Locating Transitions from Cycles to Chaos Behind Invasions in Reaction-Diffusion Equations

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Heriot-Watt University

University of Edinburgh, 27 October 2011

This talk can be downloaded from `www.ma.hw.ac.uk/~jas`

This work is in collaboration with:

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(Microsoft Research Ltd., Cambridge)

Jens Rademacher

(CWI, Amsterdam)

Outline

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Cyclic Predator-Prey Systems

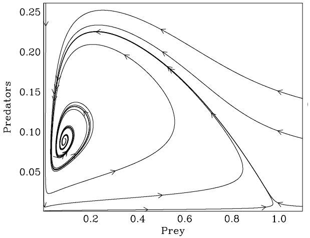
The interaction between a predator population and its prey can cause population cycles.
Example: vole – weasel interaction in Fennoscandia

vole

weasel

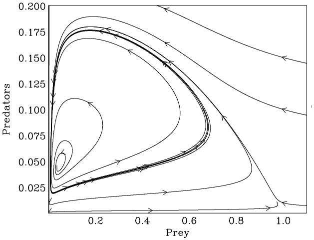
Cyclic Predator-Prey Systems

The interaction between a predator population and its prey can cause population cycles.
This has been modelled extensively using systems of two coupled ODEs



constant coexistence

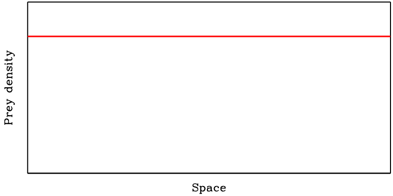
change $\longrightarrow$ parameters



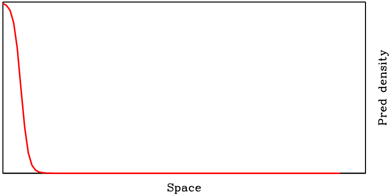
cycles

Predator-Prey Invasion

To model the invasion of a prey population by predators, one can add diffusion terms to represent dispersal.



Initially we set the prey to the prey-only equilibrium throughout the domain.



Initially we set the predators to zero except near the left hand boundary.

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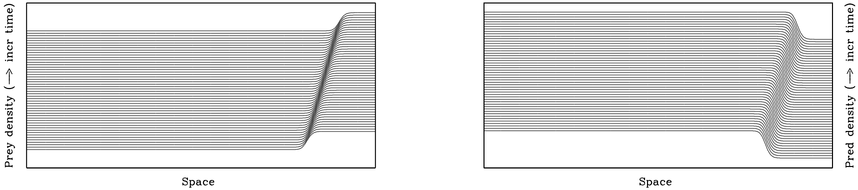

Simple invasion front (local bhr: constant)

Predator-Prey Invasion

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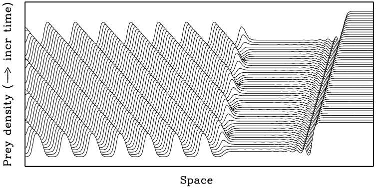

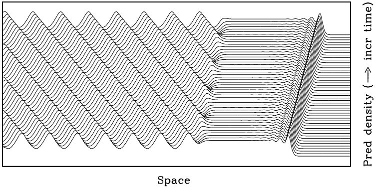

Wavetrain behind an invasion front (local bhr: cycles)

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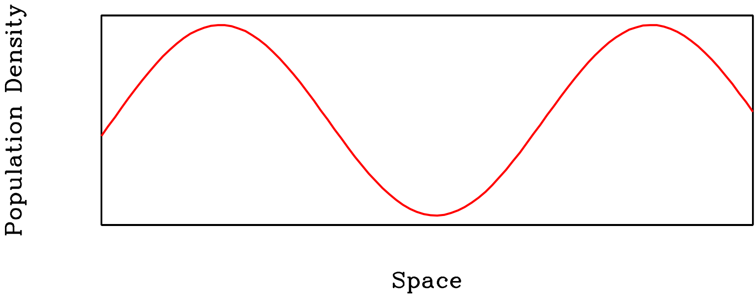

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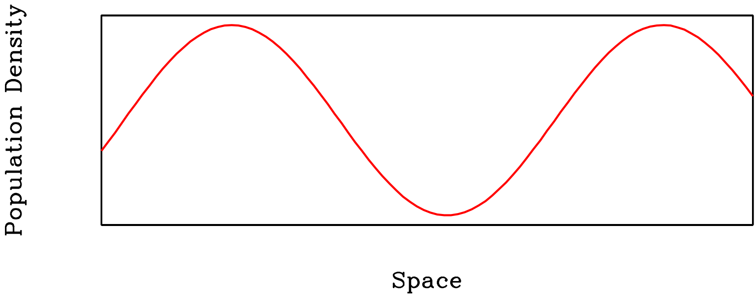

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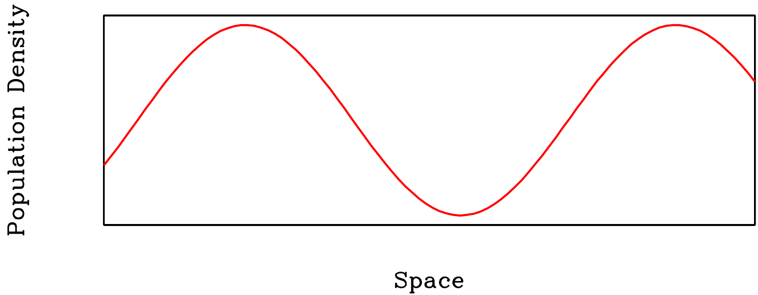

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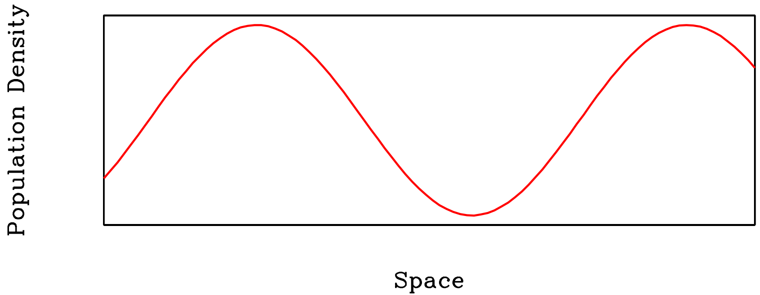

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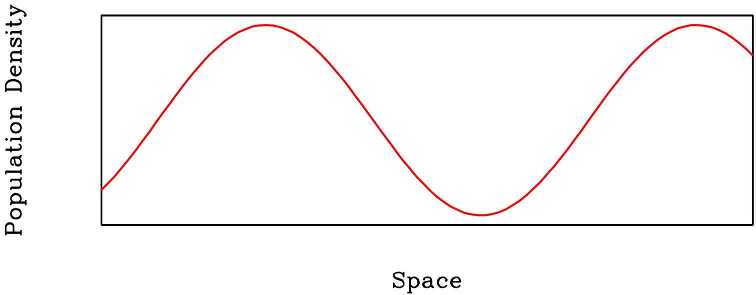

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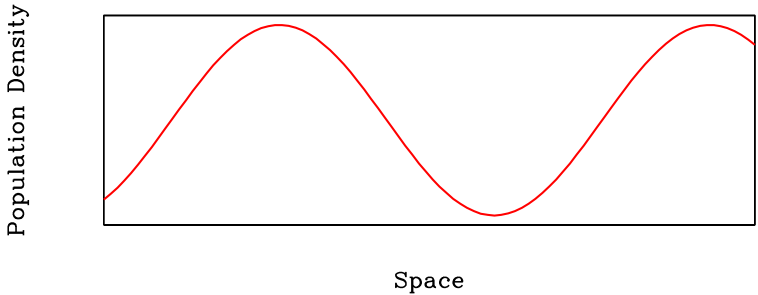

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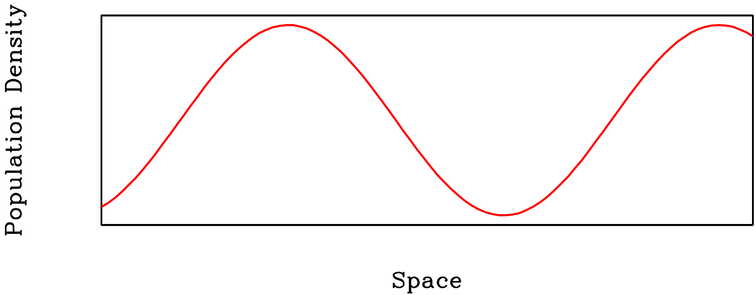

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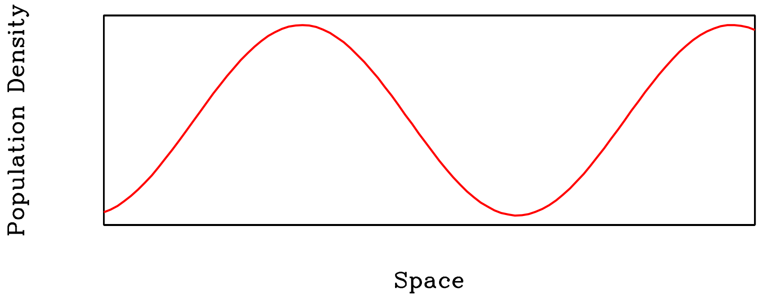

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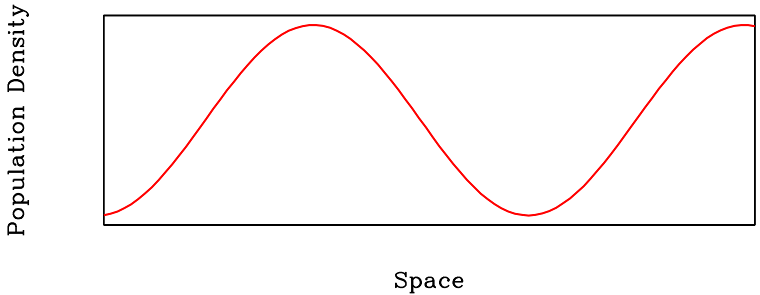

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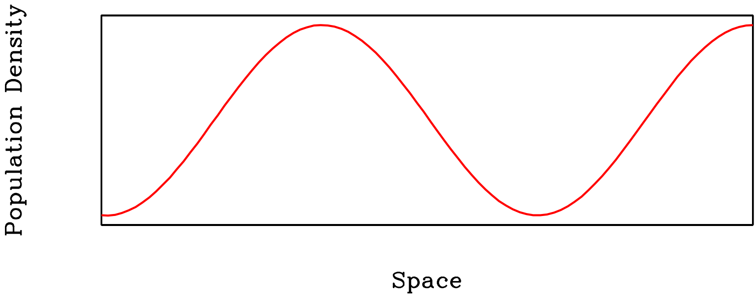

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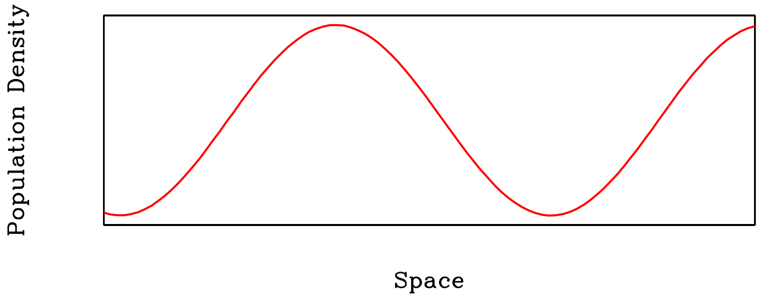

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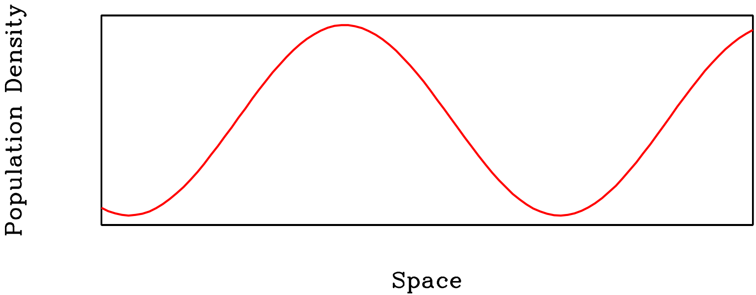

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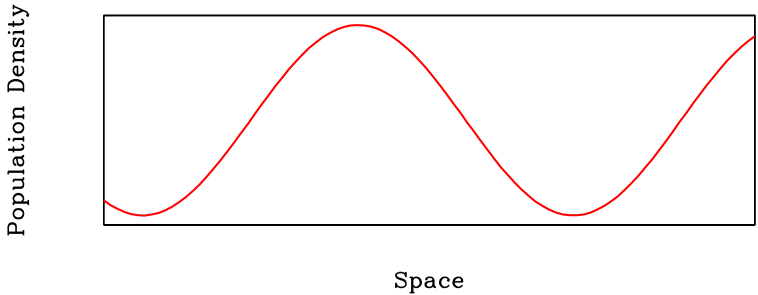

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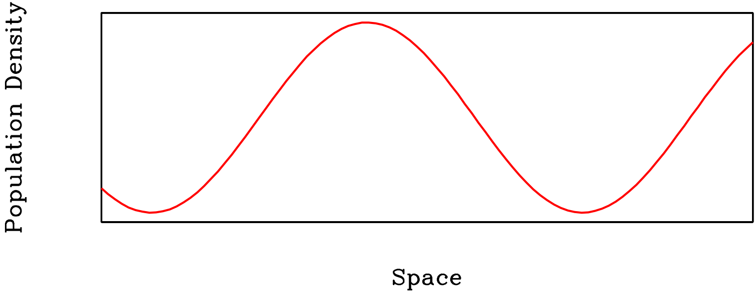

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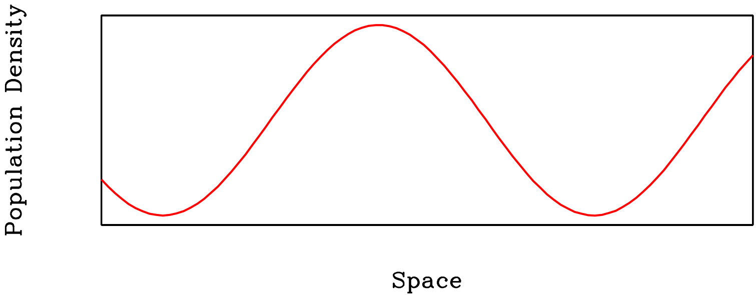

