Wavelength Selection and Hysteresis in Vegetation Patterns in Semi-Deserts

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This talk can be downloaded from my web site

www.ma.hw.ac.uk/~jas
1. Ecological Background
2. Pattern Formation in a Mathematical Model
3. Pattern Existence and Stability
4. Predictions of Pattern Wavelength vs Slope
5. Conclusions and References
Desert ecosystems provide a classic example of self-organised pattern formation.

W National Park, Niger
Average patch width is 50 m
Desert ecosystems provide a classic example of self-organised pattern formation.
Desert ecosystems provide a classic example of self-organised pattern formation.

Data from Burkina Faso
Rietkerk et al
Plant Ecology 148: 207-224, 2000

More plants ⇒ more roots and organic matter in soil
⇒ more infiltration of rainwater
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Desert ecosystems provide a classic example of self-organised pattern formation.
Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia
(Western New South Wales)

Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia, Mexico and S-W USA.
On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia
(Western New South Wales)

Wavelength can be measured via remote sensing.
Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.

Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)
Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.

Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

How does this compare with predictions of mathematical models?
Outline

1. Ecological Background
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Mathematical Model of Klausmeier

\[ \frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2} \]

\[ \frac{\partial w}{\partial t} = A - w - wu^2 + \nu \frac{\partial w}{\partial x} + D \frac{\partial^2 w}{\partial x^2} \]

(Klausmeier, Science 284: 1826-8, 1999)
The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.
The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.

Water uptake = Water density \times \text{Plant density} \times \left( \frac{\text{infiltration rate}}{} \right)
Typical Solution of the Model

![Graph of vegetation and water distribution along distance uphill](image)

- **Vegetation, u**
  - Range: 0 to 10
  - Axes: Distance uphill, x
- **Water, w**
  - Range: 0.05 to 0.15

**Typical Solution of the Model**

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

![Graph showing vegetation and water distribution along distance uphill](image-url)
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill.]
Typical Solution of the Model

![Graph showing vegetation and water patterns](image-url)
Typical Solution of the Model

![Graph showing vegetation and water profiles](attachment:typical_solution.png)

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation density and water content over distance uphill](image-url)
Typical Solution of the Model

Mathematical Model of Klausmeier

Typical Solution of the Model

Homogeneous Steady States

Predictions of Pattern Wavelength vs Slope

Predicting Pattern Wavelength: Textbook Approach

Conclusions and References

The Origin of Vegetation Patterns

Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

![Typical Solution of the Model](image-url)
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill](image)

- **Vegetation, u**: Variations in vegetation density across the distance uphill, depicting periodic patterns.
- **Water, w**: Variations in water content across the distance uphill, also showing periodic patterns.

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

![Graph showing vegetation and water profiles over distance uphill, x]

- Vegetation, \( u \)
- Water, \( w \)

Distance uphill, \( x \):

- Wavelength Selection in Vegetation Patterns

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Mathematical Model of Klausmeier
Typical Solution of the Model
Homogeneous Steady States
Predicting Pattern Wavelength: Textbook Approach
The Origin of Vegetation Patterns
Typical Solution of the Model

- Vegetation, u
- Water, w
- Distance uphill, x
Typical Solution of the Model

The graph shows variations in vegetation density and water content over distance uphill. The vegetation density $u$ and water content $w$ exhibit periodic patterns, indicating the formation of vegetation patterns. The wavelength of these patterns can be predicted using mathematical models.
Typical Solution of the Model

Ecological Background
Pattern Formation in a Mathematical Model
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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

![Graph showing typical solution of the model with vegetation and water concentration versus distance uphill.]
Typical Solution of the Model

![Graph showing the typical solution of the model with vegetation and water profiles over distance uphill.]
Typical Solution of the Model

![Graph showing vegetation and water profiles across distance uphill](image-url)

- Vegetation, $u$
  - $10$
  - $5$
  - $0.15$
  - $0.1$
  - $0.05$

- Water, $w$
  - $10$
  - $5$
  - $0.15$
  - $0.1$
  - $0.05$

Distance uphill, $x$

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

The Origin of Vegetation Patterns

Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

The graph shows the typical solution of the model, with vegetation and water content plotted against distance uphill. The vegetation content, $u$, varies from 0 to 10, while the water content, $w$, varies from 0.05 to 0.15. The distance uphill, $x$, ranges from 0 to 100.

