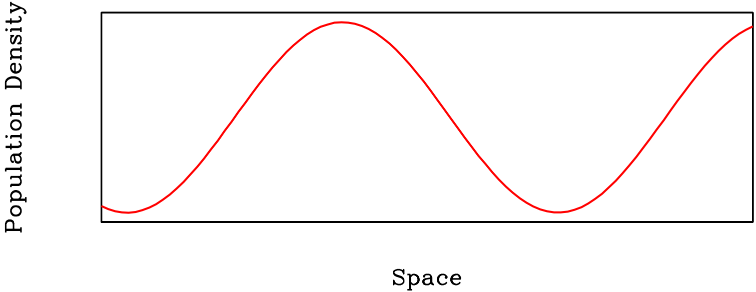
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



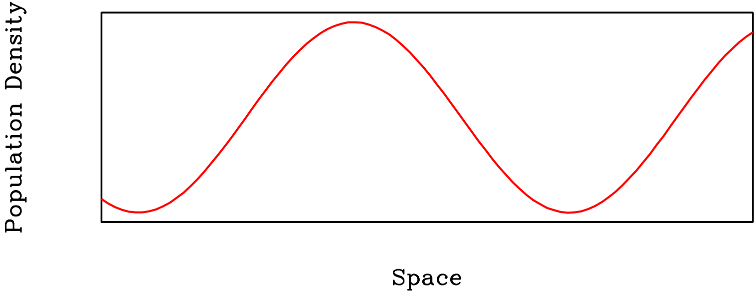
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



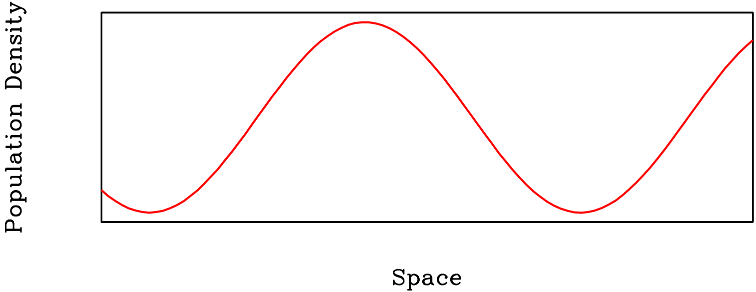
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



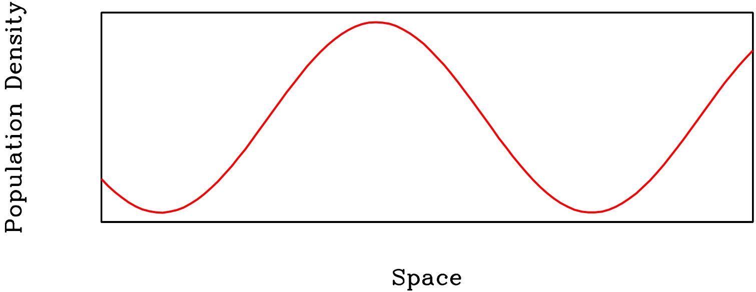
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



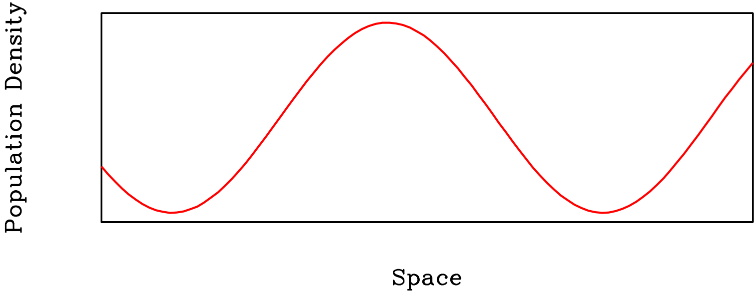
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



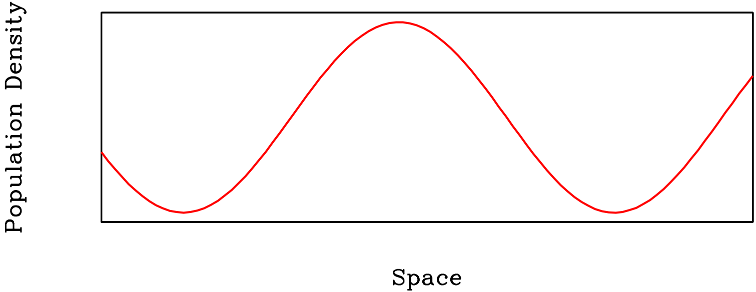
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



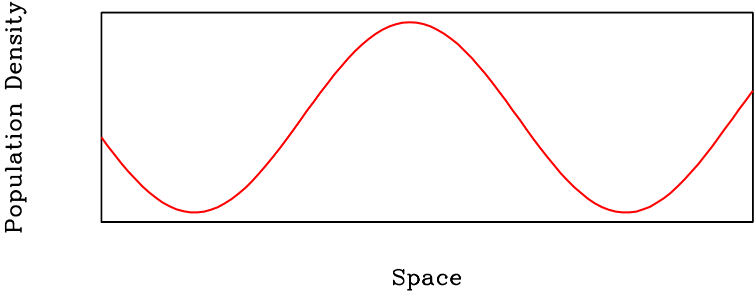
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



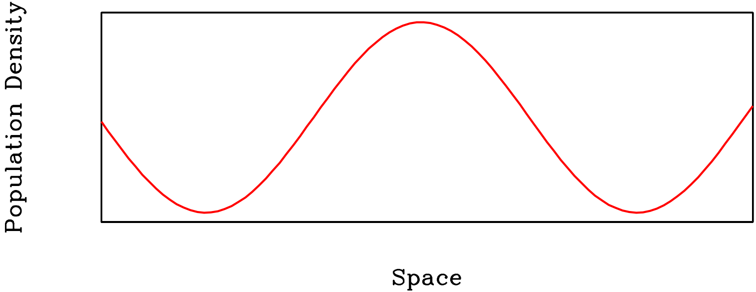
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



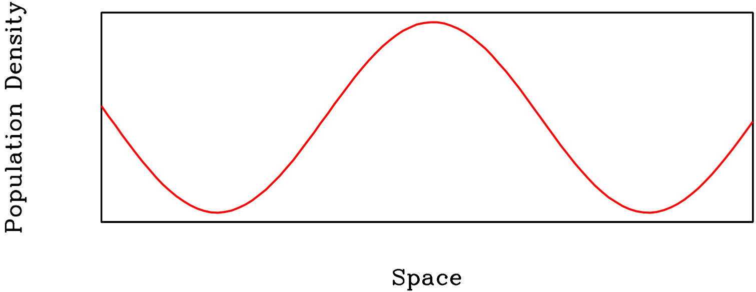
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