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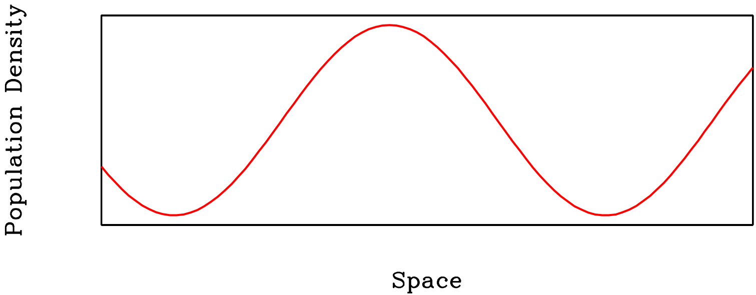

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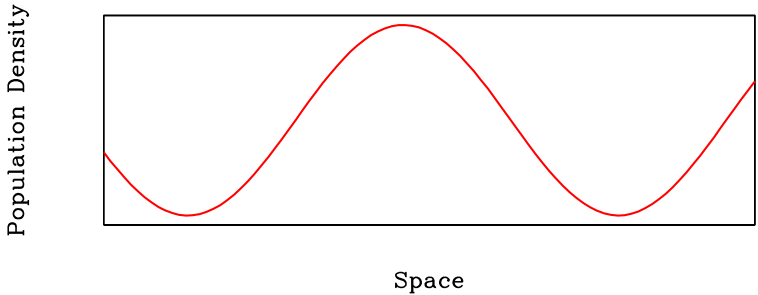

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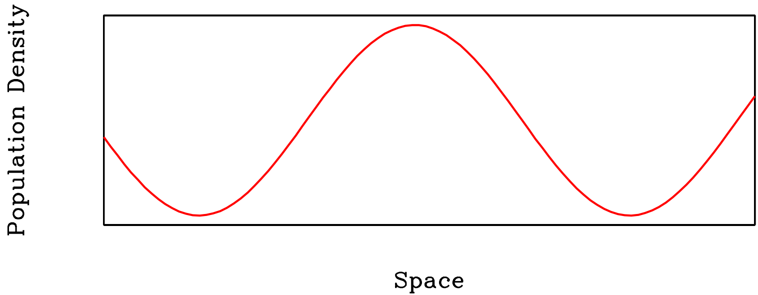

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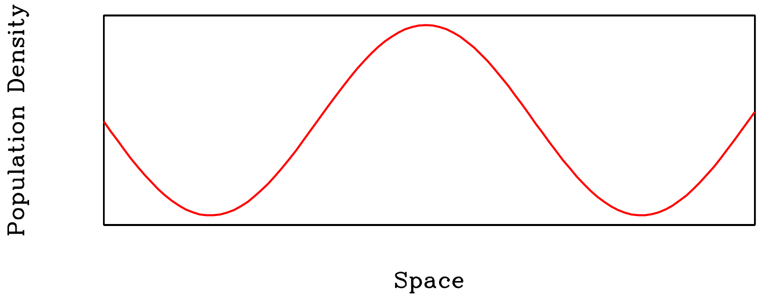

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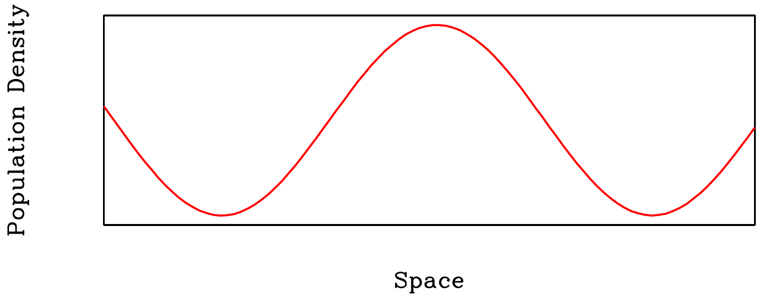

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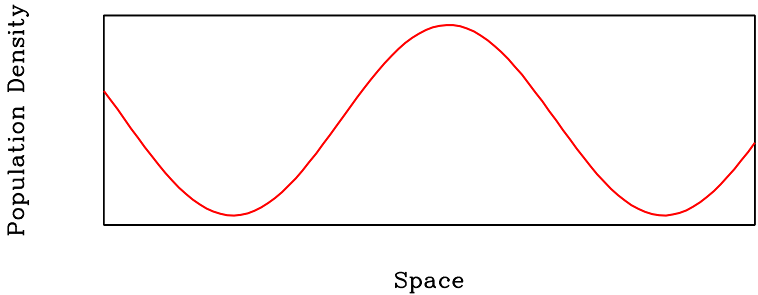

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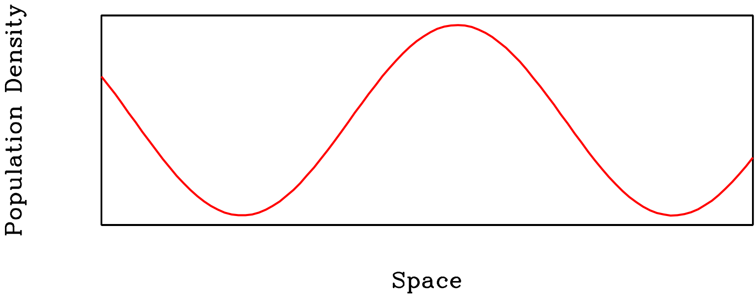

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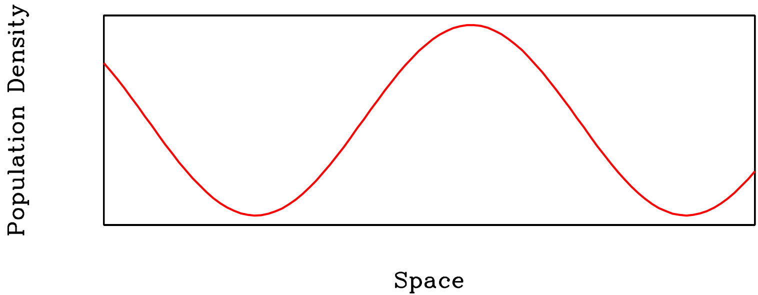

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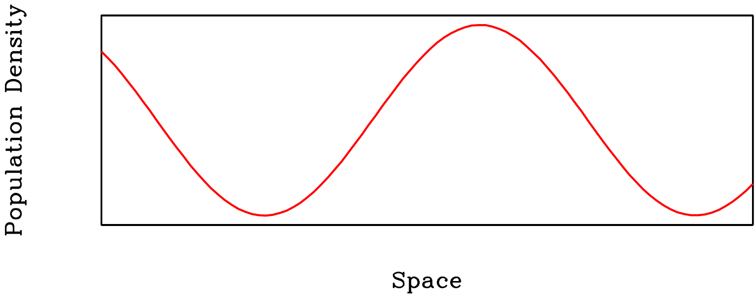

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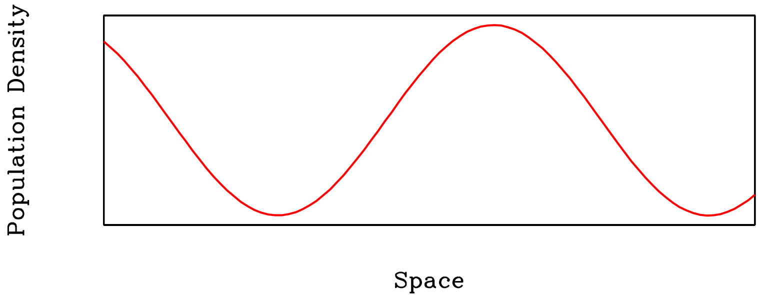

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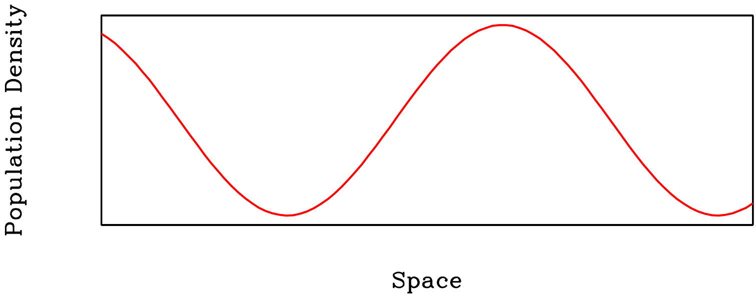

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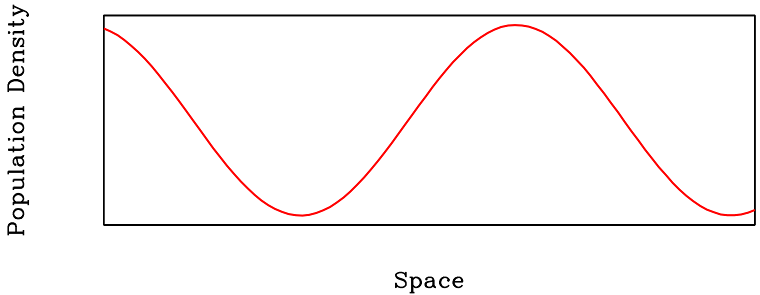

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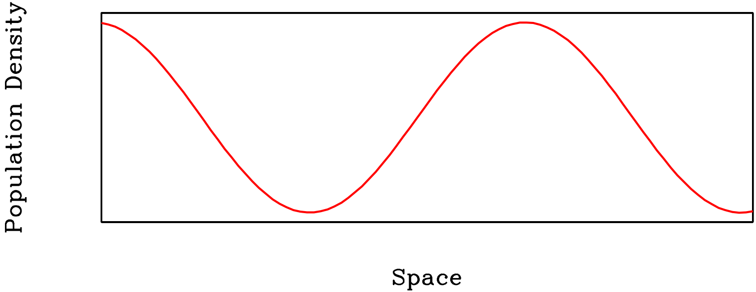

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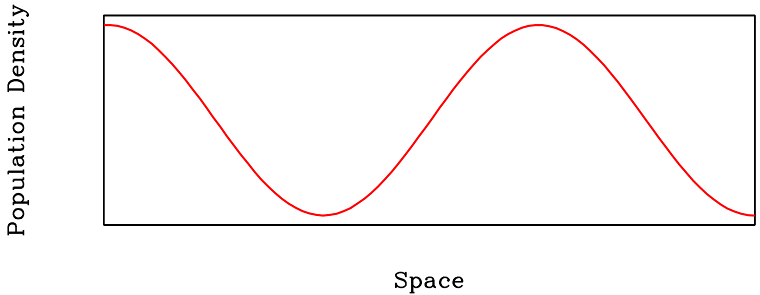

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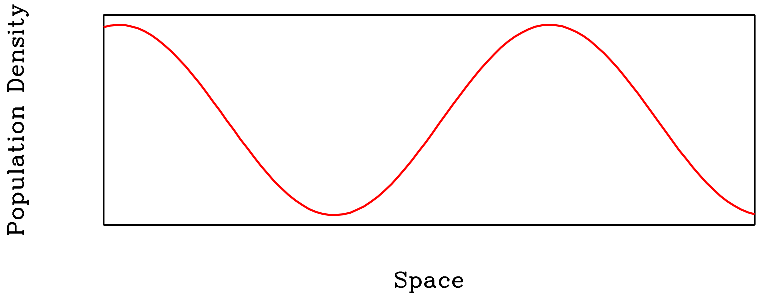

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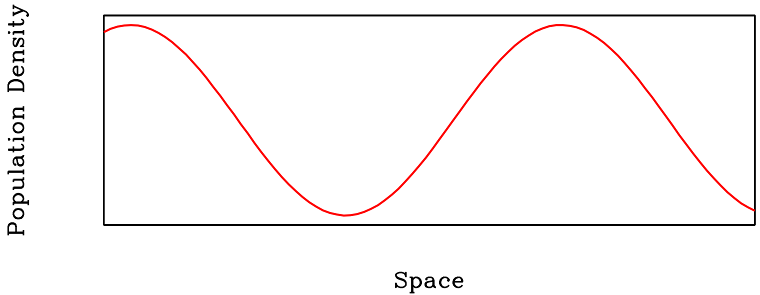

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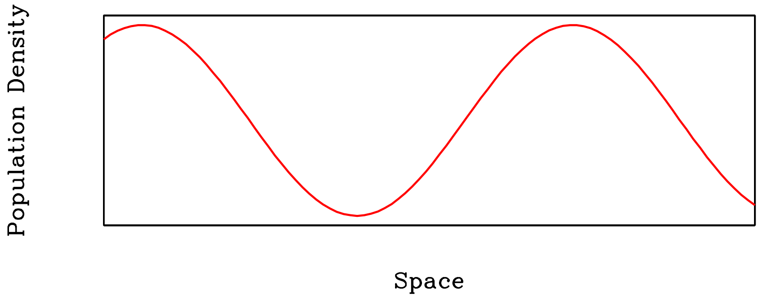

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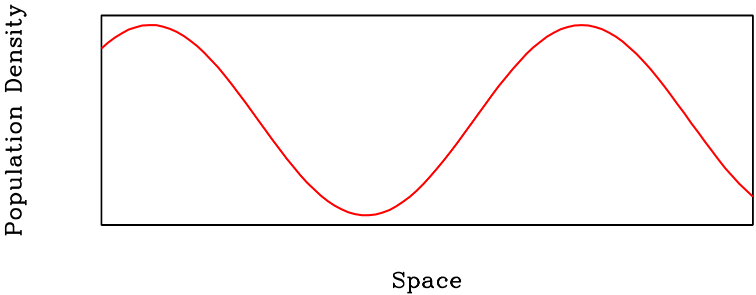

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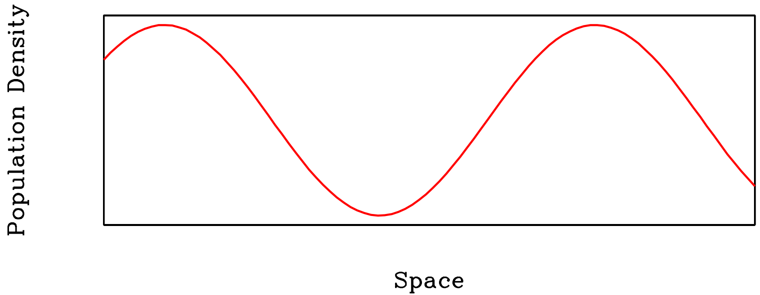