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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

Mathematical Model of Klausmeier
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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

![Graph of vegetation and water distribution over distance](image-url)
Typical Solution of the Model

The diagram shows the typical solution of the model for vegetation and water content as functions of distance uphill. The graph demonstrates periodic patterns that characterize vegetation patterns. The vegetation content, $u$, and water content, $w$, are plotted against the distance uphill, $x$. The periodicity of the patterns suggests a stable and repeatable pattern formation process in ecological systems.
Typical Solution of the Model

[Graph showing the typical solution of the model with two curves representing vegetation density (u) and water content (w) as a function of distance uphill (x).]
Typical Solution of the Model

![Graph showing vegetation and water profiles across distance uphill, x.](image)
Typical Solution of the Model
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water distribution](image-url)
Typical Solution of the Model

![Graph showing the typical solution of the model with two oscillatory waves representing vegetation and water over distance uphill.](image-url)
Typical Solution of the Model

![Graph showing the typical solution of the model for vegetation and water distribution over distance uphill.](image-url)
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill.](image-url)
Typical Solution of the Model

![Graph showing vegetation and water profiles](image-url)
Typical Solution of the Model

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$
Typical Solution of the Model
Typical Solution of the Model

![Typical Solution of the Model](image_url)
Typical Solution of the Model

![Graph showing vegetation and water profiles as a function of distance uphill. The graph displays periodic patterns with a wavelength that depends on the environmental conditions and parameters of the mathematical model.](image-url)
Typical Solution of the Model

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$

The Origin of Vegetation Patterns

Mathematical Model of Klausmeier
- Typical Solution of the Model
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Wavelength Selection in Vegetation Patterns
Typical Solution of the Model

- Vegetation, $u$
- Water, $w$

Distance uphill, $x$
Homogeneous Steady States

For all parameter values, there is a stable “desert” steady state $u = 0, w = A$
For all parameter values, there is a stable “desert” steady state \( u = 0, \ w = A \)

When \( A \geq 2B \), there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations.
For all parameter values, there is a stable “desert” steady state $u = 0, \ w = A$

- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations

- The other steady state $(u_s, w_s)$ is stable to homogeneous perturbations but can be unstable to inhomogeneous perturbations $\Rightarrow$ pattern formation
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

The expected wavelength \(\leftrightarrow\) the frequency of noise giving the fastest growth rate.
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

This implies a positive correlation between wavelength and slope, contrary to data.
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

“To date, no model of vegetation band formation has been shown to reproduce this inverse relationship between spacing and slope.” (Pelletier et al, J. Geophys. Res. 117, F04026, 2012)
“Most unstable frequency” assumes that patterns develop from a pre-existing unstable uniform state.

Vegetation patterns develop via
   either degradation of uniform vegetation
   or colonisation of bare ground
Outline

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The patterns move at constant shape and speed
⇒ \( u(x, t) = U(z), \ w(x, t) = W(z), \ z = x - ct \)

\[
d^2 U/dz^2 + c \frac{dU}{dz} + WU^2 - BU = 0
\]
\[
D d^2 W/dz^2 + (\nu + c) \frac{dW}{dz} + A - W - WU^2 = 0
\]

The patterns are periodic (limit cycle) solutions of these equations
Wavelength Selection in Vegetation Patterns
Bifurcation Diagram for Travelling Wave Equations

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Wavelength Selection in Vegetation Patterns
When do Patterns Form?

Min rainfall for patterns

Turing bifurcation

Locus of homoclinic solns

Locus of Hopf bifurcation points

Parameter region giving patterns

Min rainfall for uniform veg

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Not all of the possible patterns are stable as solutions of the model equations.
The boundary between stable and unstable patterns can be calculated by numerical continuation of the essential spectrum.

