Patterns of Sources and Sinks in the Complex Ginzburg-Landau Equation

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This talk can be downloaded from my web site
www.ma.hw.ac.uk/~jas
This work is in collaboration with:

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(Microsoft Research Ltd., Cambridge)

Jens Rademacher
(CWI, Amsterdam)
Outline

1. Ecological Motivation
2. A Generic Oscillator Equation
3. Wavetrain Stability
4. Sources and Sinks
5. Analytical Study of Source-Sink Patterns
Ecological Motivation

A Generic Oscillator Equation

Wavetrain Stability

Sources and Sinks

Analytical Study of Source-Sink Patterns

Habitat Boundaries in Ecology
Example: Red Grouse on Kerloch Moor
Second Example: Field Voles in Kielder Forest
Wavetrains in Red Grouse & Field Voles

Outline

1. Ecological Motivation
2. A Generic Oscillator Equation
3. Wavetrain Stability
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Habitat Boundaries in Ecology

- Ecological habitats are often surrounded by unfavourable environments.
- Examples: a wood surrounded by open terrain, moorland surrounded by farmland, marsh surrounded by dry ground.
- An appropriate boundary condition is “population density=0”.
Red grouse is a cyclic population (period 4-6 years)
The study site is moorland, with farmland at its Northern edge
Farmland is very hostile for red grouse
Field voles in Kielder Forest are also cyclic (period 4 years)
Boundary Condition at the Reservoir Edge

- Voles are an important prey species for owls and kestrels
- The open expanse of Kielder Water will greatly facilitate hunting at its edge

Short eared owl

Common kestrel

Voles are an important prey species for owls and kestrels.

The open expanse of Kielder Water will greatly facilitate hunting at its edge.
Voles are an important prey species for owls and kestrels.
The open expanse of Kielder Water will greatly facilitate hunting at its edge.
Therefore we expect very high vole loss at the reservoir edge, implying that a suitable boundary condition is “vole density=0”
Spatiotemporal data shows that both red grouse cycles on Kerloch Moor and field vole cycles in Kielder Forest are spatially organised into a wavetrain (also known as plane wave, periodic travelling wave).
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Population Density

Space
Wavetrains in Red Grouse & Field Voles

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![Red Grouse](image1.jpg)  
![Field Vole](image2.jpg)
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[Graph showing a periodic wave pattern]

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[Diagram of population density over space with a periodic wave pattern]
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Wavetrain Generation Question

Question

Does the Dirichlet condition at the habitat boundary play a role in generating the wavetrains?
Outline

1. Ecological Motivation
2. A Generic Oscillator Equation
3. Wavetrain Stability
4. Sources and Sinks
5. Analytical Study of Source-Sink Patterns
I consider a generic oscillator model, the complex Ginzburg-Landau equation:

\[ A_t = (1 + ib)A_{xx} + A - (1 + ic)|A|^2A. \]

I will look exclusively at \( b = 0 \). Then writing

\[ A(x, t) = e^{-iat}[u(x, t) + iv(x, t)] \]

gives a reaction-diffusion system of "λ–ω" type:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1 - r^2)u - (a + cr^2)v \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (a + cr^2)u + (1 - r^2)v
\end{align*}
\]

where \( r = \sqrt{u^2 + v^2} \).
Amplitude and Phase Equations

To study these equations, it is helpful to use the variables $r(x, t) = \sqrt{u^2 + v^2}$ and $\theta(x, t) = \tan^{-1}(v/u)$, giving

$$
\begin{align*}
  r_t & = r_{xx} - r \theta_x^2 + r(1 - r^2) \\
  \theta_t & = \theta_{xx} + \frac{2r_x \theta_x}{r} + a - cr^2
\end{align*}
$$

There is a family of wavetrain solutions ($0 < r^* < 1$):

$$
\begin{cases}
  r = r^* \\
  \theta = [(a + cr^*^2)t \pm \sqrt{(1 - r^*^2)x}]
\end{cases}
$$

$$
\begin{cases}
  u = r^* \cos\left[(a + cr^*^2)t \pm \sqrt{(1 - r^*^2)x}\right] \\
  v = r^* \sin\left[(a + cr^*^2)t \pm \sqrt{(1 - r^*^2)x}\right]
\end{cases}
$$

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Patterns of Sources and Sinks in the CGLE
Typical Model Solutions

Equations:

\[ u_t = u_{xx} + (1 - r^2)u - (a + cr)v \]
\[ v_t = v_{xx} + (a + cr^2)u + (1 - r^2)v \]

where \( r = \sqrt{u^2 + v^2} \)