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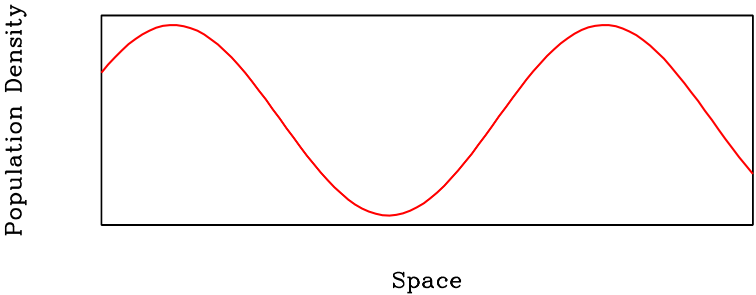

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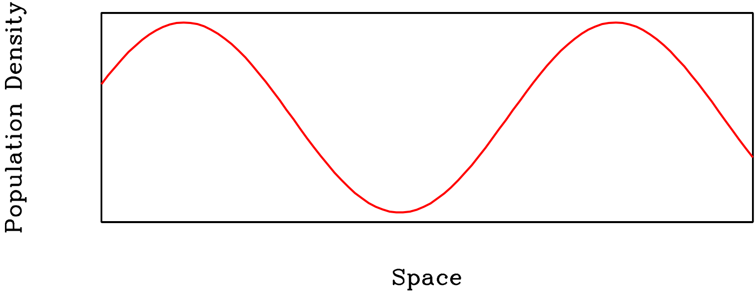

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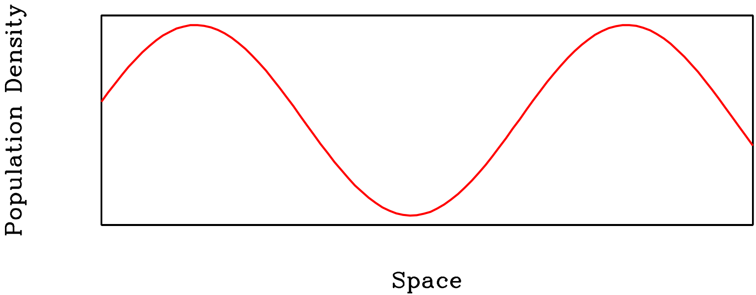

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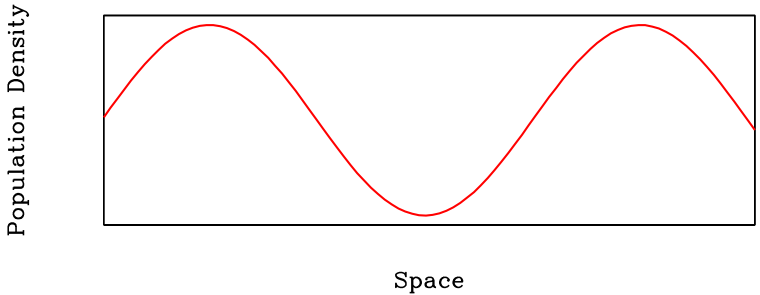

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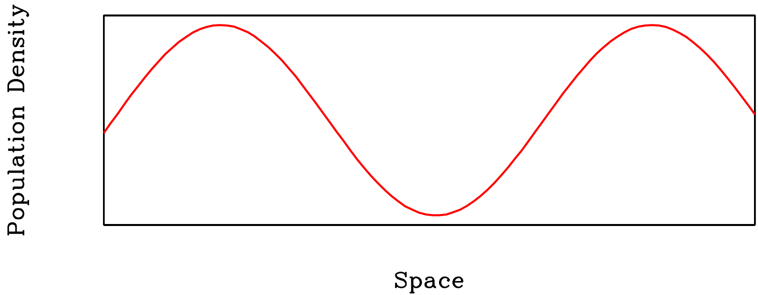

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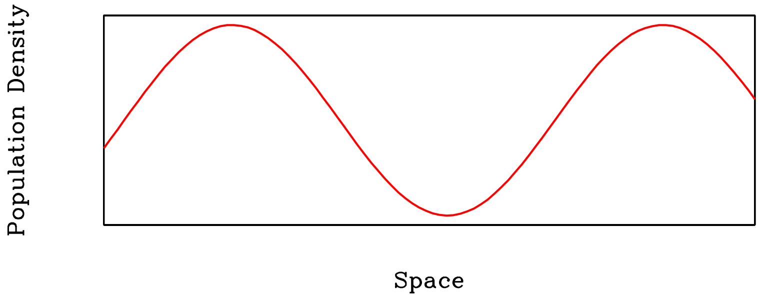

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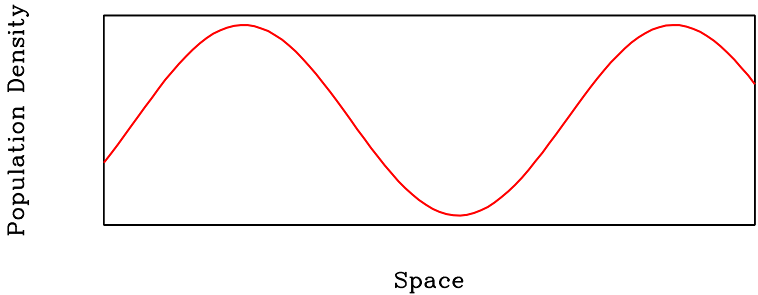

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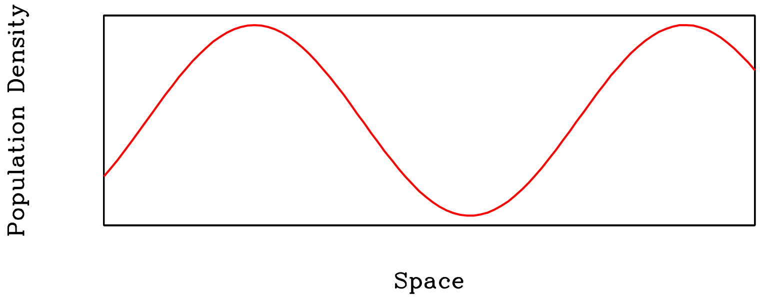

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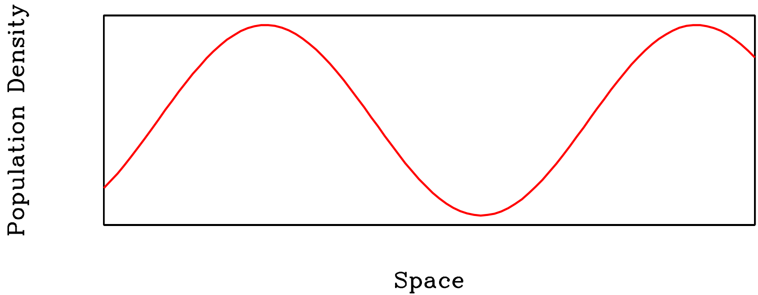

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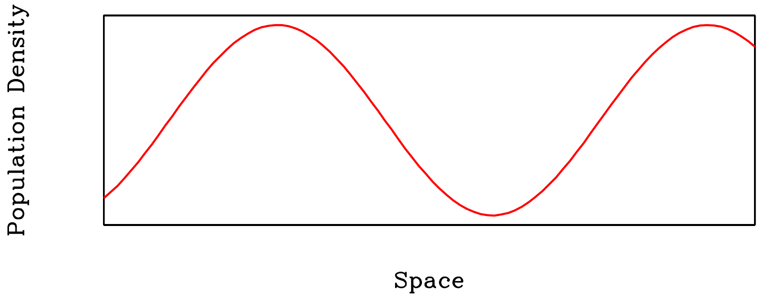

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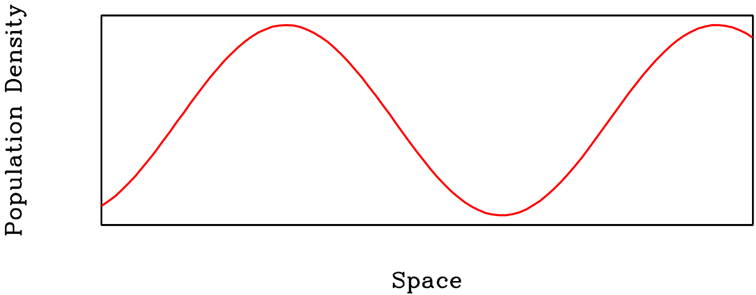

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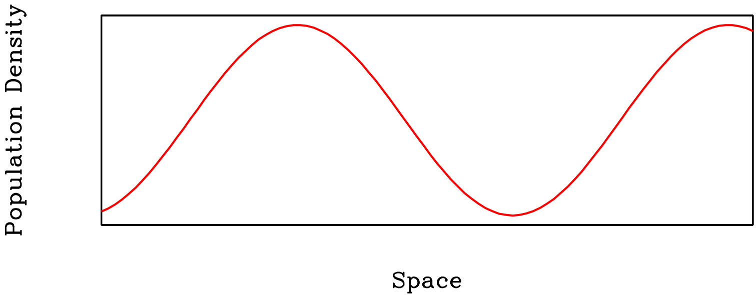

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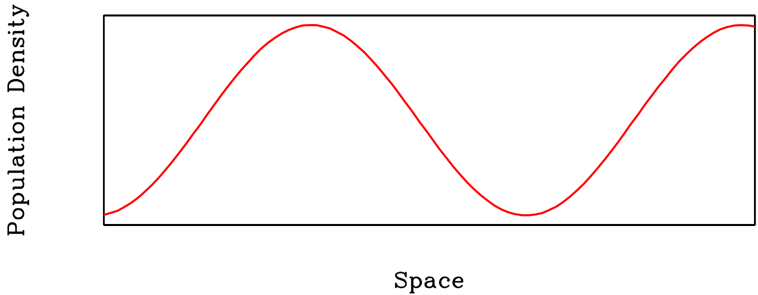

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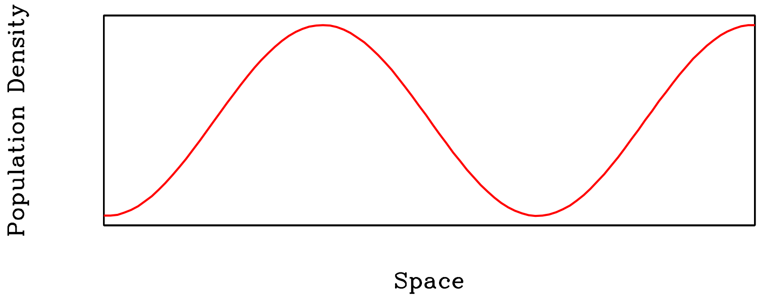

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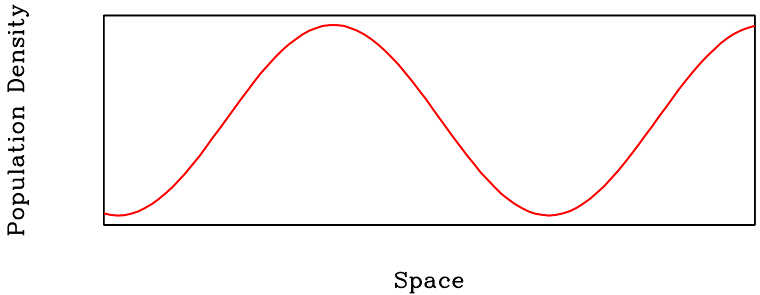

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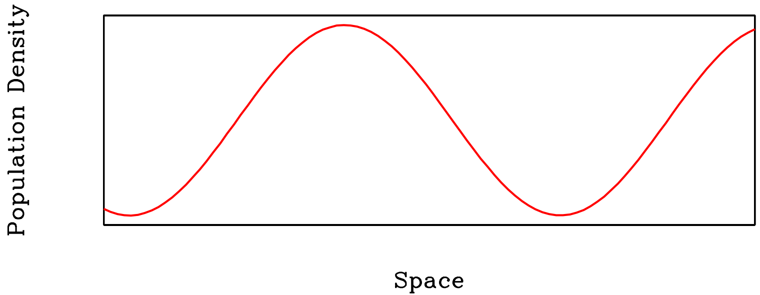

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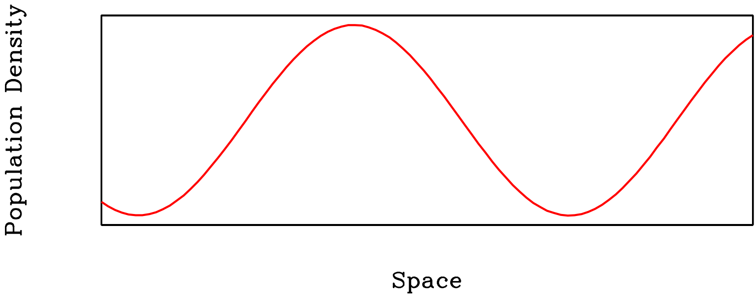

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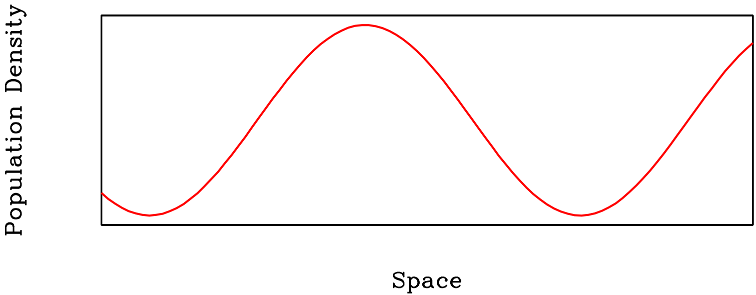

