Spatiotemporal Patterns Behind Invasions in Reaction-Diffusion Equations

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Heriot-Watt University

ICMS, Edinburgh, 10 October 2014

This talk can be downloaded from `www.ma.hw.ac.uk/~jas`

This work is in collaboration with:

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(Microsoft Research Ltd., Cambridge)

Jens Rademacher

(Universität Bremen)

Outline

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Cyclic Predator-Prey Systems

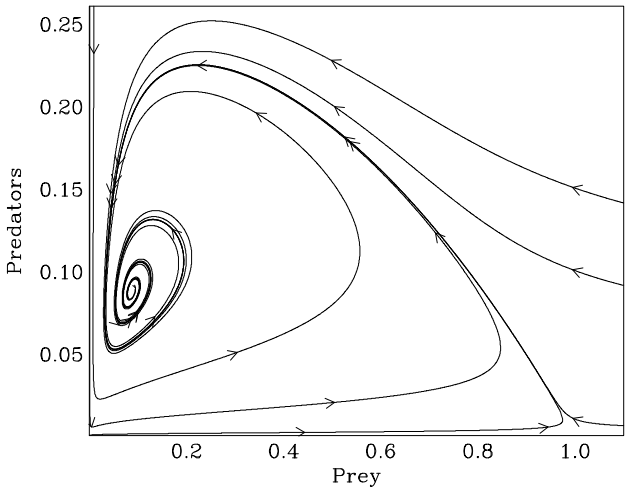
The interaction between a predator population and its prey can cause population cycles.
Example: vole – weasel interaction in Fennoscandia

vole

weasel

Cyclic Predator-Prey Systems

The interaction between a predator population and its prey can cause population cycles.
This has been modelled extensively using systems of two coupled ODEs