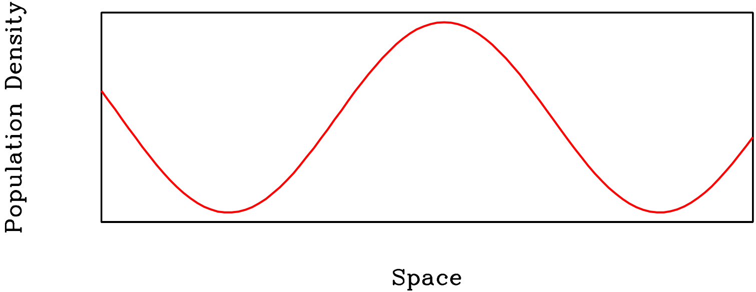
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



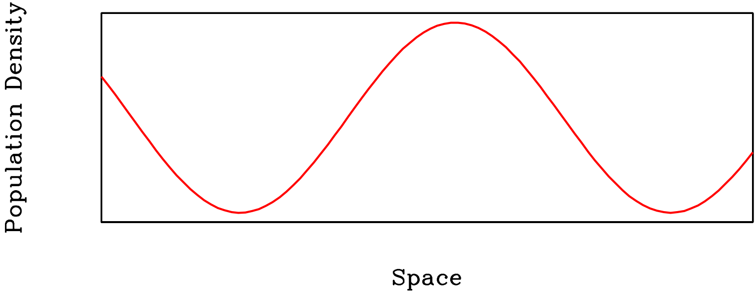
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



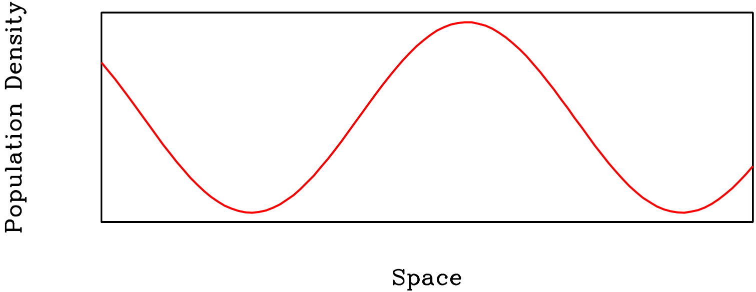
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



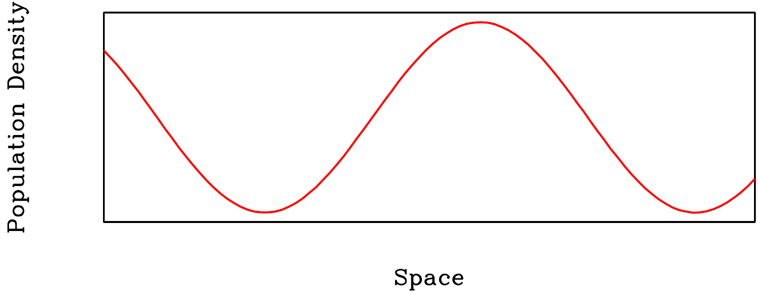
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



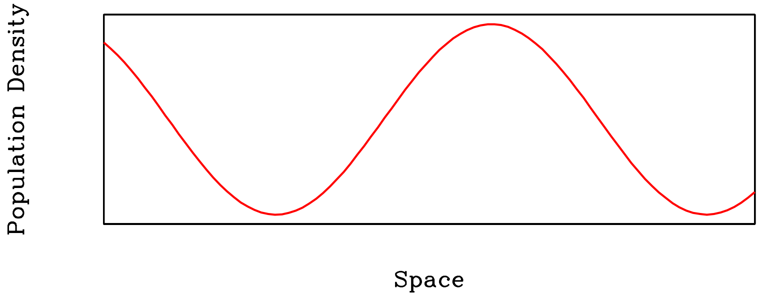
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



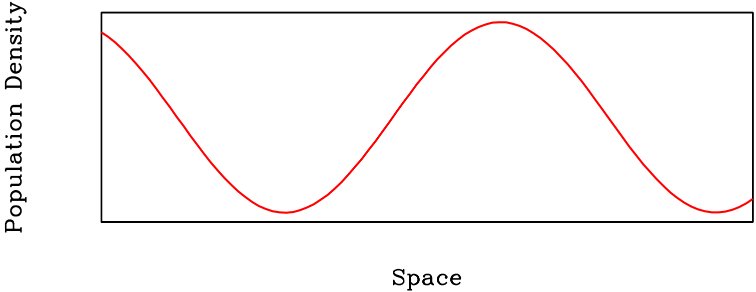
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



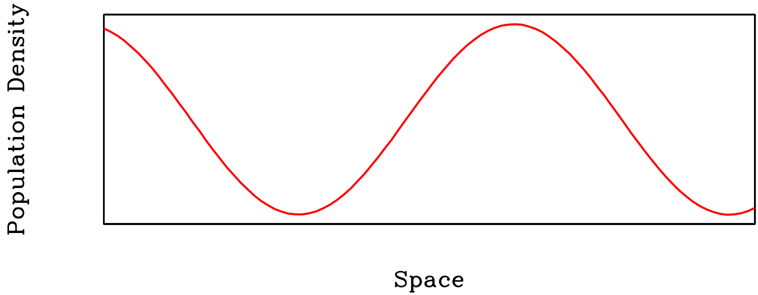
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



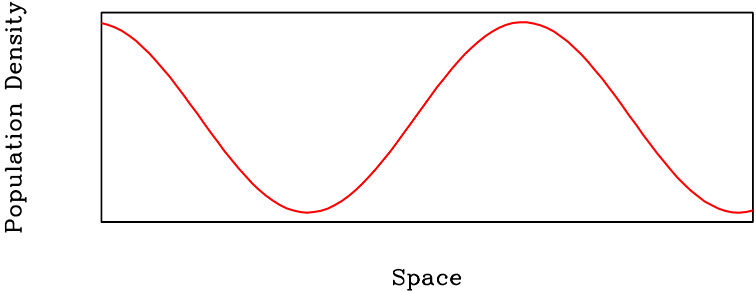
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



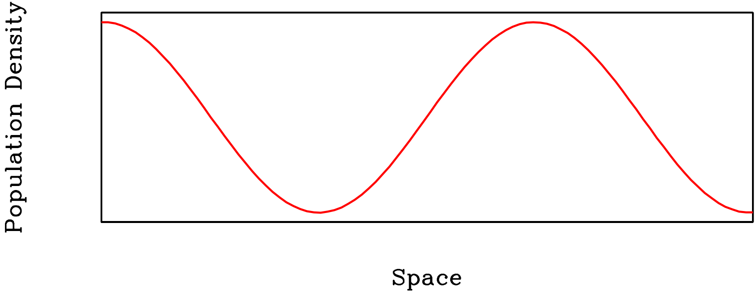
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



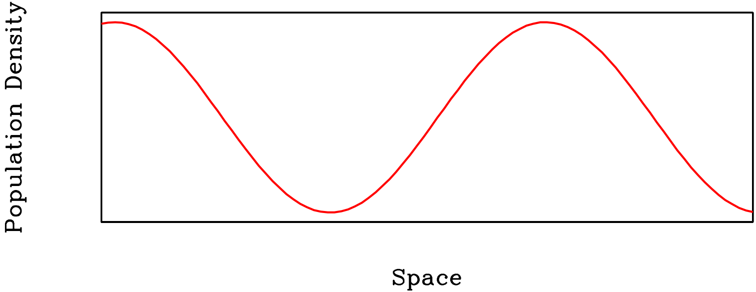
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