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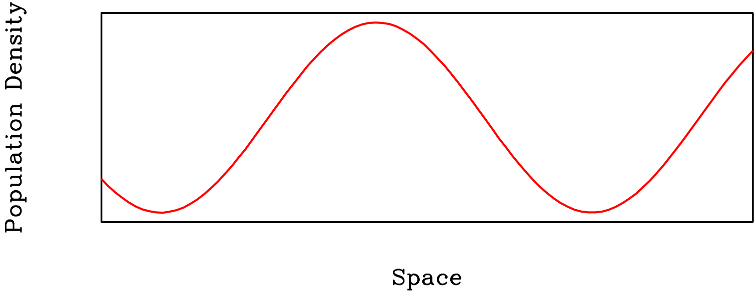

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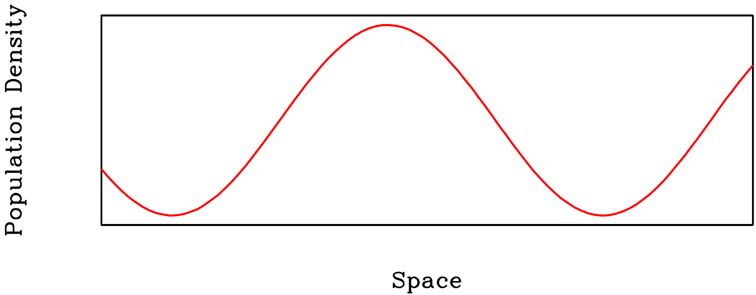

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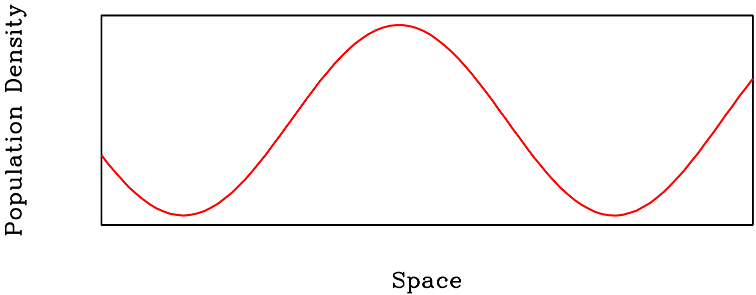

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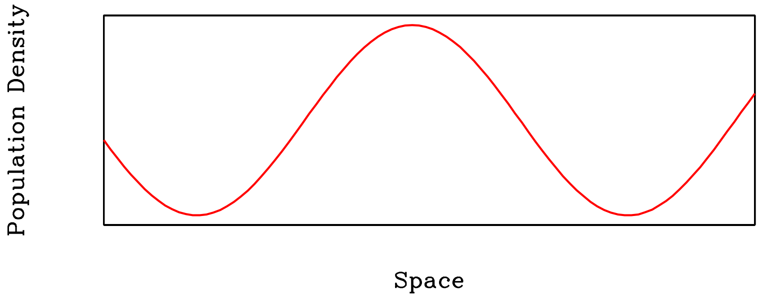

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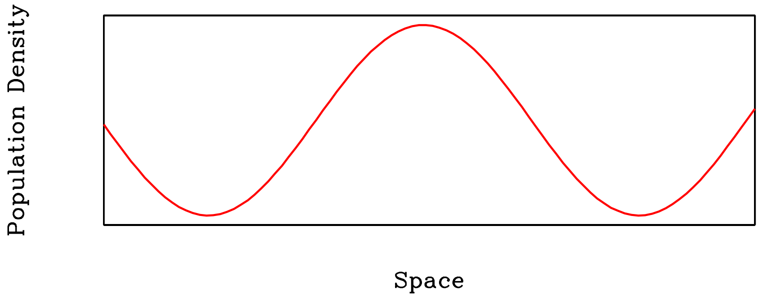

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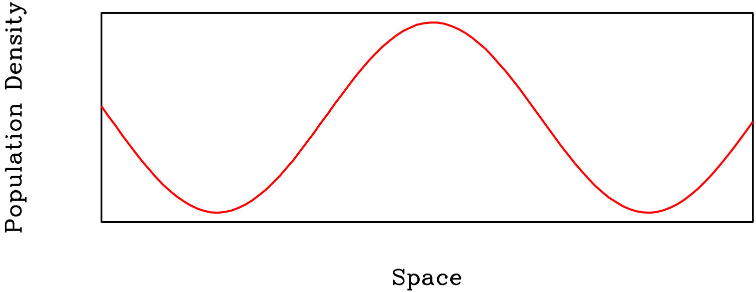

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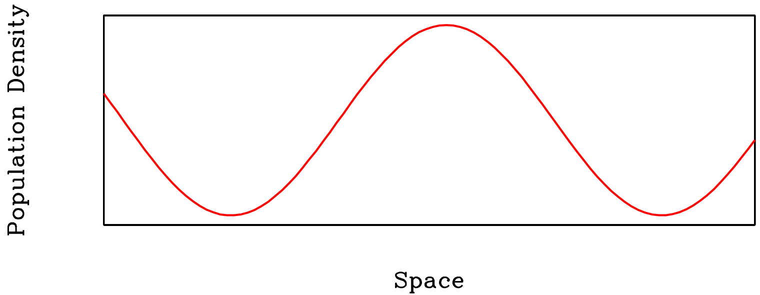

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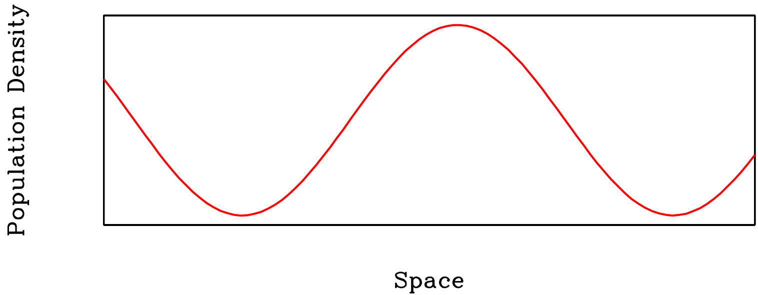

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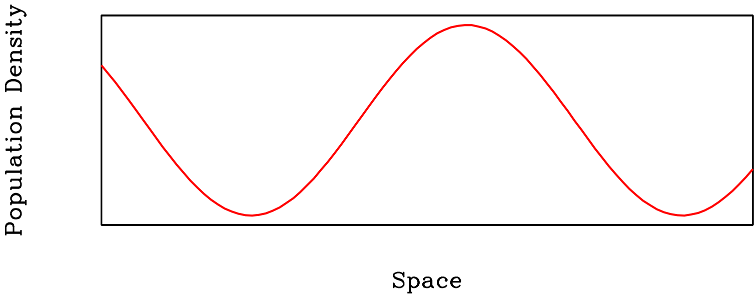

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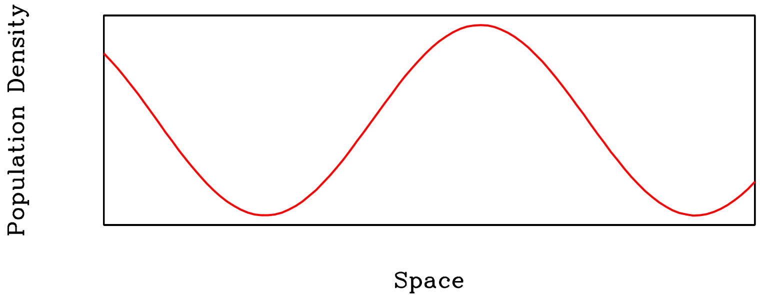

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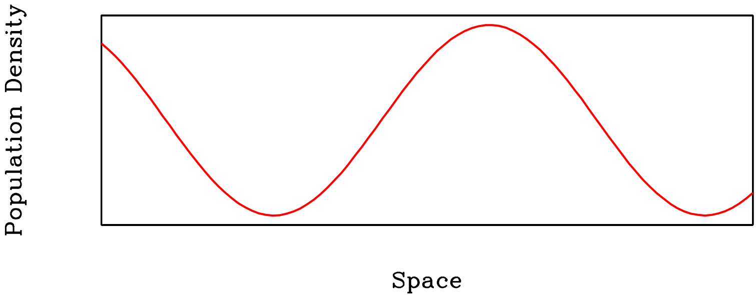

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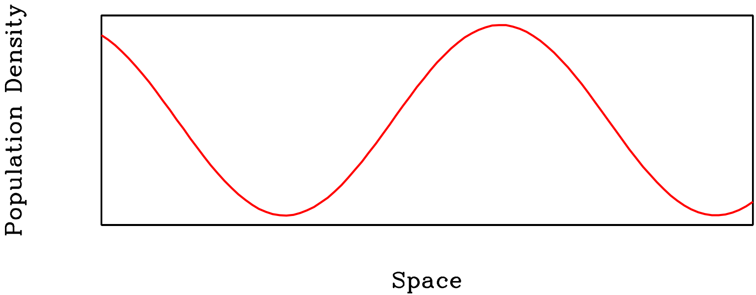

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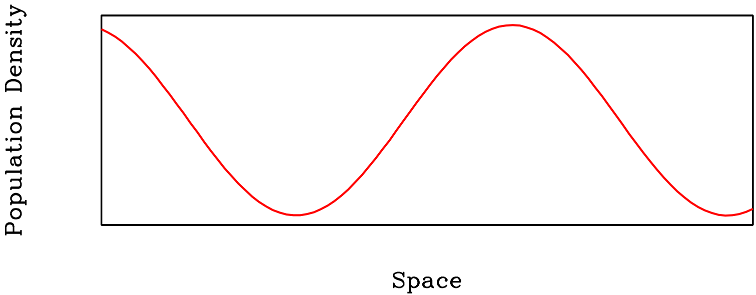

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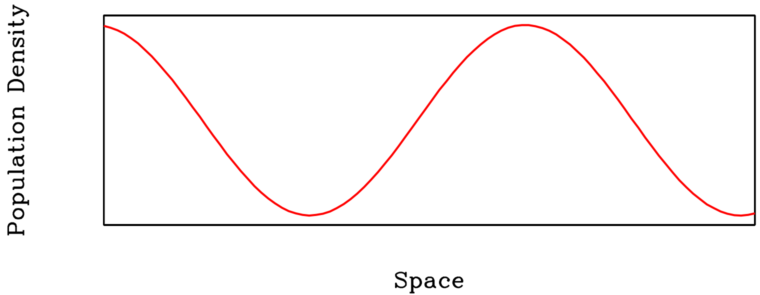

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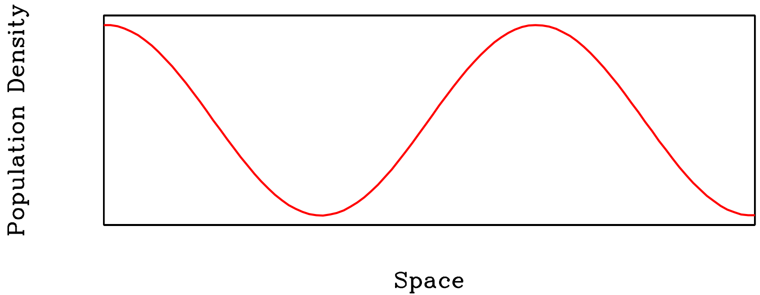

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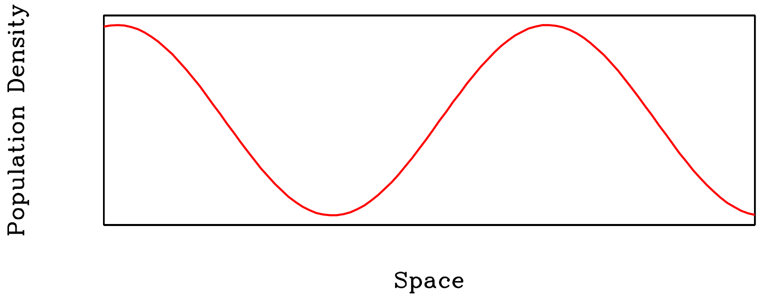

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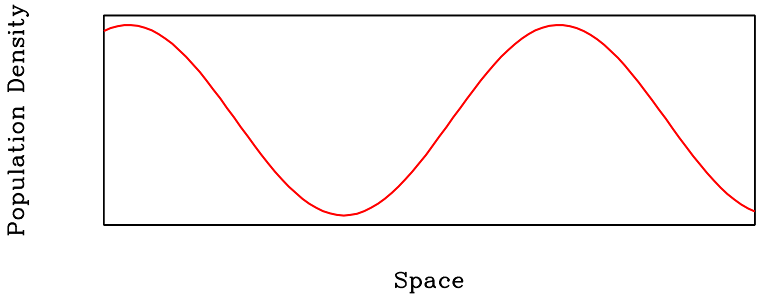

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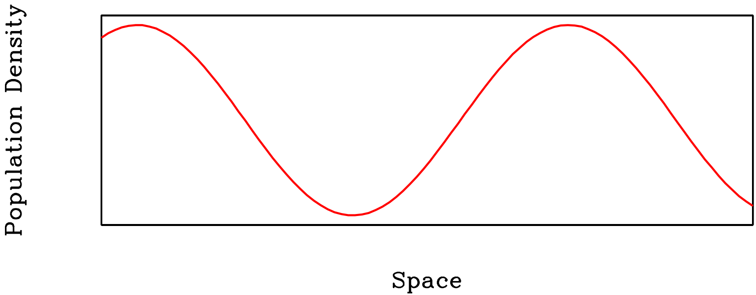

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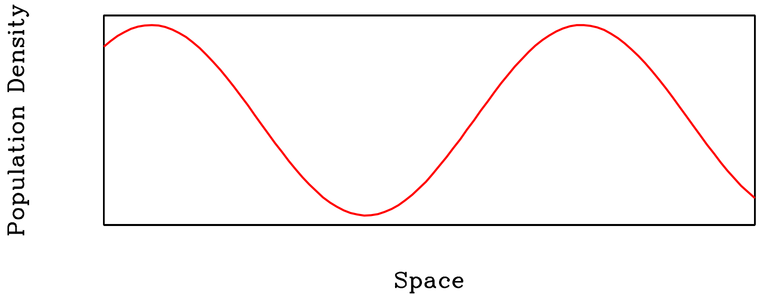

