Vegetation Stripes in Semi-Arid Environments

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In collaboration with
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Outline

1. Ecological Background
2. The Mathematical Model
3. Model Predictions: When Do Patterns Occur?
4. Conclusions
Vegetation patterns are found in semi-arid areas of Africa, Australia and Mexico (rainfall 100-700 mm/year)
- First identified by aerial photos in 1950s
- Plants vary from grasses to shrubs and trees
More Pictures of Vegetation Patterns

Labyrinth of bushy vegetation in Niger
More Pictures of Vegetation Patterns

Striped pattern of bushy vegetation in Niger
More Pictures of Vegetation Patterns

Labyrinth of grass in Israel
On flat ground, irregular mosaics of vegetation are typical.

On slopes, the patterns are stripes, parallel to contours ("Tiger bush").
Mechanisms for Vegetation Patterning

- Basic mechanism: competition for water
Mechanisms for Vegetation Patterning

- Basic mechanism: competition for water
- Possible detailed mechanism: water flow downhill causes stripes
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This mechanism suggests that the stripes would move uphill; this remains controversial.
Mathematical Model of Klausmeier

Rate of change = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change = Growth, proportional to water uptake – Mortality + Random dispersal

\[ \frac{\partial w}{\partial t} = A - w - wu^2 + v \frac{\partial w}{\partial x} \]

\[ \frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2} \]
Mathematical Model of Klausmeier

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Rate of change = Growth, proportional to water uptake – Mortality + Random dispersal

\[
\frac{\partial w}{\partial t} = A - w - wu^2 + ν w \frac{\partial w}{\partial x} \\
\frac{\partial u}{\partial t} = wu^2 - Bu + w^2 \frac{\partial^2 u}{\partial x^2}
\]

The nonlinearity in \( wu^2 \) arises because the presence of roots increases water infiltration into the soil.
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$

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Typical Solution of the Model

- Vegetation, $u$
  - Vertical axis: 0.05 to 10
- Water, $w$
  - Vertical axis: 0.05 to 0.15
- Distance uphill, $x$
  - Horizontal axis: 0 to 100

The graph shows oscillations in vegetation and water content as a function of distance uphill.
Typical Solution of the Model

Vegetation, u

Water, w

Distance uphill, x
Typical Solution of the Model
Typical Solution of the Model
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance](graph.png)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water distribution](image-url)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Diagram showing the variation of vegetation and water content with distance uphill. The graph plots vegetation and water content against distance uphill, illustrating periodic patterns typical of semi-arid environments.
Typical Solution of the Model

The graph shows the typical solution of the Mathematical Model of Klausmeier in Vegetation Stripes in Semi-Arid Environments.

- **Vegetation, $u$:** The graph plots the vegetation distribution over distance uphill, $x$, with peaks indicating areas of vegetation density.
- **Water, $w$:** The water distribution is shown below the vegetation plot, also varying over distance uphill, $x$.

The model predictions illustrate how vegetation and water patterns develop over the terrain, providing insights into the ecological dynamics in semi-arid environments.
Typical Solution of the Model

![Graph showing vegetation and water distribution](image-url)

- Vegetation, $u$
- Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$ vs. Distance uphill, $x$

Water, $w$ vs. Distance uphill, $x$
Typical Solution of the Model
Typical Solution of the Model
Typical Solution of the Model

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Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

The graph shows two plots:

1. **Vegetation, u**: This plot has a y-axis range from 0 to 10, with values representing vegetation density. The graph displays a periodic pattern that repeats every 50 units of distance uphill, x.

2. **Water, w**: This plot has a y-axis range from 0 to 0.15, with values representing water content. The graph displays a periodic pattern that repeats every 50 units of distance uphill, x.

The patterns in both graphs are consistent and reinforce the ecological patterns observed in semi-arid environments.
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Vegetation Stripes in Semi-Arid Environments

Typical Solution of the Model

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$
Typical Solution of the Model

- Vegetation, u
- Water, w

Distance uphill, x
Typical Solution of the Model

![Graph showing vegetation and water distribution](image-url)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

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Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, \( u \)

Water, \( w \)

Distance uphill, \( x \)
Typical Solution of the Model

![Graph showing the typical solution of the model with two axes: one for vegetation density (u) and another for water content (w) vs. distance uphill (x).]
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill](image-url)
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill](image-url)
Typical Solution of the Model
Typical Solution of the Model

![Graph showing the typical solution of the model with two graphs: one for vegetation and one for water. The vegetation graph shows oscillations with peaks at 10 and troughs at 5, while the water graph shows similar oscillations but with peaks at 0.15 and troughs at 0.05. The x-axis represents distance uphill, ranging from 0 to 100.](image-url)
Typical Solution of the Model

Graph showing the typical solution of the mathematical model with two oscillating graphs representing vegetation and water content as a function of distance uphill.
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill](image-url)
Typical Solution of the Model
Vegetation Stripes in Semi-Arid Environments

Typical Solution of the Model

![Graph showing vegetation and water profiles over distance uphill]

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill](image-url)
Typical Solution of the Model
1. Ecological Background
2. The Mathematical Model
3. Model Predictions: When Do Patterns Occur?
4. Conclusions
For all parameter values, there is a stable “desert” steady state $u = 0, w = A$. 
Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0, \ w = A$.
- When $A \geq 2B$, there are also non-trivial steady states. If $A$ is relatively small, this steady state destabilises, giving patterns.
An Illustration of Conditions for Patterning

\[ \nu = 182.5 \]