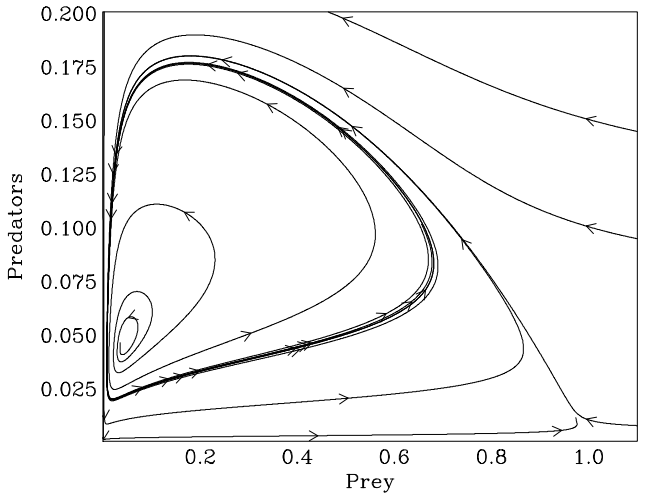


constant coexistence

change

$\longrightarrow$

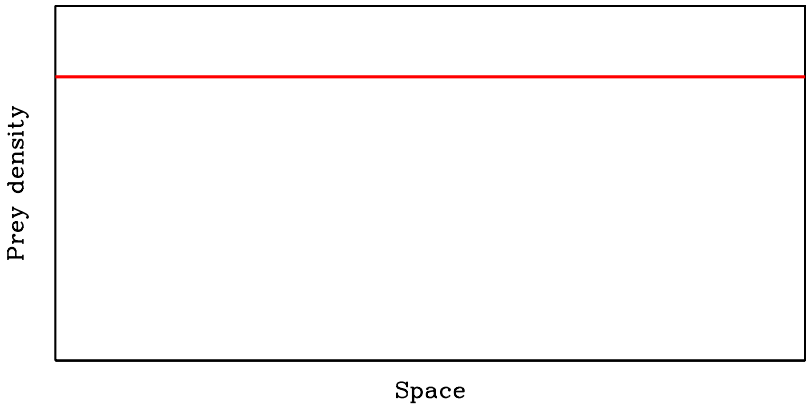
parameters



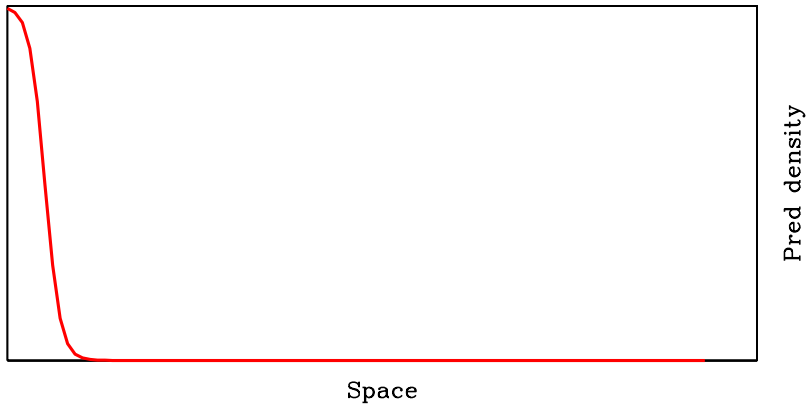
cycles

Predator-Prey Invasion

To model the invasion of a prey population by predators, one can add diffusion terms to represent dispersal.



Initially we set the prey to the prey-only equilibrium throughout the domain.



Initially we set the predators to zero except near the left hand boundary.

Predator-Prey Invasion

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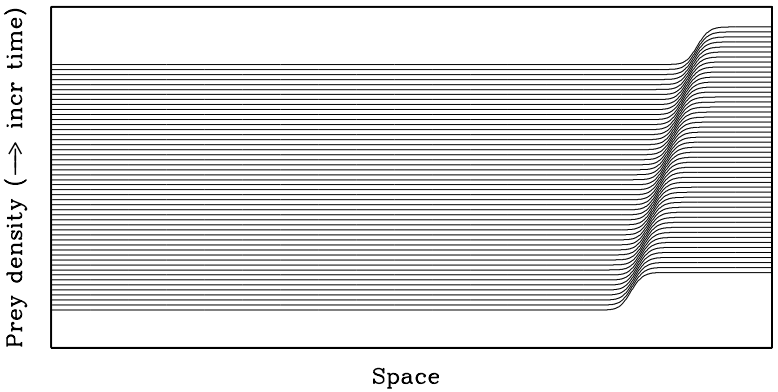

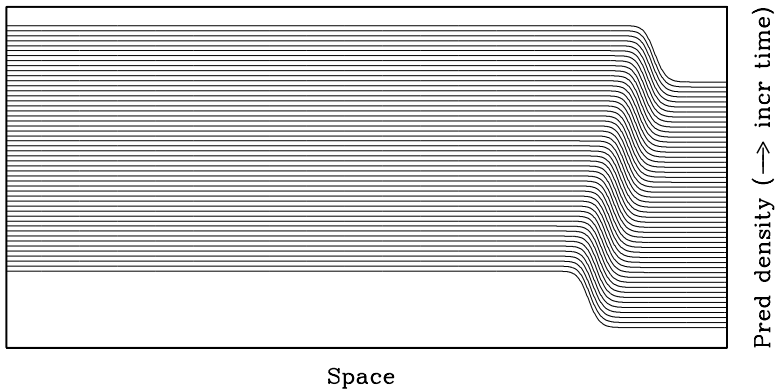

Simple invasion front (local bhr: constant)