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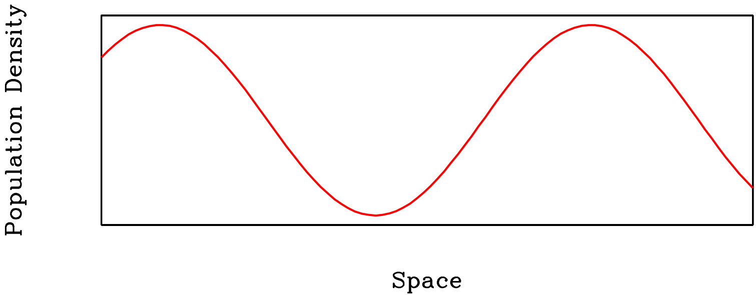

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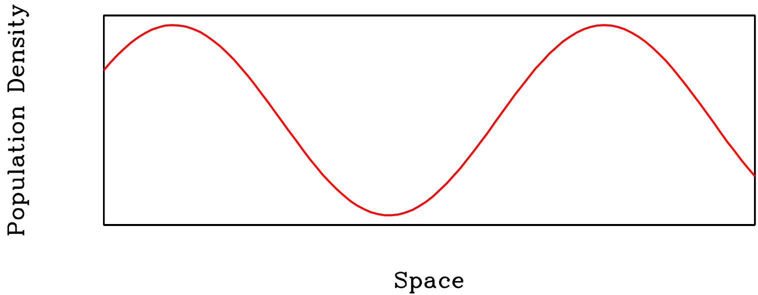

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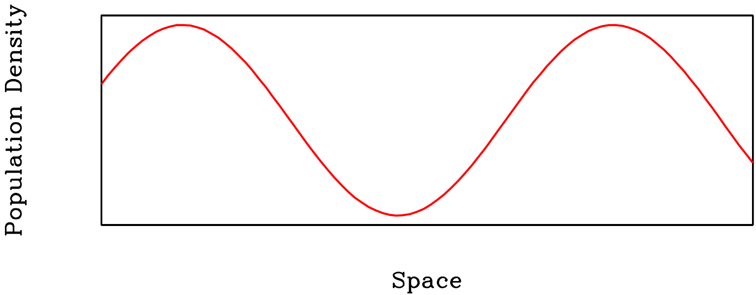

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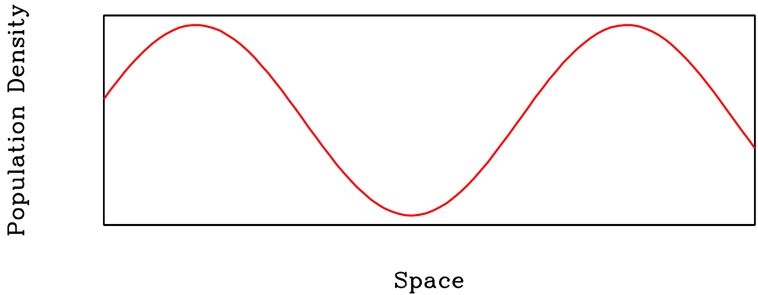

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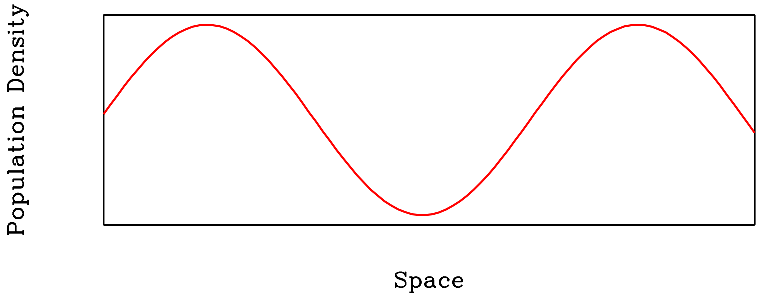

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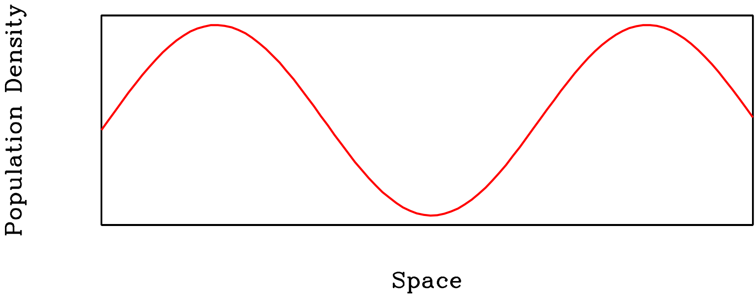

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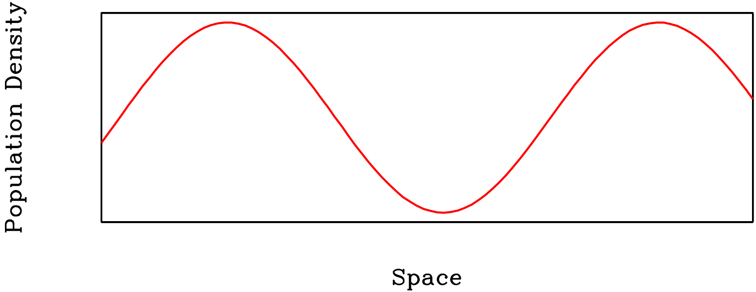

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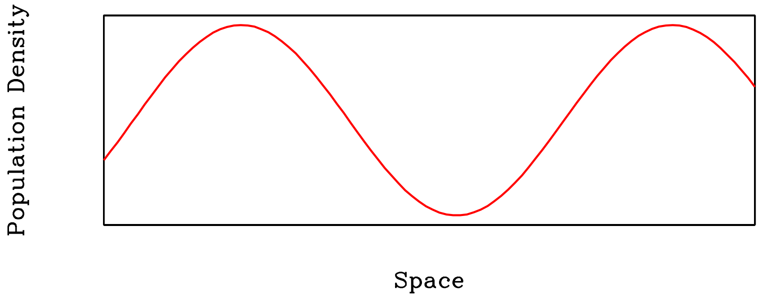


What is a Wavetrain?

A wavetrain is a soln of form $f(x \pm st)$, with $f(.)$ periodic.

There is an extensive literature on wavetrains
in oscillatory reaction-diffusion equations

$$\begin{aligned}\partial u/\partial t &= D_u\,\partial^2 u/\partial x^2 + f(u,v)\\ \partial v/\partial t &= D_v\,\partial^2 v/\partial x^2 + \underbrace{g(u,v)}_{\text{kinetics have a stable limit cycle}}\end{aligned}$$

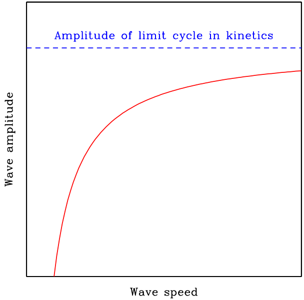
What is a Wavetrain?

A wavetrain is a soln of form $f(x \pm st)$, with $f(.)$ periodic.

An oscillatory reaction-diffusion system has a one-parameter family of wavetrain solutions

(if the diffusion coefficients are sufficiently close to one another)

(Kopell, Howard (1973) *Stud Appl Math* 52:291)



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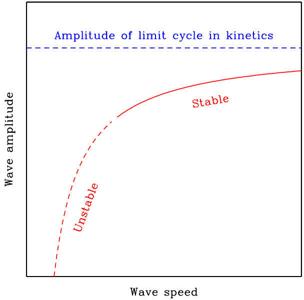
Some members of the wavetrain family are stable as solutions of the partial differential equations, while others are unstable.



The Wavetrain Band

The invasion process selects a particular member of the wavetrain family (Sherratt (1998) *Physica D* 117:145).

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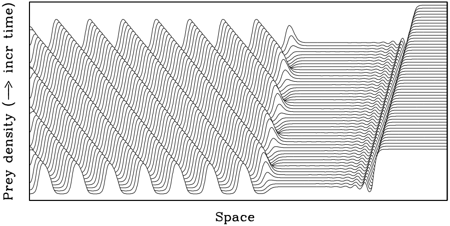
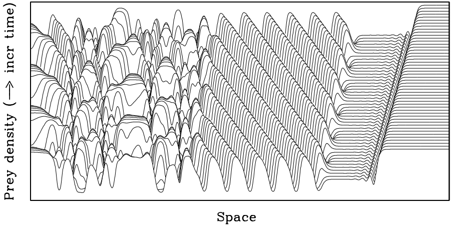
For these parameters, the selected wavetrain is stable.



The Wavetrain Band

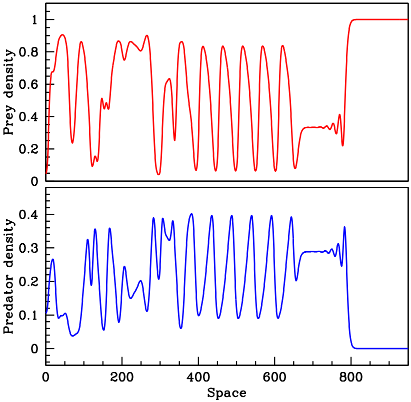
The invasion process selects a particular member of the wavetrain family (Sherratt (1998) *Physica D* 117:145).

A "wavetrain band" occurs when the selected wavetrain is unstable.



Question: what is the wavetrain band width?

The Wavetrain Band: Animation



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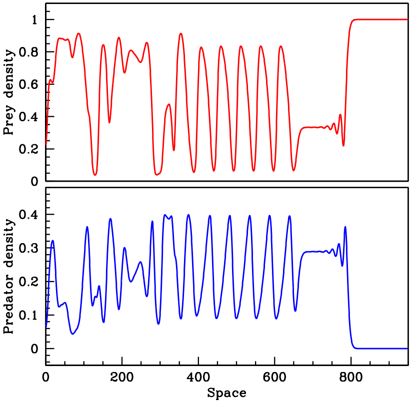

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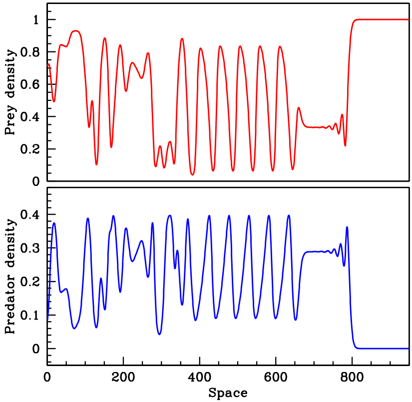

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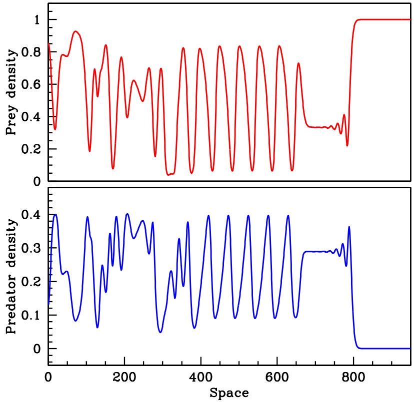

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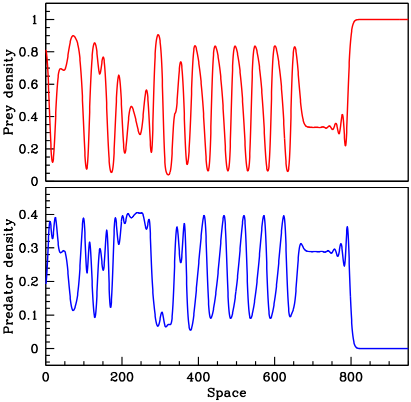

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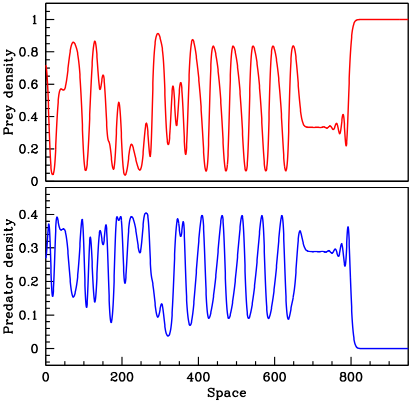

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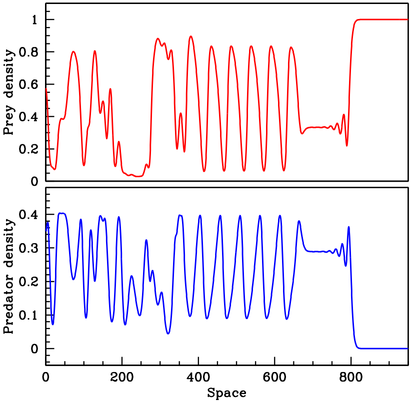

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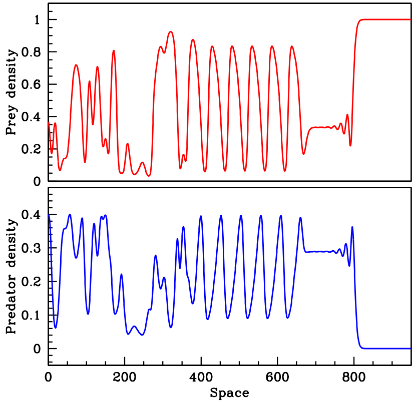

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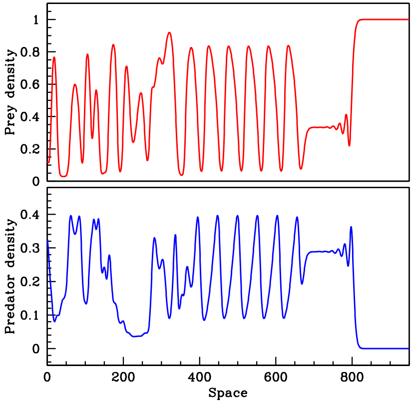

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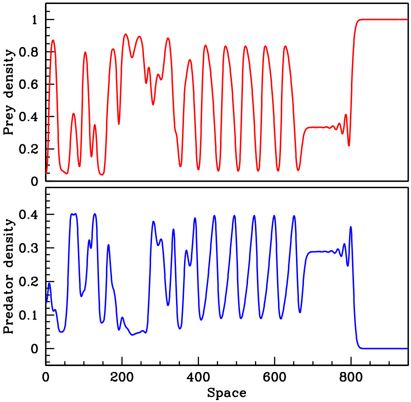