- Homogeneous vegetation
- Stripes
- No vegetation

Rainfall, A

Plant loss, B
Pattern wavelength is the most accessible property of vegetation stripes in the field, via aerial photography. Mathematical prediction of wavelength as a function of parameters (rainfall, plant loss, slope) is difficult because there are multiple pattern solutions.
To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

\[ \frac{\partial u_i}{\partial t} = w_i u_i^2 - Bu_i + \frac{(u_{i+1} - 2u_i + 2u_{i-1})}{\Delta x^2} \]

\[ \frac{\partial w_i}{\partial t} = A - w_i - w_i u_i^2 + \frac{\nu(w_{i+1} - w_i)}{\Delta x} \]

\( (i = 1, \ldots, N) \).
To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= w_i u_i^2 - Bu_i + \frac{(u_{i+1} - 2u_i + 2u_{i-1})}{\Delta x^2} \\
\frac{\partial w_i}{\partial t} &= A - w_i - w_i u_i^2 + \frac{\nu(w_{i+1} - w_i)}{\Delta x}
\end{align*}
\]

\((i = 1, \ldots, N)\).

We use upwinding for the convective term.
Discretizing the PDEs

To investigate pattern stability, we must work with the model PDEs. We discretize these in space and then use AUTO to study the resulting ODE system:

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\frac{\partial w_i}{\partial t} = A - w_i - w_i u_i^2 + \nu(w_{i+1} - w_i)/\Delta x
\]

\((i = 1, \ldots, N)\).

We use upwinding for the convective term.
Most of our work has used \(N = 40\) and \(\Delta x = 2\).
We assume periodic boundary conditions.
Bifurcation Diagram for Discretized PDEs
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Bifurcation Diagram for Discretized PDEs

- Soln amplitude
- Rainfall, A
- Space point, i
- Mode 5
- Homog st st

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Ecological Background

The Mathematical Model

Model Predictions: When Do Patterns Occur?

Conclusions

Homogeneous Steady States

An Illustration of Conditions for Patterning

Predicting Pattern Wavelength

Discretizing the PDEs

Key Result

Hysteresis

Bifurcation Diagram for Discretized PDEs

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Vegetation Stripes in Semi-Arid Environments
We determine **pattern existence** via numerical bifurcation analysis of the pattern ordinary differential equations, and **pattern stability** via numerical bifurcation analysis of the discretized model partial differential equations.
Pattern Selection

- For a range of rainfall levels, there is more than one stable pattern. Which will be selected?
- We consider initial conditions that are small perturbations of the coexistence steady state \((u_s, v_s)\).
- All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.
Pattern Selection

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![Graph showing pattern selection with rainfall on the x-axis and wavelength on the y-axis.](image-url)
Pattern Selection

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For a range of rainfall levels, there is more than one stable pattern. Which will be selected?

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All such initial conditions give a pattern, but the wavelength depends on the initial perturbation.
Key Result

For a wide range of rainfall levels, there are multiple stable patterns.
The existence of multiple stable patterns raises the possibility of hysteresis.

We consider slow variations in the rainfall parameter $A$.

Parameters correspond to grass, and the rainfall range corresponds to 130–930 mm/year.
Hysteresis

Rainfall

Space

Time

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Hysteresis

[Graph showing rainfall over time and vegetation stripes in space]

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Hysteresis

Rainfall vs. Time

Space vs. Time

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Vegetation Stripes in Semi-Arid Environments
Hysteresis

Rainfall

Space

Time

Homogeneous Steady States
An Illustration of Conditions for Patterning
Predicting Pattern Wavelength
Discretizing the PDEs
Key Result
Hysteresis

Vegetation Stripes in Semi-Arid Environments
Hysteresis

Rainfall vs. Time

Space

Vegetation Stripes in Semi-Arid Environments

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Predictions of Pattern Wavelength

- In general, pattern wavelength depends on initial conditions.
- When vegetation stripes arise from homogeneous vegetation via a decrease in rainfall, pattern wavelength will remain at a constant value.

\[
\text{Wavelength} = \sqrt{\frac{8\pi^2}{B\nu}}
\]
Other Potential Mechanisms for Vegetation Patterns

- **Rietkirk**: Klausmeier model with diffusion of water in the soil
- **van de Koppel**: Klausmeier model with grazing
- **Maron**: two variable model (plant density and water in the soil) with water transport based on porous media theory
- **Lejeune**: short range activation (shading) and long range inhibition (competition for water)

All of these models predict patterns. To discriminate between them requires a detailed understanding of each model.
Mathematical Moral

Predictions based only on linear stability analysis are misleading for this model
Ecological Background

Vegetation Pattern Formation
More Pictures of Vegetation Patterns
Vegetation Pattern Formation (contd)
Mechanisms for Vegetation Patterning

The Mathematical Model

Mathematical Model of Klausmeier
Typical Solution of the Model

Model Predictions: When Do Patterns Occur?

Homogeneous Steady States
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Predictions of Pattern Wavelength
Other Potential Mechanisms for Vegetation Patterns
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