Predator-Prey Invasion

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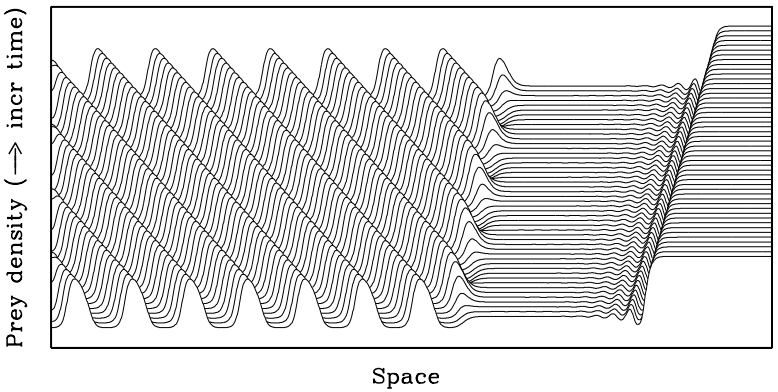

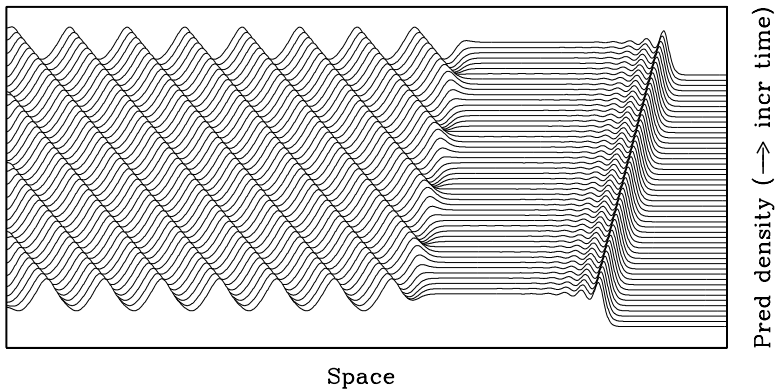

Wavetrain behind an invasion front (local bhr: cycles)

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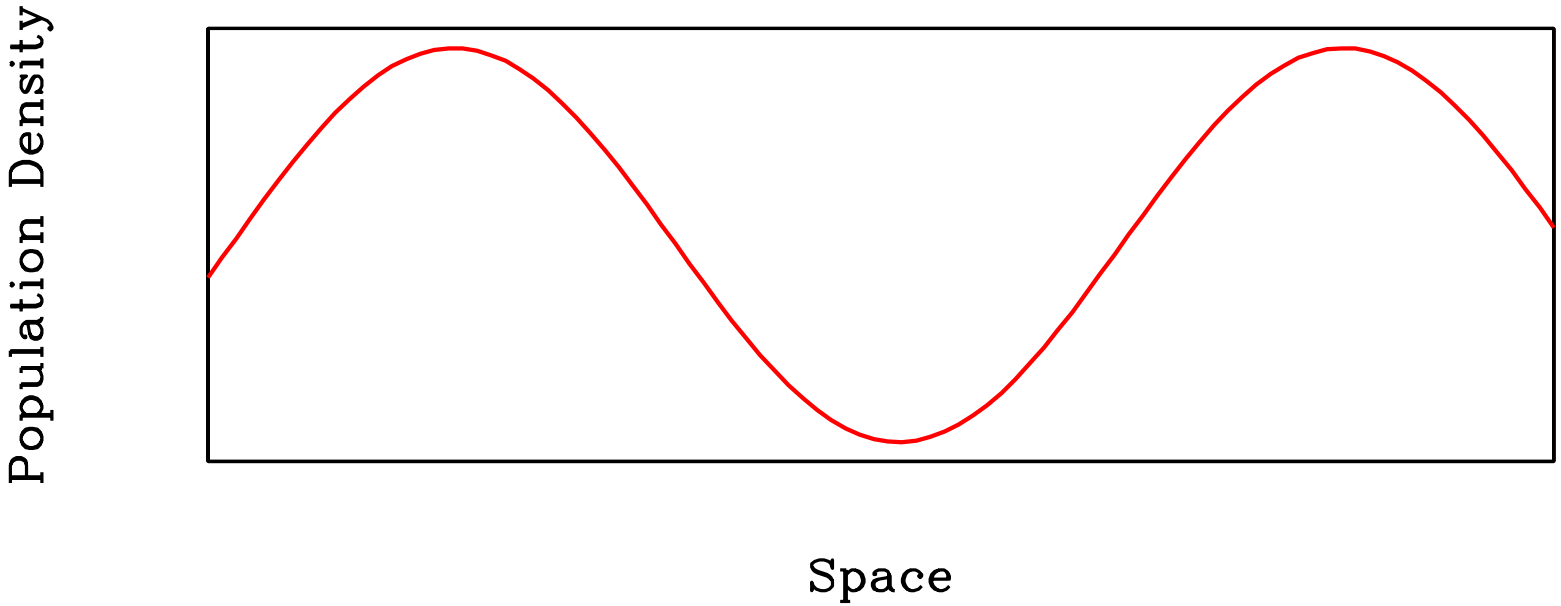