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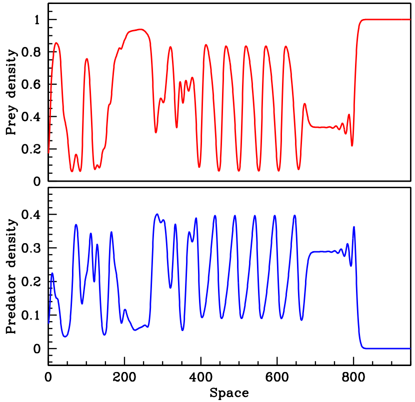

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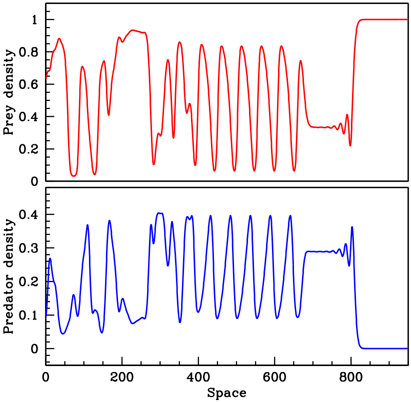

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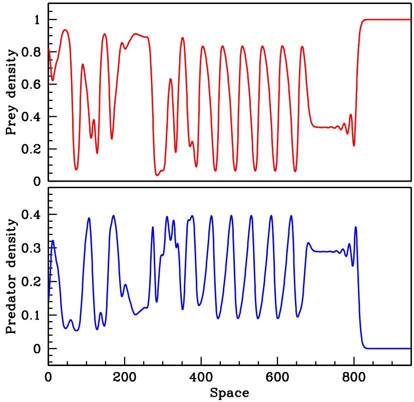

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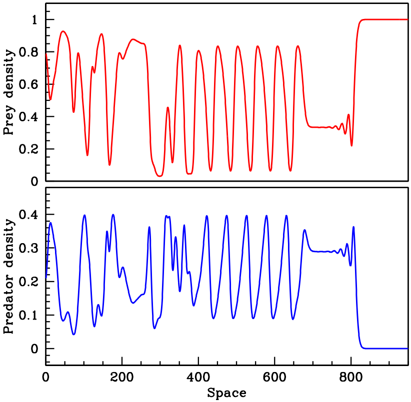

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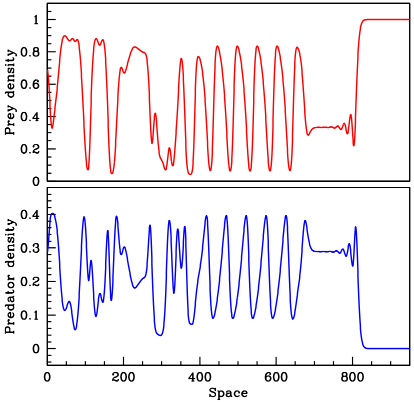

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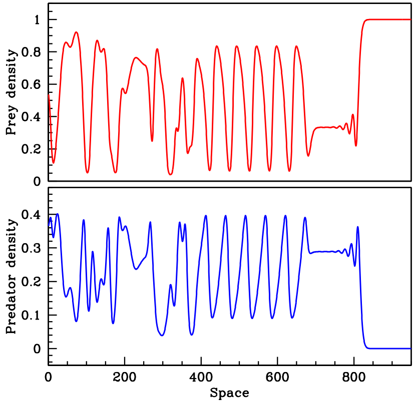

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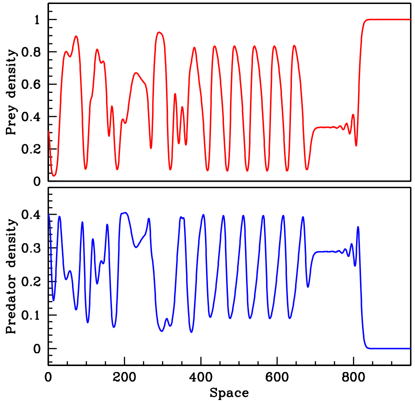

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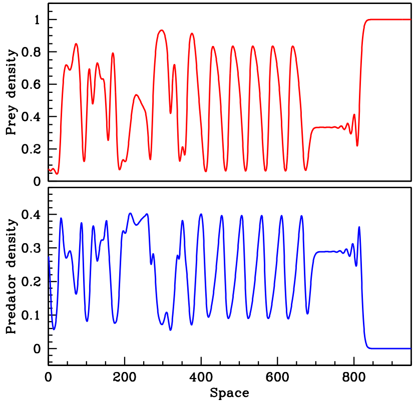

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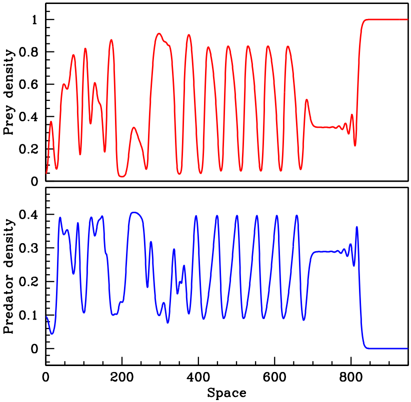

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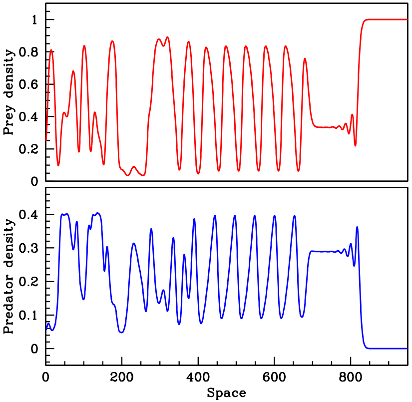

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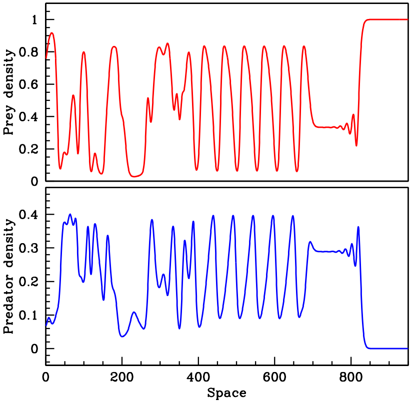

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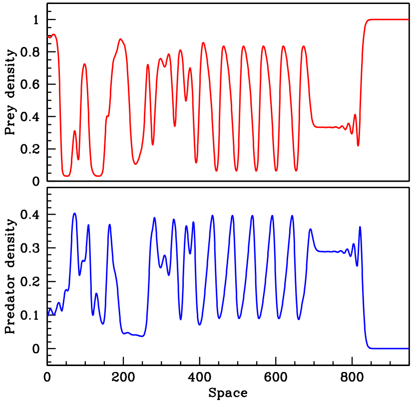

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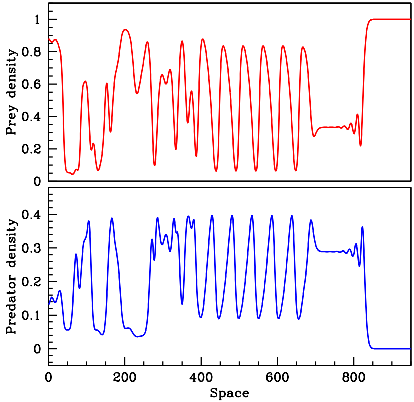

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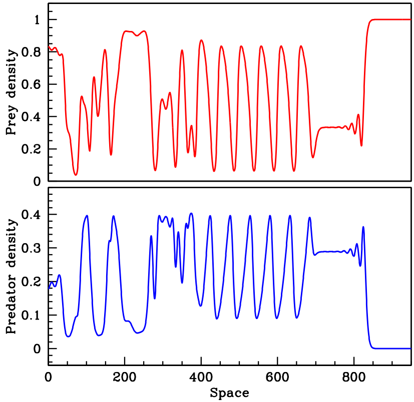

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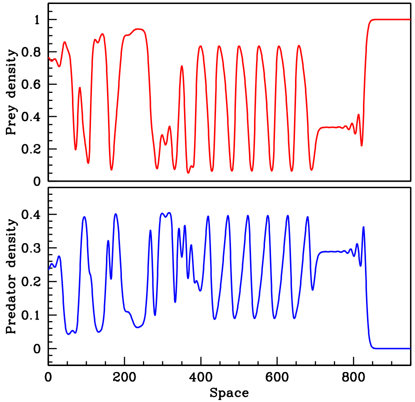

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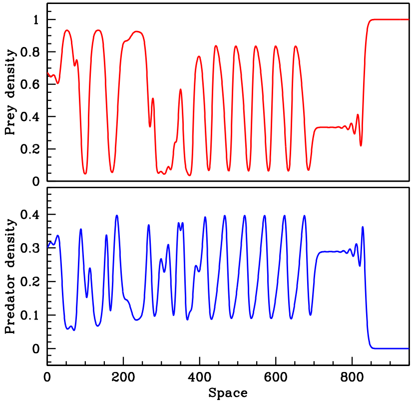

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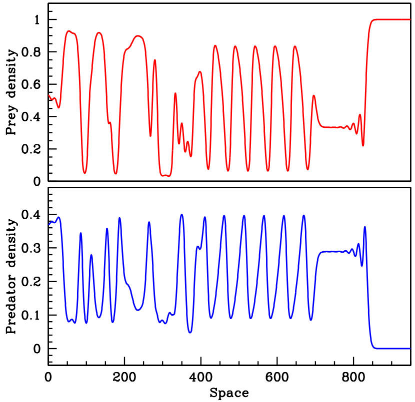

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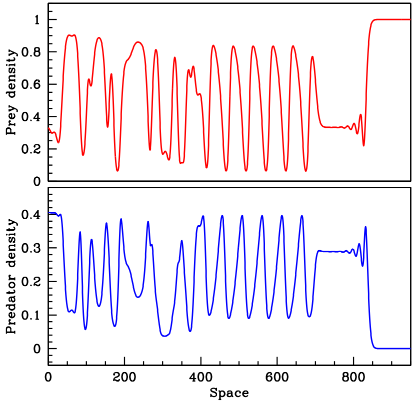

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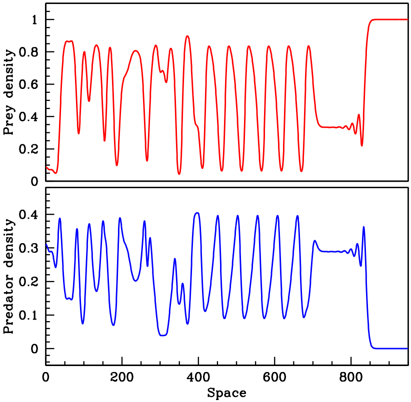

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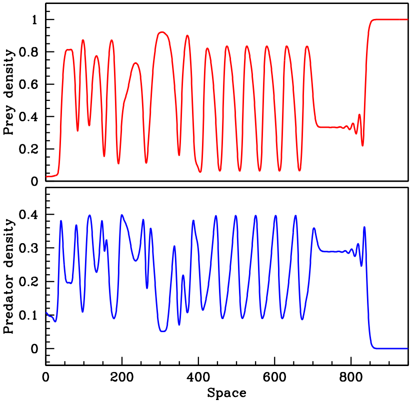

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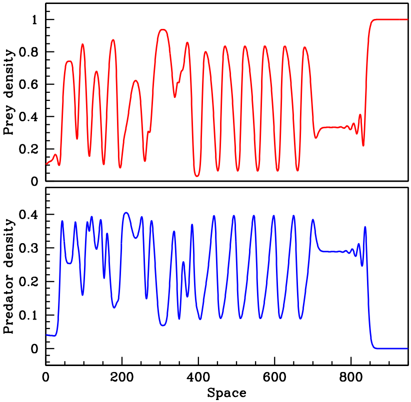

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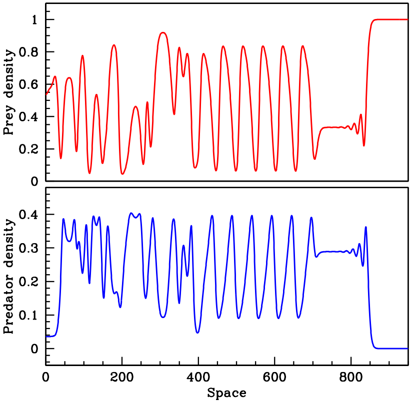

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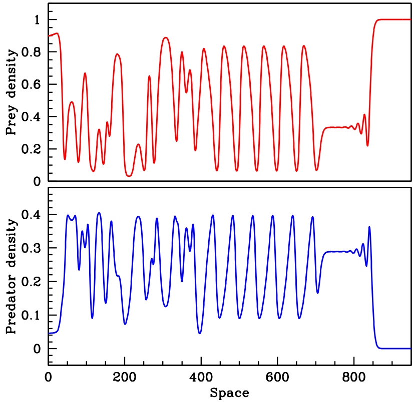

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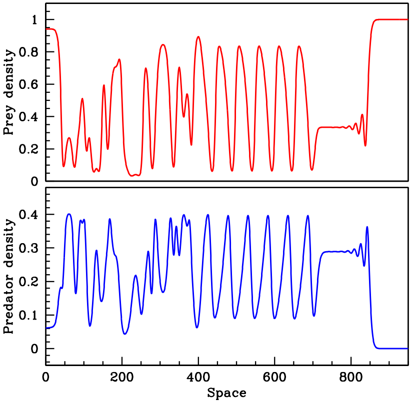

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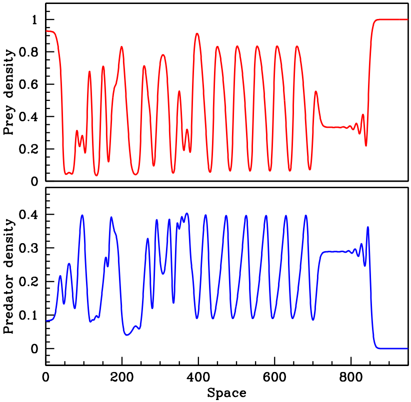

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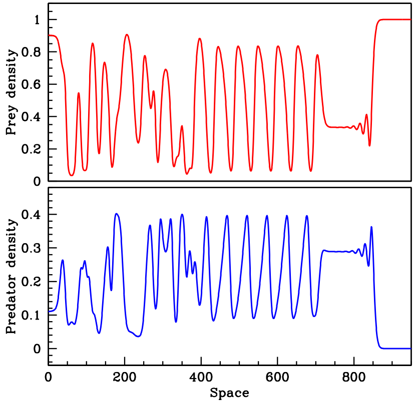