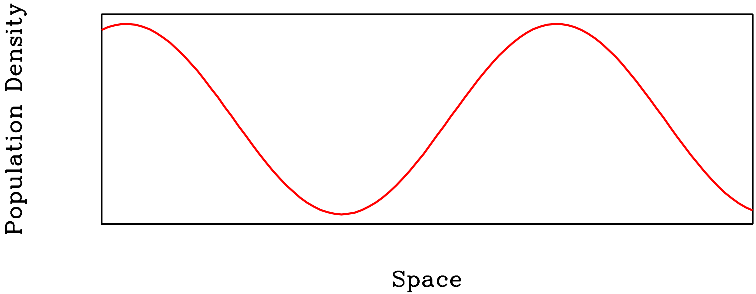
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



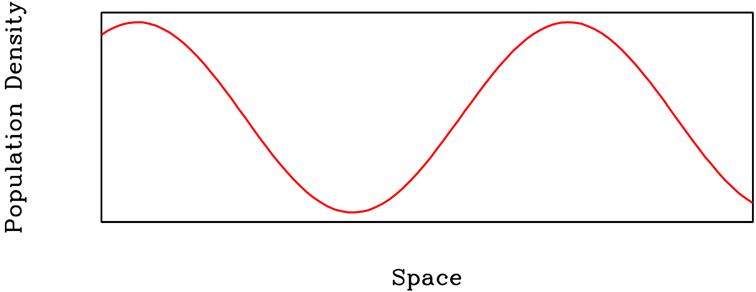
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



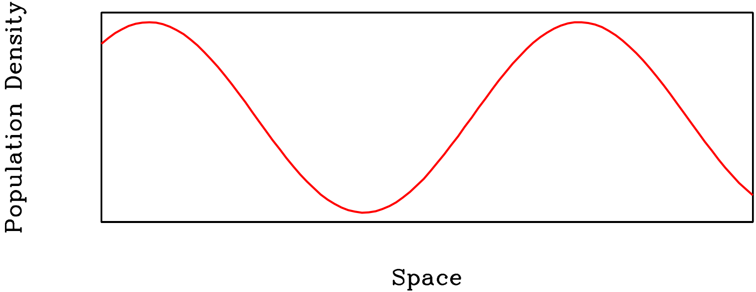
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



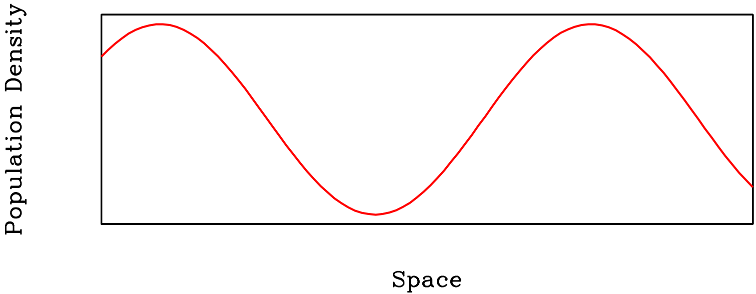
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



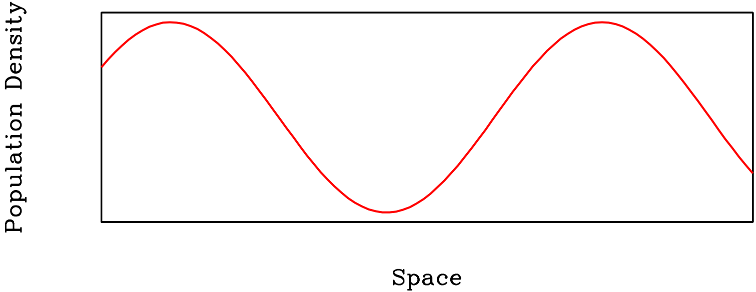
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



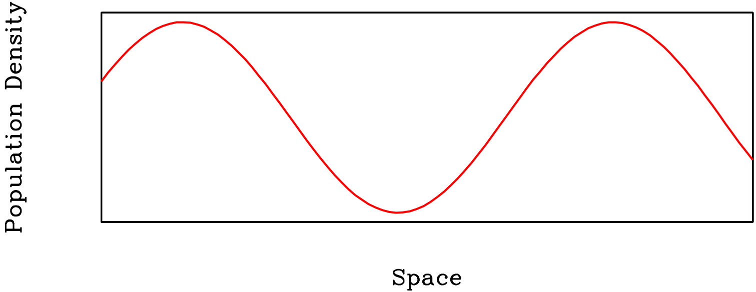
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



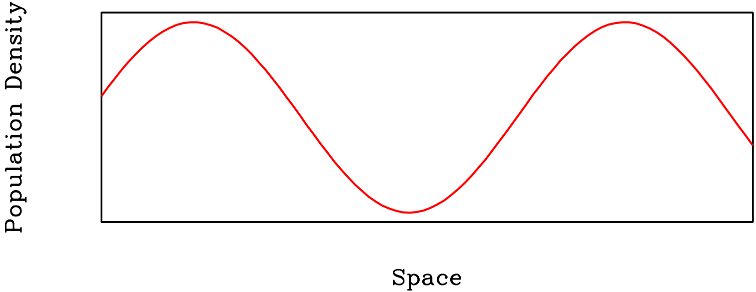
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



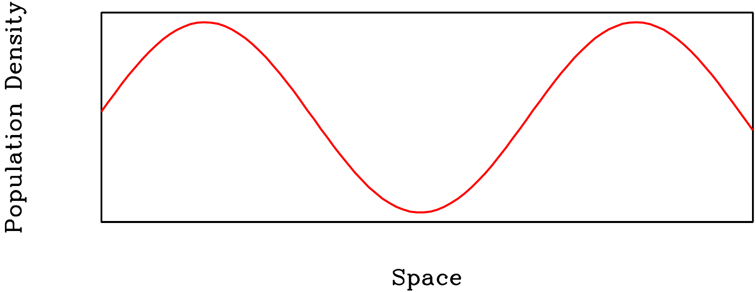
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



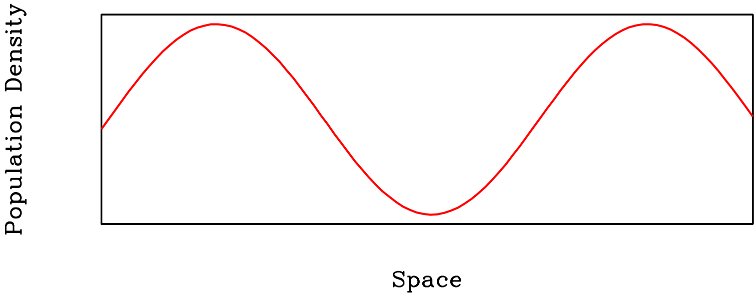
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