This is a typical numerical solution when \( u = v = 0 \) is imposed at \( x = 0 \)
Typical Model Solutions

Dirichlet boundary conditions generate a wavetrain

\[ R(x) = R^* \tanh \left( x / \sqrt{2} \right) \]
\[ \psi(x) = \psi^* \tanh \left( x / \sqrt{2} \right) \]
\[ R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} c^2} \right] \right\}^{-1/2} \]
\[ \psi^* = -\text{sign}(c) \left( 1 - R^*^2 \right)^{1/2} \]
<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
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<tr>
<td>Does the Dirichlet condition at the habitat boundary play a role in generating the wavetrains?</td>
<td>Yes (at least potentially)</td>
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In any oscillatory reaction-diffusion system, some members of the wavetrain family are stable as solutions of the partial differential equations, while others are unstable.

For our $\lambda-\omega$ system, the stability condition is

$$r^* > \left( \frac{2 + 2c^2}{3 + 2c^2} \right)^{1/2}$$
Stability of the Selected Wave

The stability of the selected wave depends on $c$:

$$R^* = \left\{ \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{8}{9} c^2} \right] \right\}^{-1/2}$$

This is stable

$$\Leftrightarrow \quad R^* > \left( \frac{2 + 2c^2}{3 + 2c^2} \right)^{1/2}$$

$$\Leftrightarrow \quad |c| < 1.110468 \ldots$$
There are two types of solution for $|c| > 1.110468\ldots$
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There are two types of solution for $|c| > 1.110468 \ldots$

The key concept for distinguishing these is “absolute stability”.

In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is “convective instability”.

![Image of two solutions with one growing and moving, and the other not]
Convective and Absolute Stability

- There are two types of solution for $|c| > 1.110468 \ldots$
- The key concept for distinguishing these is “absolute stability”.
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is “convective instability”.
- Alternatively, a solution can be unstable with perturbations growing without moving. This is “absolute instability”.

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Patterns of Sources and Sinks in the CGLE
Numerical simulations show distinct behaviours in the absolutely stable and unstable parameter regimes.

- Convectively unstable, absolutely stable
- Absolutely unstable
Outline

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The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.

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Patterns of Sources and Sinks in the CGLE
The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.

Note: sources and sinks are defined in terms of group velocity.
The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.

**Question:** How can an unstable wavetrain persist between the sources and sinks?
Question: How can an unstable wavetrain persist between the sources and sinks?

Answer: Any growing perturbations moves, and is absorbed when it reaches a sink.
Absolute stability is much harder to calculate than stability.

For wavetrain solutions of the CGLE, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)
Absolute stability is much harder to calculate than stability.

For wavetrain solutions of the CGLE, we have calculated absolute stability by computing the “absolute spectrum” via numerical continuation, adapting the method of Rademacher, Sandstede & Scheel (Physica D 229: 166-183, 2007)

Our calculation shows that the stability of the selected wavetrain is:

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<tr>
<td>0.0</td>
<td>1.110468</td>
<td>1.576465</td>
</tr>
<tr>
<td></td>
<td>STABLE</td>
<td>UNSTABLE</td>
</tr>
<tr>
<td>BUT ABSOLUTELY STABLE</td>
<td></td>
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Our calculation shows that the stability of the selected wavetrain is:

$$0.0 \quad 1.110468 \quad 1.576465$$

SOURCE–SINK

PATTERNS OCCUR HERE
Experimental Observation of Sources and Sinks

Experimental systems in which sources and sinks have been observed include:

- chemical reactions
- electrochemical systems
- heated wire convection
- binary fluid convection
- convection waves generated by heating at a boundary
- the “printer’s instability”, in which the thin gap between two rotating acentric cylinders is filled with oil.
Previous Mathematical Work on Sources and Sinks

- Sources are “Nozaki–Bekki” holes (Nozaki & Bekki, Phys. Lett. A 110: 133-135, 1985), on which the literature is extensive (> 100 citations).
- Sinks are also well studied, though only numerically.
- But patterns of sources and sinks have received almost no attention.
- One open question is: are there constraints on the distances separating sources and sinks?
Numerical Study of Source-Sink Separations

Step 1: generate a source-sink pattern via a Dirichlet boundary condition

Step 2: extract a sink-source-sink triple

Step 3: transfer this part of the solution to a domain with zero Neumann boundary conditions

Step 4: translate the source and track the subsequent dynamics
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Numerical Study of Source-Sink Separations

Original solution
Numerical Study of Source-Sink Separations

Original solution

Solution with translated source
Numerical Study of Source-Sink Separations

Original solution

Solution with translated source
Numerical Study of Source-Sink Separations

Original solution

Solution with translated source
Numerical Study of Source-Sink Separations

Original solution

Solution with translated source
Conclusion: source-sink separations appear to be constrained to a discrete set of possible values.