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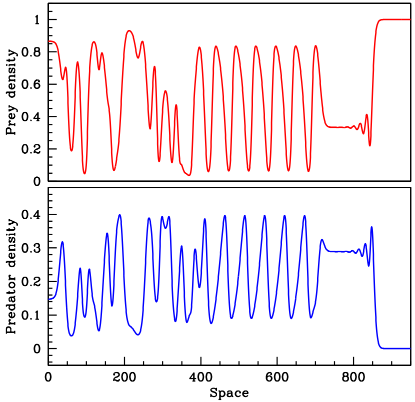

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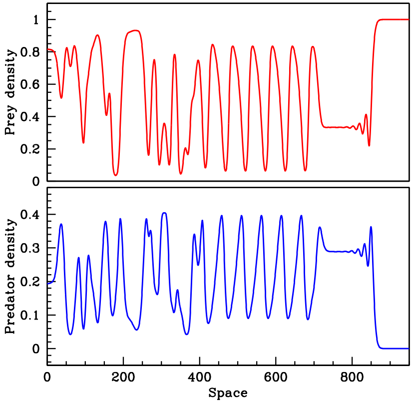

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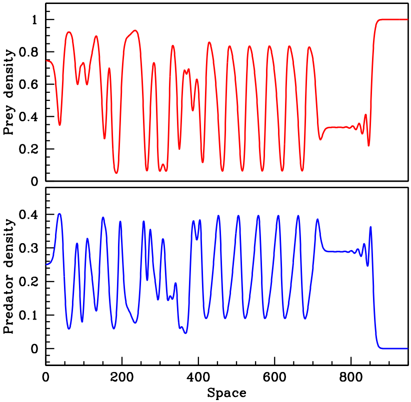

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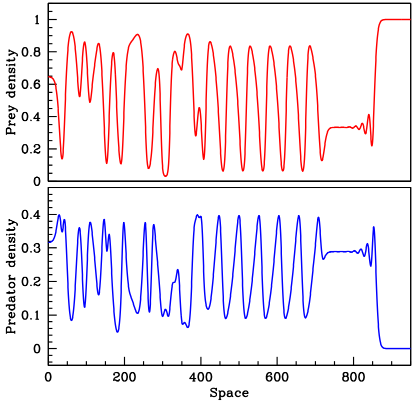

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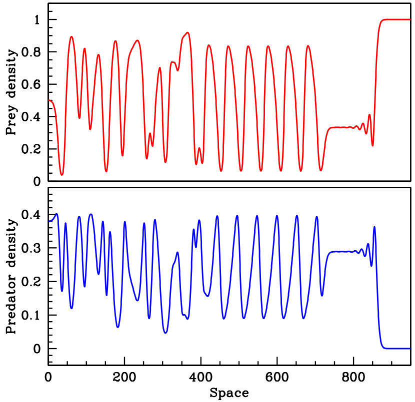

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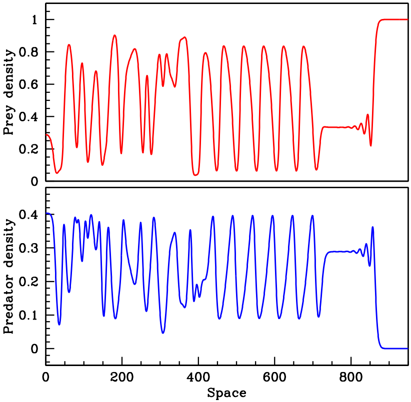

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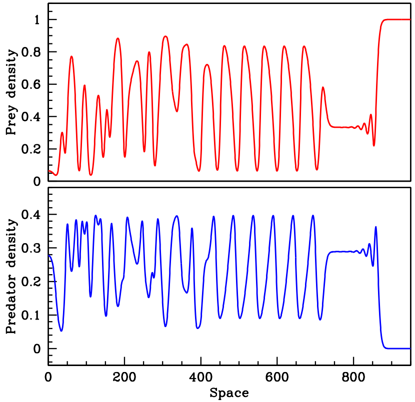

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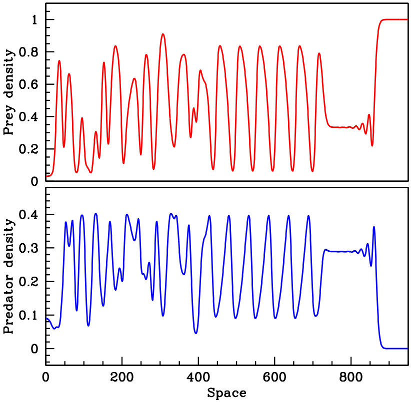

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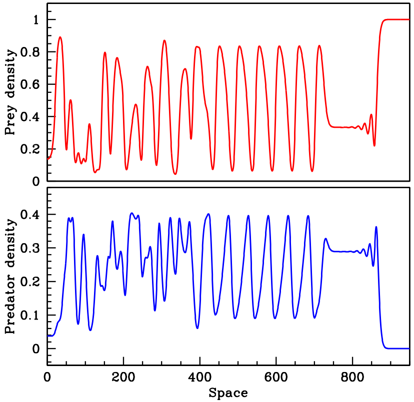

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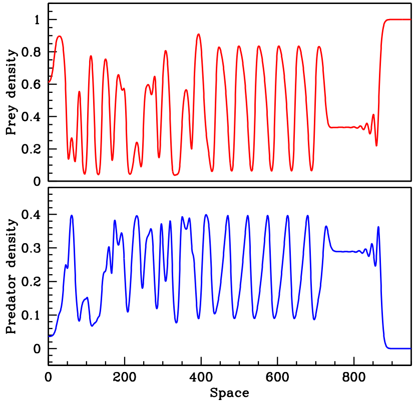

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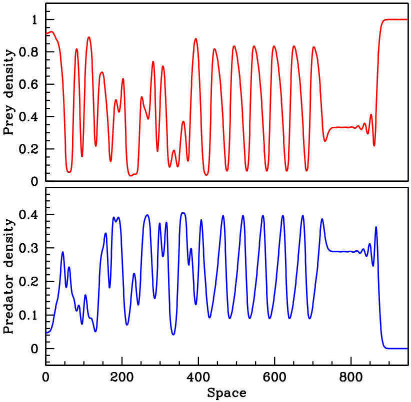

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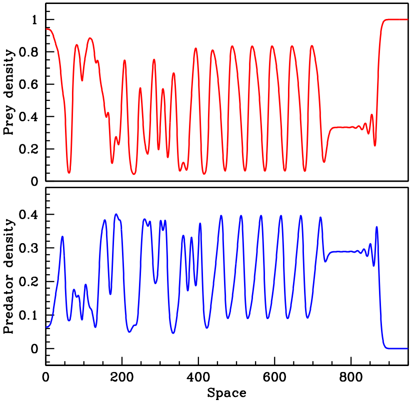

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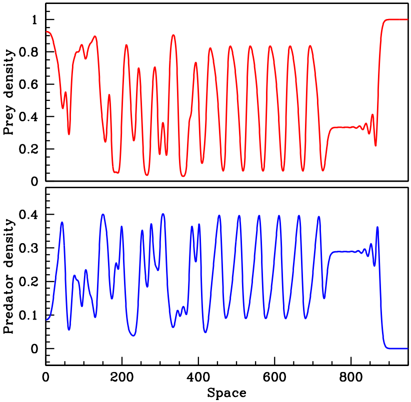
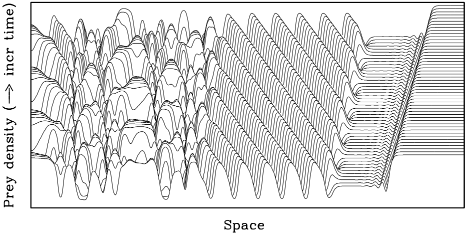


Outline

Using a Normal Form Equation

I consider parameter sets close to the Hopf bifurcation in the coexistence steady state.



Then one can use the normal form (amplitude equation).

The Complex Ginzburg-Landau Equation

The appropriate normal form (amplitude equation) is the CGLE

$$A_t = (1 + \mathrm{i}b)A_{xx} + A - (1 + \mathrm{i}c)|A|^2 A.$$

$$\begin{aligned} \text{i.e. } u_t &= u_{xx} - bv_{xx} + (1 - r^2)u + cr^2 v \\ v_t &= bu_{xx} + v_{xx} - cr^2 u + (1 - r^2)v \end{aligned}$$

Here $A = u + \mathrm{i}v$, $r = \sqrt{u^2 + v^2} = |A|$,
and b and c are functions of the ecological parameters.

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The wavetrain family is

$$A = \sqrt{1-Q^2}\exp\left[\mathrm{i}\left\{Qx + (c - bQ^2 - cQ^2)t\right\}\right] \quad (-1 < Q < 1)$$

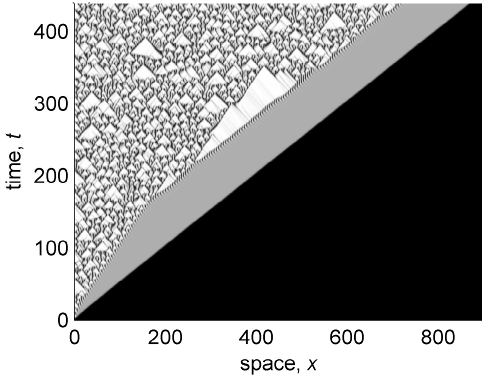
Invasion in the CGLE

Domain: 0 < x < xmax

Initial conditions: u = 0
v = 0
with a small perturbation near x=0

Boundary conditions: zero flux (i.e. zero Neumann)

Invasion in the CGLE



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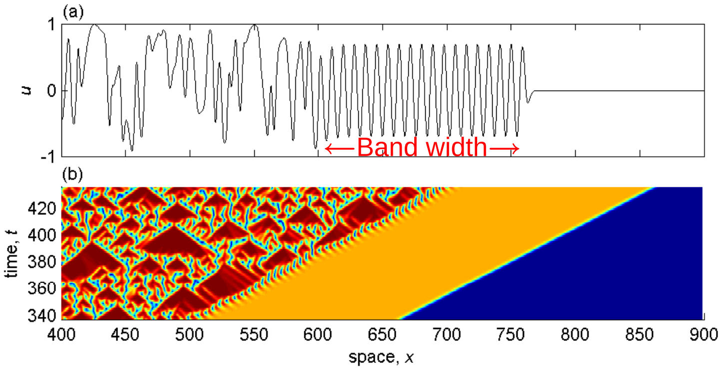
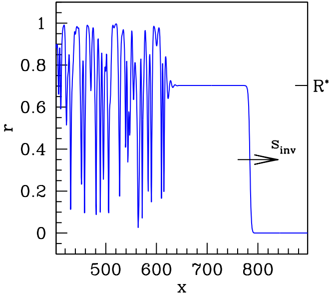


Outline

The Selected Wavetrain Amplitude

The form of the invasion solution is



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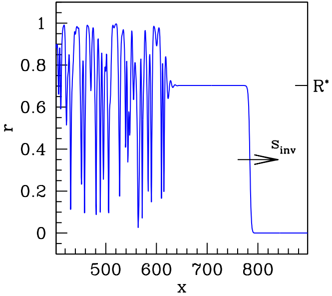
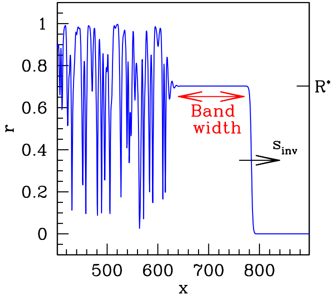


The value of R^* can be calculated exactly, as a function of b and c.

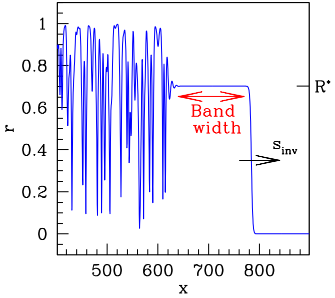
The Band Width Question

- Our question is: how wide is the region in which $r \approx R^*$?



The Band Width Question

- Our question is: how wide is the region in which $r \approx R^*$?
- We define its left-hand edge as where unstable linear modes generated by the invasion front are amplified by a factor $\mathcal{F}$



- The band width has the form $\underbrace{\log(\mathcal{F})}_{\text{arbitrary}} \cdot \underbrace{\mathcal{W}(b,c)}_{\text{"band width coefficient"}}$

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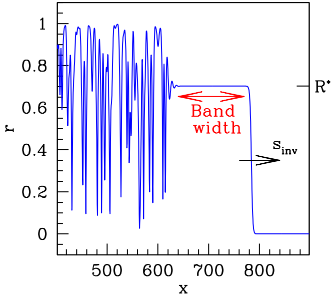

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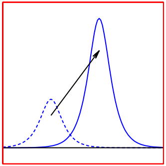
Outline

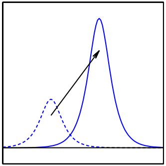
Convective and Absolute Stability

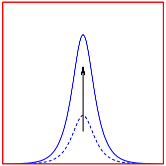
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving.
  This is “convective instability”.



Convective and Absolute Stability

- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".
- Alternatively, a solution can be unstable with perturbations growing without moving. This is "absolute instability".





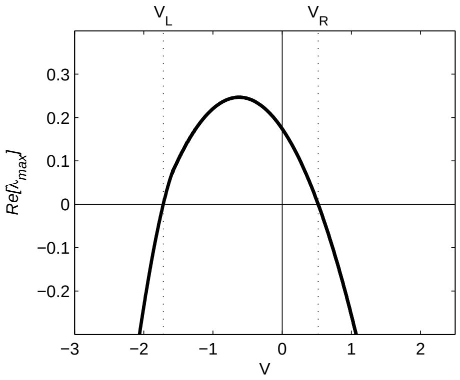
Absolute Stability in a Moving Frame of Reference

Absolute stability refers to the growth/decay of stationary perturbations.