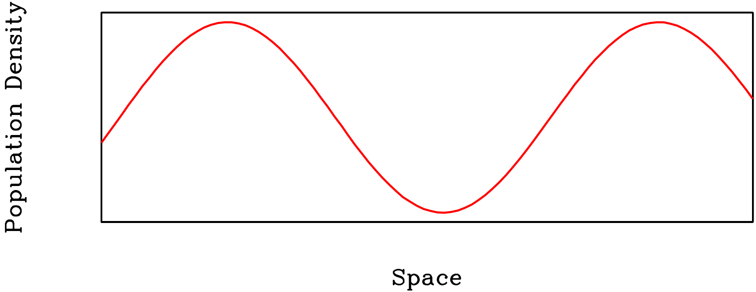
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



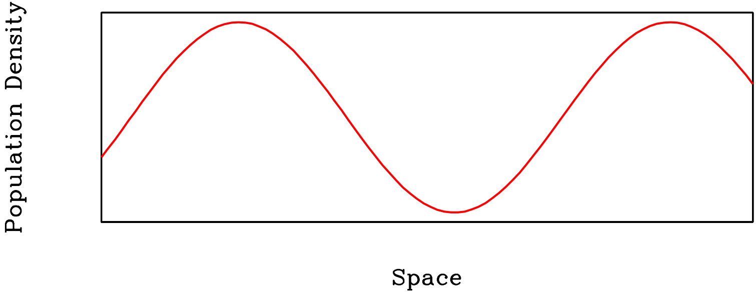
Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



Wavetrains in Red Grouse & Field Voles

Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave)



Wavetrain Generation Question

Question

Does the Dirichlet condition at the habitat boundary play a role in generating the wavetrains?

Outline

The Complex Ginzburg-Landau Equation

I consider a generic oscillator model, the complex Ginzburg-Landau equation:

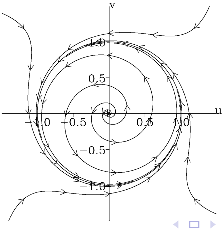
$$A_t = (1+\mathrm{i}b)A_{xx} + A - (1+\mathrm{i}c)|A|^2 A.$$

I will look exclusively at $b = 0$. Then writing

$$A(x,t) = e^{-\mathrm{i}at}[u(x,t) + \mathrm{i}v(x,t)]$$

gives a reaction-diffusion system of "λ–ω" type:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1-r^2)u - (a+cr^2)v \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (a+cr^2)u + (1-r^2)v \\ \text{where } r &= \sqrt{u^2+v^2} \end{aligned}$$



Amplitude and Phase Equations

To study these equations, it is helpful to use the variables $r(x,t) = \sqrt{u^2+v^2}$ and $\theta(x,t) = \tan^{-1}(v/u)$, giving

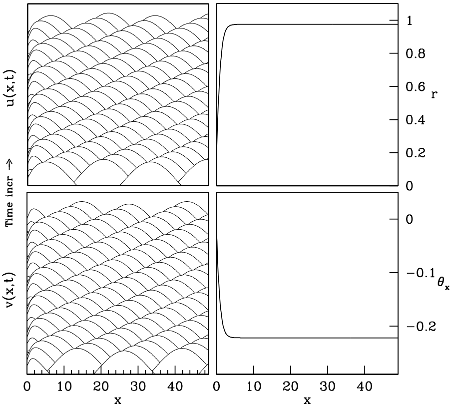
$$\begin{aligned} r_t &= r_{xx} - r\theta_x^2 + r(1-r^2) \\ \theta_t &= \theta_{xx} + \frac{2r_x\theta_x}{r} + a - cr^2 \end{aligned}$$

There is a family of wavetrain solutions ($0 < r^* < 1$):

$$\left\{ \begin{array}{l} r = r^* \\ \theta = \left[(a + c{r^*}^2)t \pm \sqrt{(1-{r^*}^2)}x \right] \end{array} \right\}$$

$$\leftrightarrow \left\{ \begin{array}{l} u = r^* \cos \left[(a + c{r^*}^2)t \pm \sqrt{(1-{r^*}^2)}x \right] \\ v = r^* \sin \left[(a + c{r^*}^2)t \pm \sqrt{(1-{r^*}^2)}x \right] \end{array} \right\}$$

Typical Model Solutions



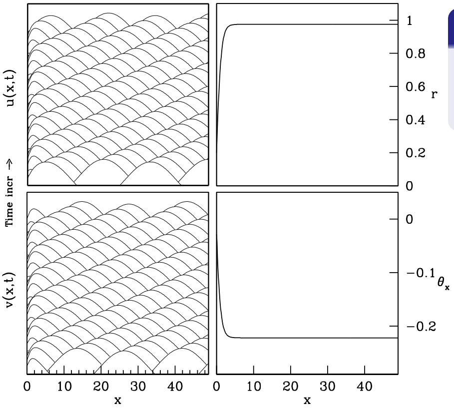
Equations:

$$u_t = u_{xx} + (1 - r^2)u - (a + cr)v$$
$$v_t = v_{xx} + (a + cr^2)u + (1 - r^2)v$$

where $r = \sqrt{u^2 + v^2}$

This is a typical numerical solution when $u = v = 0$ is imposed at $x = 0$

Typical Model Solutions



Conclusion

Dirichlet boundary conditions generate a wavetrain

$$
\begin{aligned}
R(x) &= R^* \tanh\left(x/\sqrt{2}\right) \\
\Psi(x) &= \Psi^* \tanh\left(x/\sqrt{2}\right) \\
R^* &= \left\{\tfrac{1}{2}\left[1+\sqrt{1+\tfrac{8}{9}c^2}\right]\right\}^{-1/2} \\
\Psi^* &= -\mathrm{sign}(c)\left(1-R^{*\,2}\right)^{1/2}
\end{aligned}
$$

Wavetrain Generation Question

Question

Does the Dirichlet condition at the habitat boundary play a role in generating the wavetrains?

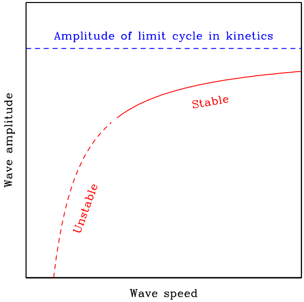
Answer

Yes
(at least potentially)

Outline

Stability in the Wavetrain Family

In any oscillatory reaction-diffusion system, some members of the wavetrain family are stable as solutions of the partial differential equations, while others are unstable.