Calculations of this type can be performed using the software package WAVETRAIN (www.ma.hw.ac.uk/wavetrain).
Pattern Stability: Wavelength vs Rainfall

![Graph showing the relationship between wavelength and rainfall.](image-url)
Pattern Stability: Wavelength vs Rainfall

The diagram illustrates the relationship between wavelength and rainfall. The x-axis represents rainfall (A), while the y-axis represents wavelength. The graph shows several lines indicating different patterns of wavelength stability over varying rainfall amounts.

References:
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- www.ma.hw.ac.uk/~jas

Wavelength Selection in Vegetation Patterns
Pattern Stability: The Key Result

Key Result

Many of the possible patterns are unstable and thus will never be seen.

However, for a wide range of rainfall levels, there are multiple stable patterns.
The existence of multiple stable patterns suggests the possibility of hysteresis.

Domain length 150, periodic bc's
Data on the Effects of Changing Rainfall

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5. Conclusions and References
Pattern wavelength is history-dependent.

We must focus on the onset of patterning.

Degradation of uniform vegetation

Colonisation of bare ground

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How to Predict Pattern Wavelength

Wavelength vs Slope for Degradation of Uniform Vegetation

When Does Vegetation Colonise Bare Ground?

Wavelength vs Slope for Colonisation
Wavelength vs Slope for Degradation of Uniform Vegetation

![Graph showing the relationship between wavelength and slope for degradation of uniform vegetation. The graph depicts a curve where wavelength increases with increasing slope up to a certain point and then decreases.]
For realistic parameters, wavelength increases with slope, contrary to data.
When Does Vegetation Colonise Bare Ground?

- **Downhill ↔ Uphill**
- **Very low rainfall**: an isolated vegetation patch dies out
- **Slightly larger rainfall**: both edges move uphill
- **Larger rainfall**: the patch expands both uphill and downhill
The key critical case is when the downhill edge is stationary.
Wavelength vs Slope for Colonisation

![Graph showing the relationship between rainfall, slope, and vegetation patterns.](image-url)
Wavelength vs Slope for Colonisation

![Graph showing the relationship between rainfall, slope, and vegetation patterns.](image)
Wavelength vs Slope for Colonisation

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Wavelength Selection in Vegetation Patterns
Wavelength decreases with slope, in agreement with data
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Patterned vegetation is widespread in the Sahel

Several studies of banded vegetation show wavelength ↓ as slope ↑
Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.

There is no direct data on rainfall before c. 1850

Proxy data: (i) lake levels, esp. Lake Chad; (ii) historical chronologies, e.g. Bornu Empire; (iii) memories of local peoples.
Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.
- Reasonable assumption: areas with vegetation patterns today had uniform vegetation at the end of the 17th century.
- Since wavelength decreases with slope, my results imply that vegetation must have died out and then recolonised since the end of the 17th century.
- The most severe drought since 1700 was c. 1738-1756. So today’s vegetation patterns result from recolonisation since 1760.
Wavelength is positively correlated with slope \(\Rightarrow\) vegetation pattern originated by degradation of uniform vegetation

Wavelength is negatively correlated with slope \(\Rightarrow\) vegetation pattern originated by colonisation of bare ground

Main message: combined wavelength–slope data is much more valuable than wavelength data alone.
Remote Sensing of Wavelength and Elevation

Google Earth: online satellite images, min. 15 m resolution

Google Earth

Remote Sensing of Wavelength and Elevation
Remote Sensing of Wavelength and Elevation

**WorldDEM**: online elevation data, 12 m resolution
References


Ecological Background

Pattern Formation in a Mathematical Model

Pattern Existence and Stability

Predictions of Pattern Wavelength vs Slope

Conclusions and References

Example: The African Sahel

Rainfall History in the Sahel

Conclusions

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   - Data on Wavelength vs Slope

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   - Pattern Stability
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   - How to Predict Pattern Wavelength
   - Wavelength vs Slope for Degradation of Uniform Vegetation
   - When Does Vegetation Colonise Bare Ground?
   - Wavelength vs Slope for Colonisation

5. Conclusions and References
   - Example: The African Sahel
   - Rainfall History in the Sahel
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