Next Step: analytical investigation of the separations.

A Complication: sources and sinks often move, very slowly.
These sources and sinks appear to be stationary........
These sources and sinks appear to be stationary .......... but very long simulations show that they move.
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The source-sink patterns are of travelling wave form in amplitude.

Substitute \( r(x, t) = \hat{r}(z), \theta_x(x, t) = \hat{\psi}(z), z = x - st \)

\[
\Rightarrow \quad \frac{d^2\hat{r}}{dz^2} + s \frac{d\hat{r}}{dz} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) = 0
\]

\[
\frac{d\hat{\psi}}{dz} + s \hat{\psi} + K - c\hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) = 0
\]

\((K\) is a constant of integration).
Solution Structure

Between the source and the neighbouring sinks,
\[-s(1 - R^{-2})^{1/2} + cR^{-2} = K = +s(1 - R^{+2})^{1/2} + cR^{+2}\]

\(\implies\) velocity \(s\) has the same sign as \(R^- - R^+\).

Based on source-sink patterns seen in numerical simulations, we consider large separations and small velocities.
Isolated sources and sinks satisfy

\[
\begin{align*}
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) & = 0 \\
\frac{d\hat{\psi}}{dz} + K - c\hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) / \hat{r} & = 0.
\end{align*}
\]
Isolated sources and sinks satisfy

\[
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left( 1 - \hat{r}^2 - \hat{\psi}^2 \right) = 0
\]

\[
\frac{d\hat{\psi}}{dz} + K - c\hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) / \hat{r} = 0.
\]

Linearise about the wavetrain

\Rightarrow \text{isolated sources decay to the wavetrain at rate } \sqrt{2}

\& \text{isolated sinks decay to the wavetrain at rate } 1/\sqrt{2} \pm i\delta/4

(\delta = \sqrt{11 - 12R^*^2} \in \mathbb{R})

\Rightarrow \text{in patterns, the effect of sinks on sources dominates}

the effect of sources on sinks, for large separations

\Rightarrow \text{when the velocity } s \text{ is small, we can just consider the correction}

\text{to an isolated source}
Isolated sources and sinks satisfy

\[
\frac{d^2 \hat{r}}{dz^2} + \hat{r} \left(1 - \hat{r}^2 - \hat{\psi}^2\right) = 0
\]

\[
\frac{d \hat{\psi}}{dz} + K - c\hat{r}^2 + 2\hat{\psi} \left(\frac{d\hat{r}}{dz}\right)/\hat{r} = 0.
\]

Linearise about the wavetrain

\Rightarrow \text{isolated sources decay to the wavetrain at rate } \sqrt{2}

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Isolated sources and sinks satisfy
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\[
\frac{d\hat{\psi}}{dz} + K - c\hat{r}^2 + 2\hat{\psi} \left( \frac{d\hat{r}}{dz} \right) / \hat{r} = 0.
\]

Linearise about the wavetrain
\[
\Rightarrow \text{isolated sources decay to the wavetrain at rate } \sqrt{2}
\]
\& \text{isolated sinks decay to the wavetrain at rate } 1 / \sqrt{2} \pm i\delta / 4
\]
\[
(\delta = \sqrt{11 - 12R^*^2} \in \mathbb{R})
\]
\[
\Rightarrow \text{in patterns, the effect of sinks on sources dominates}
\]
\text{the effect of sources on sinks, for large separations}
\]
\[
\Rightarrow \text{when the velocity } s \text{ is small, we can just consider the correction}
\]
\text{to an isolated source}

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Patterns of Sources and Sinks in the CGLE
Isolated sources and sinks satisfy
\[
\begin{align*}
    d^2 \hat{r} / dz^2 + \hat{r} \left(1 - \hat{r}^2 - \hat{\psi}^2\right) &= 0 \\
    d\hat{\psi} / dz + K - c\hat{r}^2 + 2\hat{\psi} (d\hat{r} / dz) / \hat{r} &= 0.
\end{align*}
\]