For our λ–ω system, the stability condition is

$$r^* > \left(\frac{2 + 2c^2}{3 + 2c^2} \right)^{1/2}$$

Stability of the Selected Wave

The stability of the selected wave depends on c:

$$R^* = \left\{ \frac{1}{2} \left[1 + \sqrt{1 + \frac{8}{9} c^2} \right] \right\}^{-1/2}$$

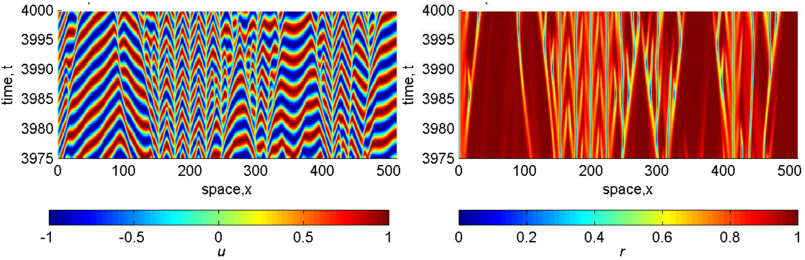
This is stable

$$\Leftrightarrow \quad R^* > \left(\frac{2 + 2c^2}{3 + 2c^2} \right)^{1/2}$$

$$\Leftrightarrow \quad |c| < 1.110468\ldots$$

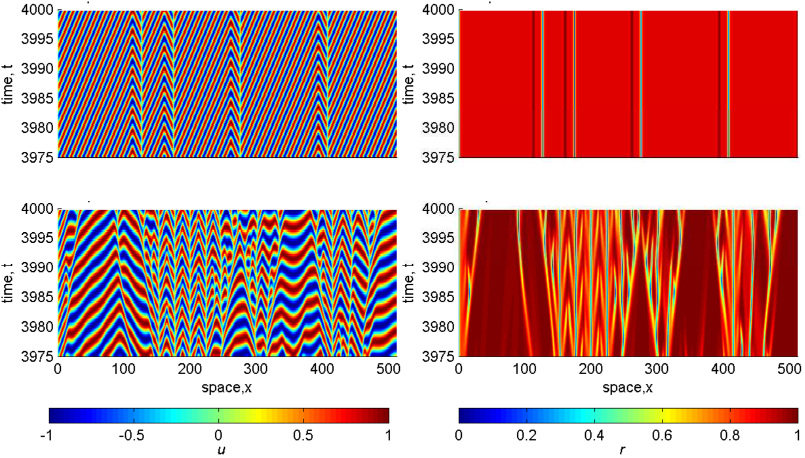
Convective and Absolute Stability

- There are two types of solution for $|c| > 1.110468\ldots$



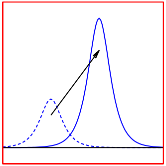
Convective and Absolute Stability

- There are two types of solution for $|c| > 1.110468\ldots$



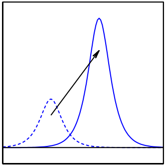
Convective and Absolute Stability

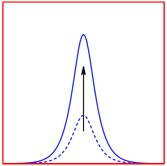
- There are two types of solution for $|c| > 1.110468\ldots$
- The key concept for distinguishing these is "absolute stability".
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".



Convective and Absolute Stability

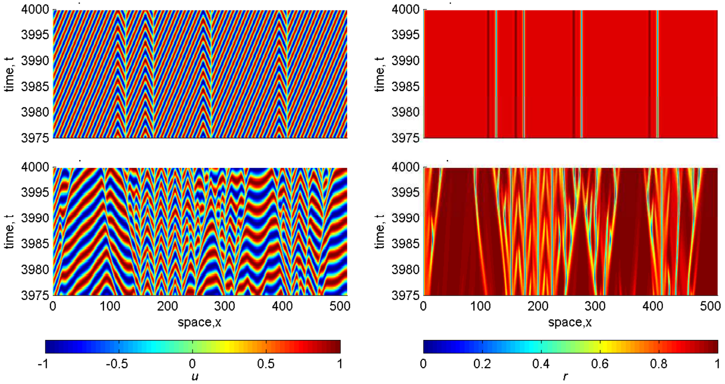
- There are two types of solution for $|c| > 1.110468\ldots$
- The key concept for distinguishing these is "absolute stability".
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".
- Alternatively, a solution can be unstable with perturbations growing without moving. This is "absolute instability".





Generation of Absolutely Stable and Unstable Wavetrains by Dirichlet Boundary Conditions

Numerical simulations show distinct behaviours in the absolutely stable and unstable parameter regimes



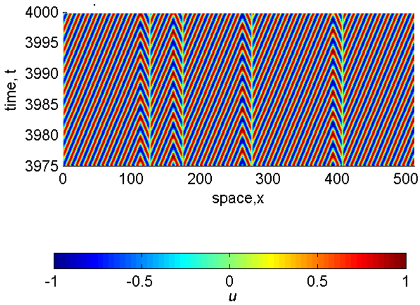
Convectively unstable, absolutely stable

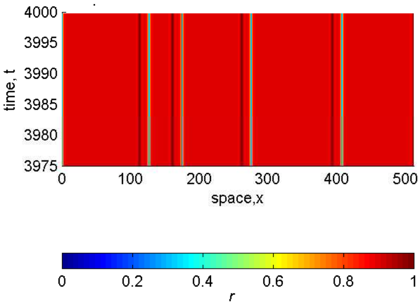
Absolutely unstable

Outline

Sources, Sinks, and Convective Instability

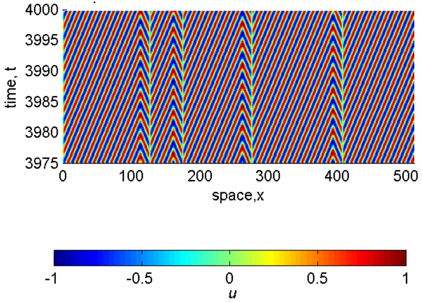
The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.

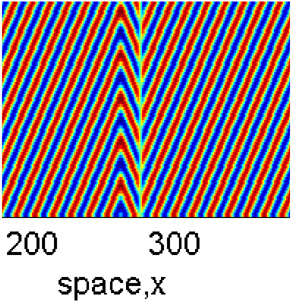




Sources, Sinks, and Convective Instability

The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.

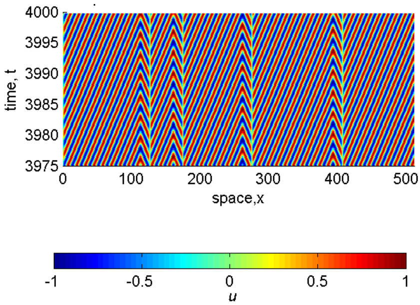


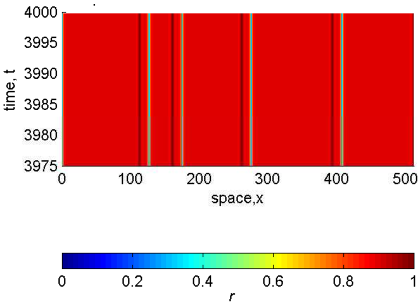


Note: sources and sinks are defined in terms of group velocity.

Sources, Sinks, and Convective Instability

The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.



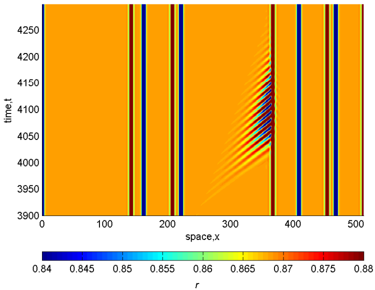


Question: How can an unstable wavetrain persist between the sources and sinks?

Sources, Sinks, and Convective Instability

Question: How can an unstable wavetrain persist between the sources and sinks?

Answer: Any growing perturbations moves, and is absorbed when it reaches a sink.