We must consider the growth/decay of perturbations moving with a specified velocity V, i.e. absolute stability in a frame of reference moving with velocity V.

$$
\begin{aligned}
\text{Define } \lambda_{max}(V) &= \text{temporal eigenvalue of the most unstable linear mode} \\
\nu_{max}(V) &= \text{the corresponding spatial eigenvalue}
\end{aligned}
$$

Absolute Stability in a Moving Frame of Reference



Calculation of $\lambda_{max}(V)$

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2. $\mathcal{D}$ is a quartic polynomial in ν, roots $\nu_1, \ldots, \nu_4$ with $\mathrm{Re}\,\nu_1 \geq \mathrm{Re}\,\nu_2 \geq \mathrm{Re}\,\nu_3 \geq \mathrm{Re}\,\nu_4$
3. “Absolute spectrum” $:= \{\lambda | \mathrm{Re}\,\nu_2 = \mathrm{Re}\,\nu_3\}$
   $\lambda_{max}(V) = \lambda$ with max Re in the absolute spectrum

The Significance of $\mathrm{Re}\,\nu_2 = \mathrm{Re}\,\nu_3$

(Worledge, Knobloch, Tobias, Proctor (1997) *Proc. R. Soc. Lond. A* 453:119)

- Consider the linearised r–θ PDEs on $-\ell < x < +\ell$, ℓ large.
- For given λ, these equations have the solution

$$\underbrace{(\tilde{r},\tilde{\theta})}_{\text{Linearisation variables}} = e^{\lambda t} \sum_{j=1}^{4} \underbrace{(\bar{r}_j,\bar{\theta}_j)}_{\text{eigen-vector}} k_j e^{\nu_j x}$$

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- Suppose that both boundary conditions are $\tilde{r}=0$, $\tilde{\theta}_x=0$
- If $\mathrm{Re}\,(\nu_j)$'s are distinct then since ℓ is large

$$\sum_{j=1}^{2}(\bar{r}_j,\nu_j\bar{\theta}_j)k_j e^{+\nu_j\ell} = \sum_{j=3}^{4}(\bar{r}_j,\nu_j\bar{\theta}_j)k_j e^{-\nu_j\ell} = (0,0).$$

Typically this has no non-trivial solutions for the k_j's

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- $\mathrm{Re}\,(\nu_1) = \mathrm{Re}\,(\nu_2)$ and/or $\mathrm{Re}\,(\nu_3) = \mathrm{Re}\,(\nu_4) \Rightarrow$ no change:

$$\sum_{j=1}^{2}(\bar{r}_j,\nu_j\bar{\theta}_j)k_j e^{+\nu_j\ell} = \sum_{j=3}^{4}(\bar{r}_j,\nu_j\bar{\theta}_j)k_j e^{-\nu_j\ell} = (0,0).$$

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- Suppose that both boundary conditions are $\tilde{r} = 0$, $\tilde{\theta}_x = 0$
- But if $\mathrm{Re}\,(\nu_2) = \mathrm{Re}\,(\nu_3)$ then

$$\sum_{j=1}^{3} (\bar{r}_j, \nu_j \bar{\theta}_j) k_j e^{+\nu_j \ell} = \sum_{j=2}^{4} (\bar{r}_j, \nu_j \bar{\theta}_j) k_j e^{-\nu_j \ell} = (0, 0).$$

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Calculation of $\lambda_{max}(V)$

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Tutorial:

`research.microsoft.com/en-us/projects/loptw/tutorial.aspx`

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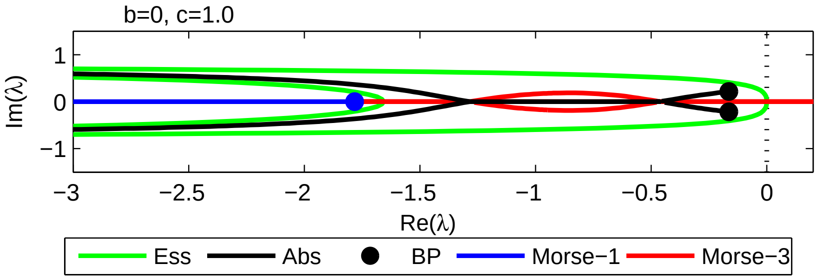

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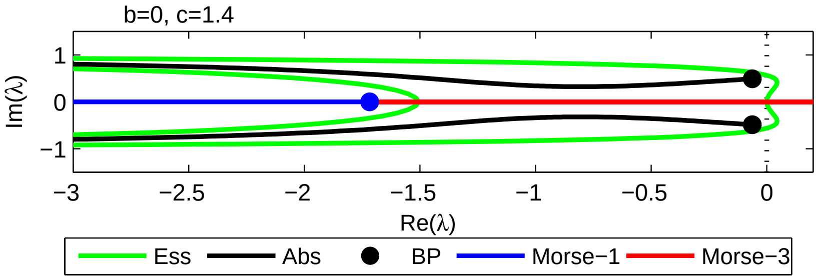

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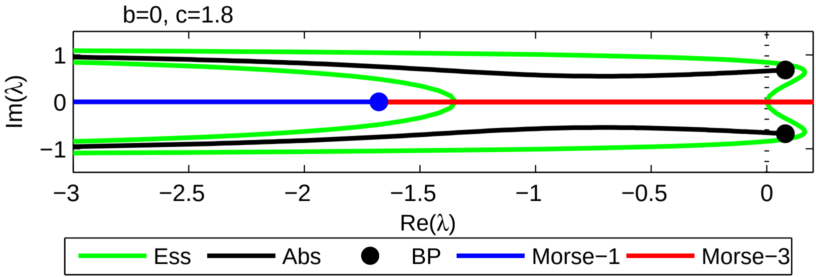

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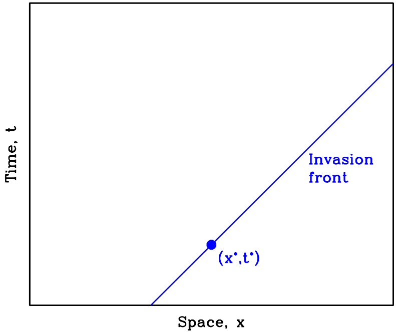
Outline

The Band Width Formula



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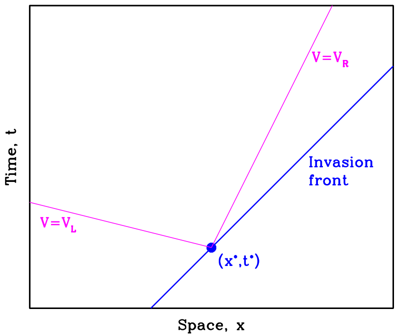

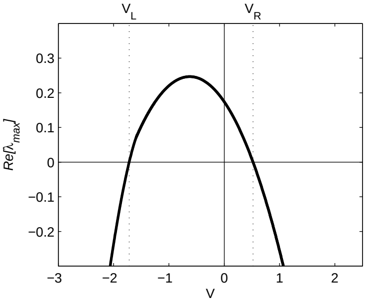

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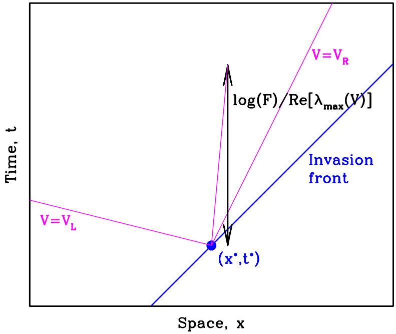
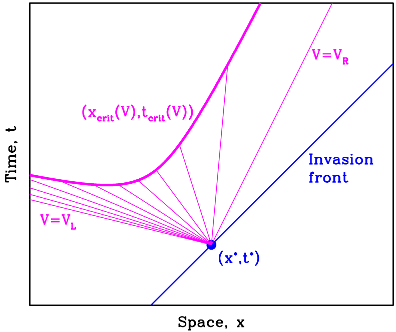


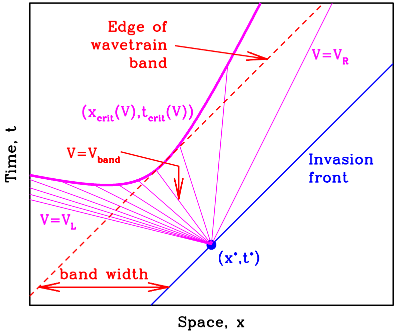
Perturbations moving with velocity V grow as $\exp[\mathrm{Re}\,(\lambda_{max}(V))\cdot t]$

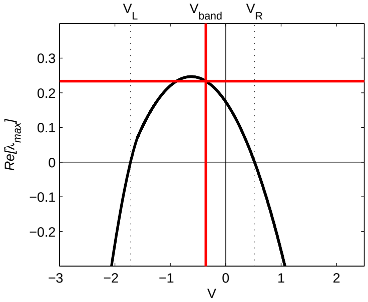
$\Rightarrow$ amplified by the factor $\mathcal{F}$ after time $\log(\mathcal{F})/\mathrm{Re}\,(\lambda_{max}(V))$

The Band Width Formula

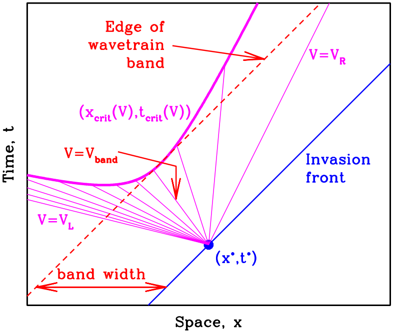


The Band Width Formula





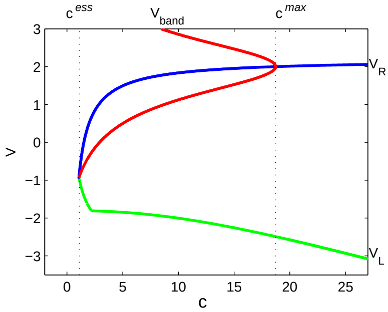
The Band Width Formula



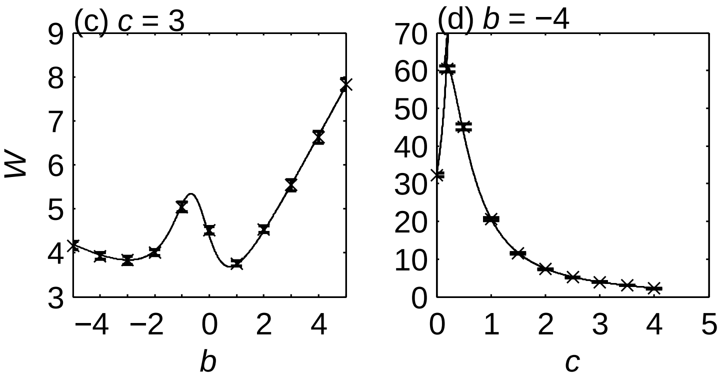
$$\mathcal{W} = 1/\mathrm{Re}\,[\nu_{max}(V_{band})]$$

where $(V_{band} - s_{inv})\mathrm{Re}\,[\nu_{max}(V_{band})] = \mathrm{Re}\,[\lambda_{max}(V_{band})]$

The Form of V_{band} and $\mathcal{W}$

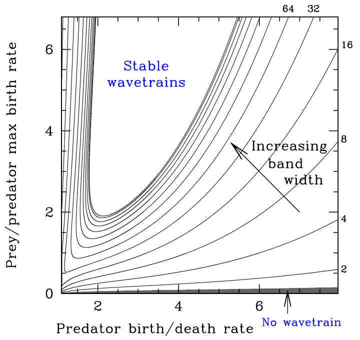


The Form of V_{band} and $\mathcal{W}$



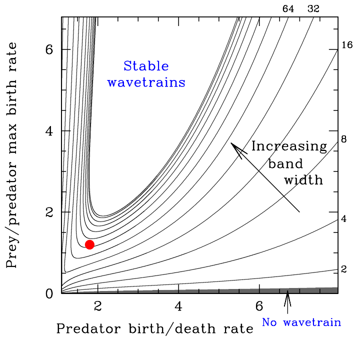
Back to Predator-Prey Invasion

Our formula gives band width vs *b* and *c*.
Normal form calculation gives *b* and *c* vs ecological parameters.



Back to Predator-Prey Invasion

Our formula gives band width vs b and c.
Normal form calculation gives b and c vs ecological parameters.

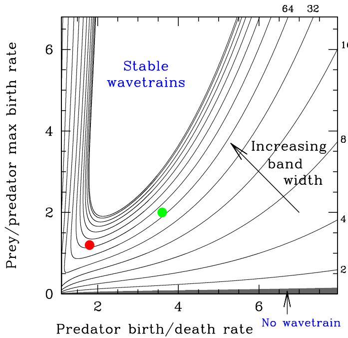


● $=$ weasel–vole demographic parameters, $b = 0$.

5%↑ in vole birth rate
$\Rightarrow$ 22%↑ in band width.

Back to Predator-Prey Invasion

Our formula gives band width vs *b* and *c*.
Normal form calculation gives *b* and *c* vs ecological parameters.



● = plankton demographic parameters, $b = 0$ (*Daphnia pulex–Chlamydomonas reinhardii*).

5.2%↓ in zooplankton birth rate
⇒ doubling of band width.

Ecological Implications of Band Width Sensitivity

- Climate change $\Rightarrow$ more frequent invasions.
- It is known that climate change is significantly affecting the parameters of oscillatory ecological systems (e.g. Ims *et al* (2008) *TREE* 23:79).
- We have shown that band width depends sensitively on ecological parameters.
- Our results suggest that the implications of climate change for *spatio*temporal dynamics may be even more dramatic than for purely temporal behaviour.

References

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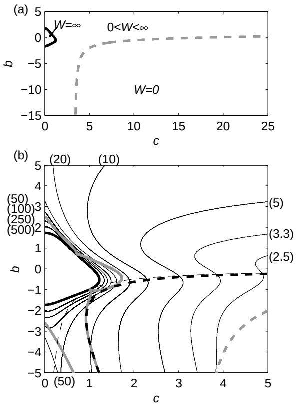
  SIAM J. Appl. Dyn. Systems 8, 1136-1159 (2009).

List of Frames

The Form of V_{band} and $\mathcal{W}$



band width contour
stability bdy for selected wavetrain
abs stab bdy for selected wavetrain in invasion frame of reference
abs stab bdy for selected wavetrain in stationary frame of reference
Benjamin-Feir-Newell curve
abs stab curve