Linearise about the wavetrain
\[
\Rightarrow \text{isolated sources decay to the wavetrain at rate } \sqrt{2} \quad \text{and isolated sinks decay to the wavetrain at rate } 1/\sqrt{2} \pm i \delta/4 \\
\quad (\delta = \sqrt{11 - 12R^*^2} \in \mathbb{R})
\]
\[
\Rightarrow \text{in patterns, the effect of sinks on sources dominates} \\
\quad \text{the effect of sources on sinks, for large separations}
\]
\[
\Rightarrow \text{when the velocity } s \text{ is small, we can just consider the correction} \\
\quad \text{to an isolated source: } r = R^* | \tanh(z/\sqrt{2}) |
\]
We use perturbation theory with the small parameter

\[ \epsilon = \left[ \frac{1}{2} \left( R^- + R^+ \right) - R^* \right] \cdot (\text{constant}) \]

where \( R^* \) is the amplitude of the stationary source.
Perturbation Theory Calculation

For $\epsilon = 0$:

$s = 0$

$K = \frac{(9 - \sqrt{81 + 72c^2})}{4c}$

$\hat{r} = R^* |\tanh(z/\sqrt{2})|$

$\hat{\psi} = -(1 - R^*^2)^{1/2} \tanh(z/\sqrt{2})$
For $\epsilon \neq 0$:

\[
\begin{align*}
    s &= \epsilon S_1 + O(\epsilon^2) \\
    K &= (9 - \sqrt{81 + 72c^2})/(4c) + \epsilon K_1 + O(\epsilon^2) \\
    \hat{r} &= R^* |\tanh(z/\sqrt{2})| + \epsilon \hat{r}_1(z) + O(\epsilon^2) \\
    \hat{\psi} &= -(1 - R^*^2)^{1/2} \tanh(z/\sqrt{2}) + \epsilon \hat{\psi}_1(z) + O(\epsilon^2)
\end{align*}
\]
Perturbation Theory Calculation

Key result (phase-locking condition):

\[
\arg \left[ \exp \left( -L_-(1 + i\delta)/\sqrt{2} \right) + \exp \left( -L_+(1 + i\delta)/\sqrt{2} \right) \right] = \text{constant}.
\]

The constant is determined explicitly.
Surprisingly, it is independent of \(s_1\).
Illustration of the Locking Condition

\[ \arg \left[ \exp \left( -L_-(1 + i\delta)/\sqrt{2} \right) + \exp \left( -L_+(1 + i\delta)/\sqrt{2} \right) \right] = \text{constant} \]
Numerical Verification of the Analysis

Patterns of Sources and Sinks in the CGLE
Conclusions

Main Results:

- For behaviour induced by Dirichlet boundary conditions, the transition from a wavetrain to spatiotemporal disorder occurs via source-sink patterns.

- The separations between a source and its neighbouring sinks, $L_-$ and $L_+$, are constrained to lie on one of a discrete infinite sequence of curves in the $L_-L_+$ plane (to leading order as velocity $\to 0$ and separations $\to \infty$).
Conclusions

Implications for Real Systems:

**Physics**
- Experiments are sufficiently precise that the prediction of discrete spacings are testable.

**Ecology**
- Empirical testing is not feasible.
- In the convectively unstable parameter regime, wavetrains will only be detected in field data if the spatial scale of the data is small compared to source-sink separations.
Future Work

- Selection of source-sink separations from the discrete family by initial and boundary conditions
- Stability of source-sink patterns
- Higher order terms (sink-sink coupling)
- Extension to $b \neq 0$
Ecological Motivation

Habitat Boundaries in Ecology
Example: Red Grouse on Kerloch Moor
Second Example: Field Voles in Kielder Forest
Wavetrains in Red Grouse & Field Voles

A Generic Oscillator Equation
The Complex Ginzburg-Landau Equation
Amplitude and Phase Equations
Typical Model Solutions

Wavetrain Stability
Stability in the Wavetrain Family
Convective and Absolute Stability

Sources and Sinks
Sources, Sinks, and Convective Instability
Calculation of Absolute Stability of Wavetrains
Literature on Sources and Sinks
Numerical Study of Source-Sink Separations
Movement of Sources and Sinks

Analytical Study of Source-Sink Patterns
Travelling Waves of Amplitude
Solution Structure
Numerical Verification of the Analysis
Conclusions
Future Work

Patterns of Sources and Sinks in the CGLE
Dependence of Source-Sink Separations on $c$
Detailed form of a Source-Sink Pair

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Patterns of Sources and Sinks in the CGLE