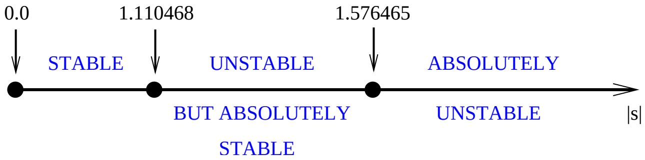


Calculation of Absolute Stability of Wavetrains

- Absolute stability is much harder to calculate than stability.
- For wavetrain solutions of the CGLE, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)

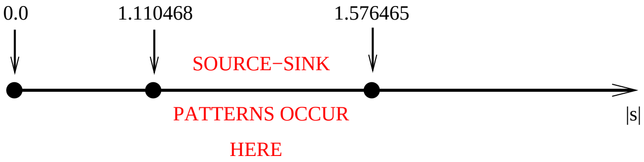
Calculation of Absolute Stability of Wavetrains

- Absolute stability is much harder to calculate than stability.
- For wavetrain solutions of the CGLE, we have calculated absolute stability by computing the "absolute spectrum" via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)
- Our calculation shows that the stability of the selected wavetrain is:



Calculation of Absolute Stability of Wavetrains

- Absolute stability is much harder to calculate than stability.
- For wavetrain solutions of the CGLE, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)
- Our calculation shows that the stability of the selected wavetrain is:



Experimental Observation of Sources and Sinks

Experimental systems in which sources and sinks have been observed include:

- chemical reactions
- electrochemical systems
- heated wire convection
- binary fluid convection
- convection waves generated by heating at a boundary
- the “printer’s instability”, in which the thin gap between two rotating acentric cylinders is filled with oil.

Previous Mathematical Work on Sources and Sinks

- Sources are "Nozaki–Bekki" holes (Nozaki & Bekki, Phys. Lett. A 110: 133-135, 1985), on which the literature is extensive ($>$ 100 citations).
- Sinks are also well studied, though only numerically.
- But patterns of sources and sinks have received almost no attention.
- One open question is: are there constraints on the distances separating sources and sinks?

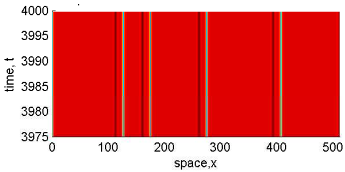
Numerical Study of Source-Sink Separations

Step 1: generate a source-sink pattern via a Dirichlet boundary condition

Step 2: extract a sink-source-sink triple

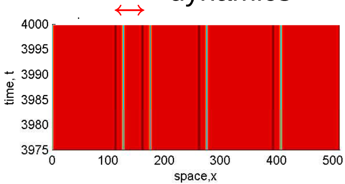
Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions

Step 4: translate the source and track the subsequent dynamics



Numerical Study of Source-Sink Separations

- Step 1: generate a source-sink pattern via a Dirichlet boundary condition
- Step 2: extract a sink-source-sink triple
- Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions
- Step 4: translate the source and track the subsequent dynamics



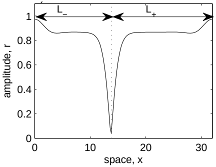
Numerical Study of Source-Sink Separations

Step 1: generate a source-sink pattern via a Dirichlet boundary condition

Step 2: extract a sink-source-sink triple

Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions

Step 4: translate the source and track the subsequent dynamics



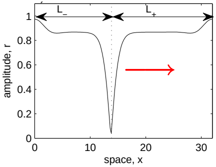
Numerical Study of Source-Sink Separations

Step 1: generate a source-sink pattern via a Dirichlet boundary condition

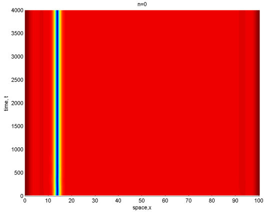
Step 2: extract a sink-source-sink triple

Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions

Step 4: translate the source and track the subsequent dynamics

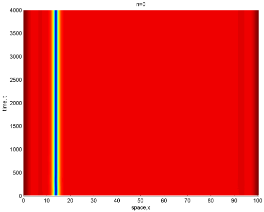


Numerical Study of Source-Sink Separations

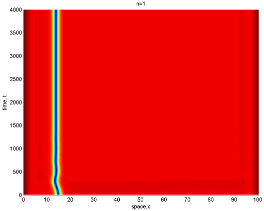


Original solution

Numerical Study of Source-Sink Separations

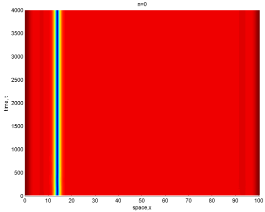


Original solution

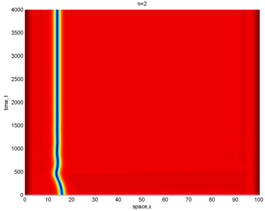


Solution with translated source

Numerical Study of Source-Sink Separations

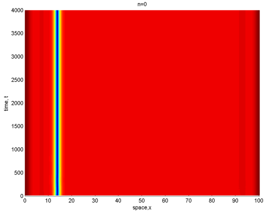


Original solution

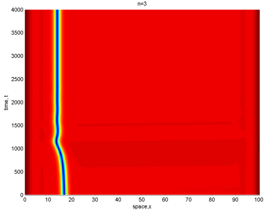


Solution with translated source

Numerical Study of Source-Sink Separations

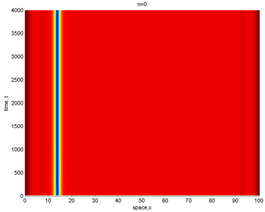


Original solution

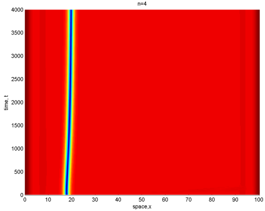


Solution with translated source

Numerical Study of Source-Sink Separations



Original solution



Solution with translated source

Numerical Study of Source-Sink Separations

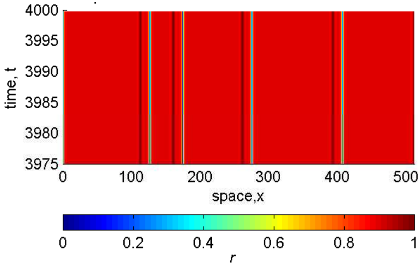
Conclusion: source-sink separations appear to be constrained to a discrete set of possible values.

Next Step: analytical investigation of the separations.

A Complication: sources and sinks often move, very slowly.

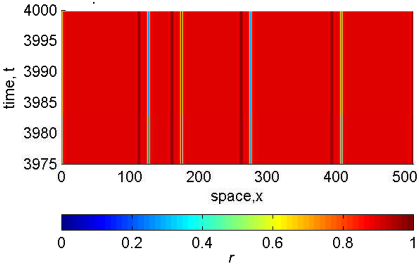
Movement of Sources and Sinks

These sources and sinks appear to be stationary........

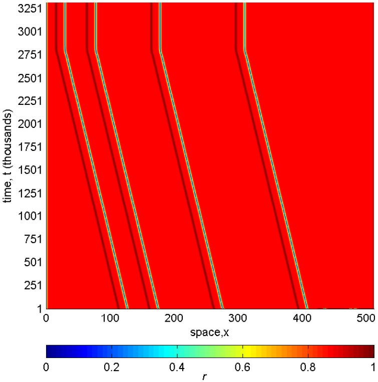


Movement of Sources and Sinks

These sources and sinks appear to be stationary........



..........but very long simulations show that they move.



Outline

Travelling Waves of Amplitude

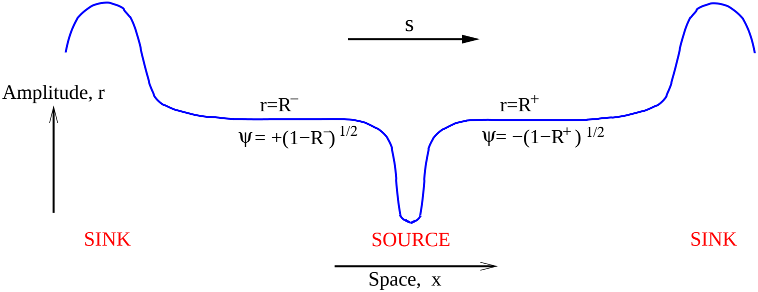
The source-sink patterns are of travelling wave form in amplitude.

Substitute $r(x,t) = \widehat{r}(z)$, $\theta_x(x,t) = \widehat{\psi}(z)$, $z = x - st$

$$
\begin{aligned}
\Longrightarrow \quad d^2\widehat{r}/dz^2 + s\, d\widehat{r}/dz + \widehat{r}\left(1 - \widehat{r}^{\,2} - \widehat{\psi}^{\,2}\right) &= 0 \\
d\widehat{\psi}/dz + s\,\widehat{\psi} + K - c\widehat{r}^{\,2} + 2\widehat{\psi}\,(d\widehat{r}/dz)/\widehat{r} &= 0
\end{aligned}
$$

(K is a constant of integration).

Solution Structure



Between the source and the neighbouring sinks,

$$-s(1-R^{-2})^{1/2}+cR^{-2}=K=+s(1-R^{+2})^{1/2}+cR^{+2}$$

$\Longrightarrow$ velocity s has the same sign as $R^{-}-R^{+}$.

Based on source-sink patterns seen in numerical simulations, we consider large separations and small velocities.

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\widehat{r}/dz^2 + \widehat{r}\left(1 - \widehat{r}^{\,2} - \widehat{\psi}^{\,2}\right) &= 0 \\ d\widehat{\psi}/dz + K - c\widehat{r}^{\,2} + 2\widehat{\psi}\,(d\widehat{r}/dz)/\widehat{r} &= 0. \end{aligned}$$

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\widehat{r}/dz^2 + \widehat{r}\left(1 - \widehat{r}^{\,2} - \widehat{\psi}^{\,2}\right) &= 0 \\ d\widehat{\psi}/dz + K - c\widehat{r}^{\,2} + 2\widehat{\psi}\,(d\widehat{r}/dz)/\widehat{r} &= 0. \end{aligned}$$

Linearise about the wavetrain

$\Rightarrow$ isolated sources decay to the wavetrain at rate $\sqrt{2}$
& isolated sinks decay to the wavetrain at rate $1/\sqrt{2} \pm i\delta/4$

$(\delta = \sqrt{11 - 12R^{*\,2}} \in \mathbb{R})$

$\Rightarrow$ in patterns,the effect of sinks on sources dominates
the effect of sources on sinks,for large separations

$\Rightarrow$ when the velocity s is small, we can just consider the correction to an isolated source

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\hat{r}/dz^2 + \hat{r}\left(1 - \hat{r}^2 - \hat{\psi}^2\right) &= 0 \\ d\hat{\psi}/dz + K - c\hat{r}^2 + 2\hat{\psi}(d\hat{r}/dz)/\hat{r} &= 0. \end{aligned}$$

Linearise about the wavetrain

$\Rightarrow$ isolated sources decay to the wavetrain at rate $\sqrt{2}$
& isolated sinks decay to the wavetrain at rate $1/\sqrt{2} \pm i\delta/4$

$(\delta = \sqrt{11 - 12R^{*2}} \in \mathbb{R})$

$\Rightarrow$ in patterns,the effect of sinks on sources dominates
the effect of sources on sinks,for large separations

$\Rightarrow$ when the velocity s is small, we can just consider the correction to an isolated source

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\hat{r}/dz^2 + \hat{r}\left(1 - \hat{r}^2 - \hat{\psi}^2\right) &= 0 \\ d\hat{\psi}/dz + K - c\hat{r}^2 + 2\hat{\psi}\,(d\hat{r}/dz)/\hat{r} &= 0. \end{aligned}$$

Linearise about the wavetrain

$\Rightarrow$ isolated sources decay to the wavetrain at rate $\sqrt{2}$
& isolated sinks decay to the wavetrain at rate $1/\sqrt{2} \pm i\delta/4$

$(\delta = \sqrt{11 - 12R^{*2}} \in \mathbb{R})$

$\Rightarrow$ in patterns,the effect of sinks on sources dominates
the effect of sources on sinks,for large separations

$\Rightarrow$ when the velocity s is small, we can just consider the correction to an isolated source

Eigenvalue Structure of Isolated Sources and Sinks

Isolated sources and sinks satisfy

$$\begin{aligned} d^2\hat{r}/dz^2 + \hat{r}\left(1 - \hat{r}^2 - \hat{\psi}^2\right) &= 0 \\ d\hat{\psi}/dz + K - c\hat{r}^2 + 2\hat{\psi}\,(d\hat{r}/dz)/\hat{r} &= 0. \end{aligned}$$

Linearise about the wavetrain

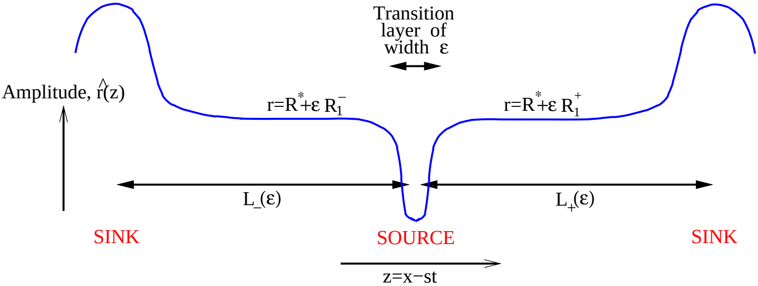
$\Rightarrow$ isolated sources decay to the wavetrain at rate $\sqrt{2}$
& isolated sinks decay to the wavetrain at rate $1/\sqrt{2} \pm i\delta/4$

$(\delta = \sqrt{11 - 12R^{*2}} \in \mathbb{R})$

$\Rightarrow$ in patterns,the effect of sinks on sources dominates
the effect of sources on sinks,for large separations

$\Rightarrow$ when the velocity s is small, we can just consider the correction to an isolated source: $r = R^*|\tanh(z/\sqrt{2})|$

Perturbation Theory Calculation

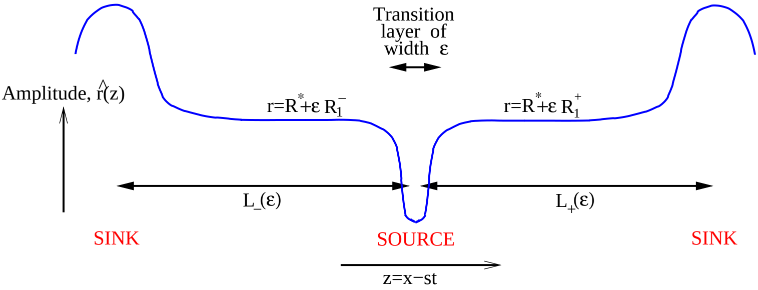


We use perturbation theory with the small parameter

$$\epsilon = \left[\tfrac{1}{2}(R^- + R^+) - R^*\right] \cdot (\text{constant})$$

where R^* is the amplitude of the stationary source.

Perturbation Theory Calculation



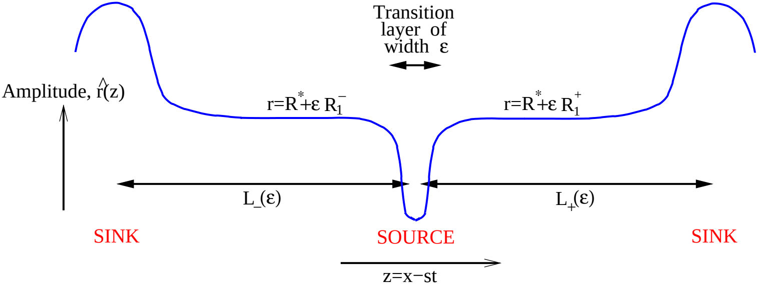
$$\text{For } \epsilon = 0: s = 0$$

$$K = (9-\sqrt{81+72c^2})/(4c)$$

$$\widehat{r} = R^*|\tanh(z/\sqrt{2})|$$

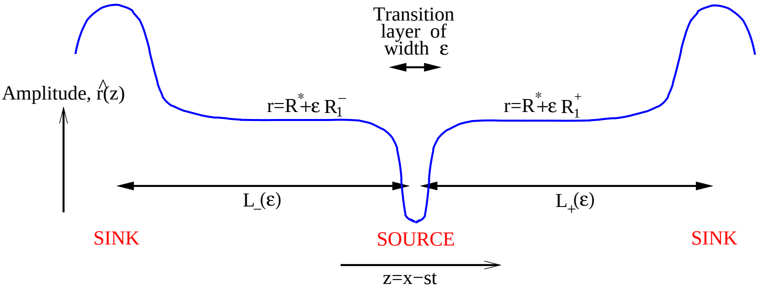
$$\widehat{\psi} = -(1-R^{*2})^{1/2}\tanh(z/\sqrt{2})$$

Perturbation Theory Calculation



$$
\begin{aligned}
\text{For } \epsilon \neq 0: s &= \epsilon s_1 + O(\epsilon^2) \\
K &= (9-\sqrt{81+72c^2})/(4c) + \epsilon K_1 + O(\epsilon^2) \\
\hat{r} &= R^*|\tanh(z/\sqrt{2})| + \epsilon \hat{r}_1(z) + O(\epsilon^2) \\
\hat{\psi} &= -(1-R^{*\,2})^{1/2}\tanh(z/\sqrt{2}) + \epsilon\hat{\psi}_1(z) + O(\epsilon^2)
\end{aligned}
$$

Perturbation Theory Calculation



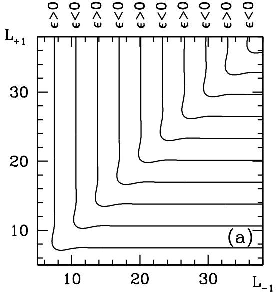
Key result (phase-locking condition):

$$\arg\left[\exp\left(-L_-(1+i\delta)/\sqrt{2}\right)+\exp\left(-L_+(1+i\delta)/\sqrt{2}\right)\right]=\text{constant}\,.$$

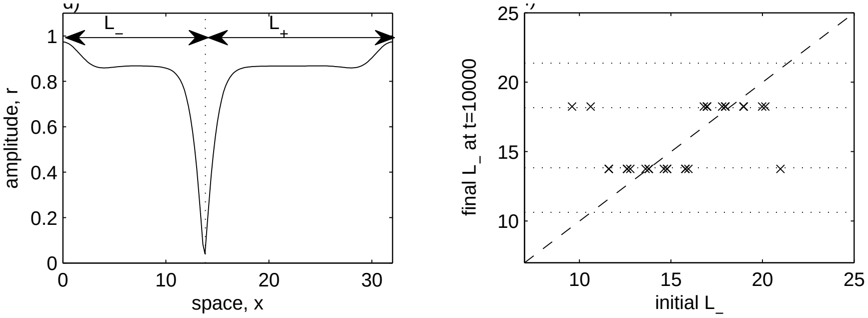
The constant is determined explicitly.
Surpisingly, it is independent of s_1.

Illustration of the Locking Condition

$$\arg\left[\exp\left(-L_-(1+i\delta)/\sqrt{2}\right)+\exp\left(-L_+(1+i\delta)/\sqrt{2}\right)\right]=\text{constant}$$



Numerical Verification of the Analysis



Conclusions

Main Results:

- For behaviour induced by Dirichlet boundary conditions, the transition from a wavetrain to spatiotemporal disorder occurs via source-sink patterns.
- The separations between a source and its neighbouring sinks, L_- and L_+, are constrained to lie on one of a discrete infinite sequence of curves in the L_-–L_+ plane (to leading order as velocity $\rightarrow 0$ and separations $\rightarrow \infty$).

Conclusions

Implications for Real Systems:

Physics
- Experiments are sufficiently precise that the prediction of discrete spacings are testable.

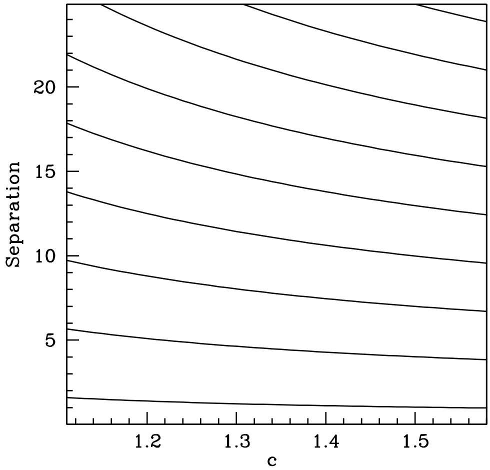
Ecology
- Empirical testing is not feasible.
- In the convectively unstable parameter regime, wavetrains will only be detected in field data if the spatial scale of the data is small compared to source-sink separations.

Future Work

- Selection of source-sink separations from the discrete family by initial and boundary conditions
- Stability of source-sink patterns
- Higher order terms (sink-sink coupling)
- Extension to $b \neq 0$

List of Frames

Dependence of Source-Sink Separations on c



Detailed form of a Source-Sink Pair