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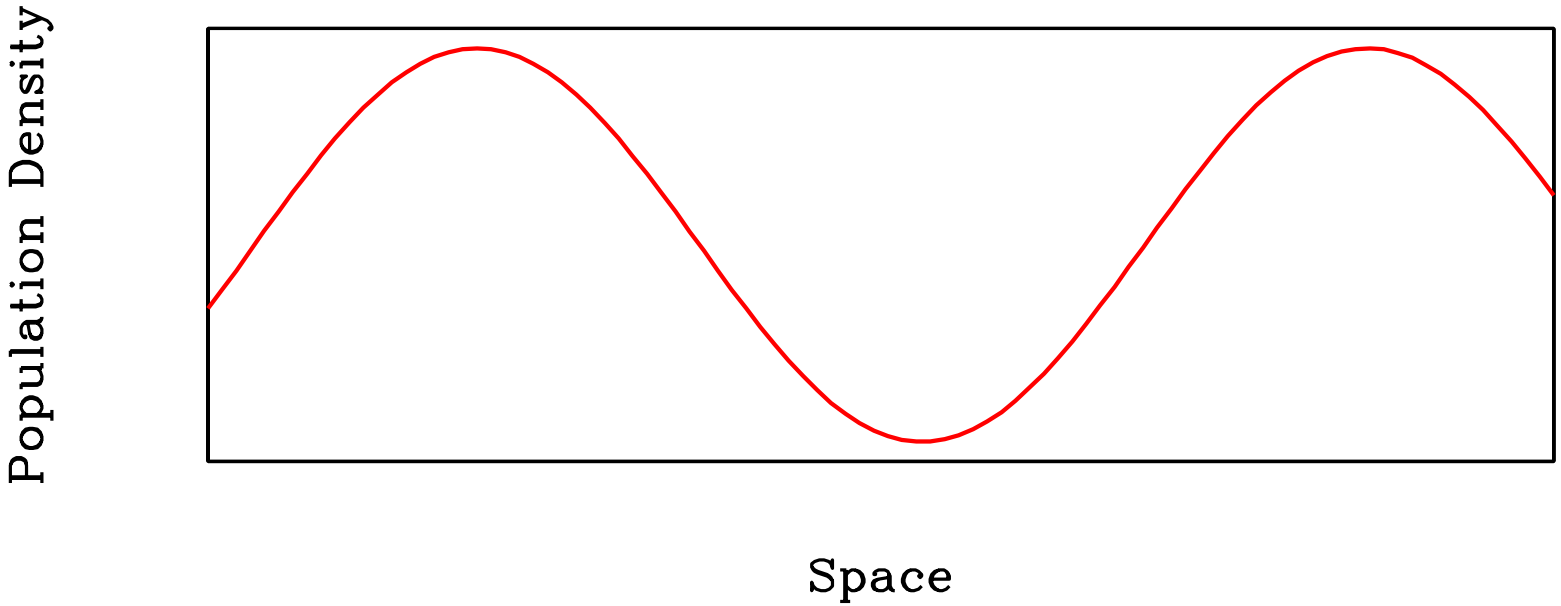

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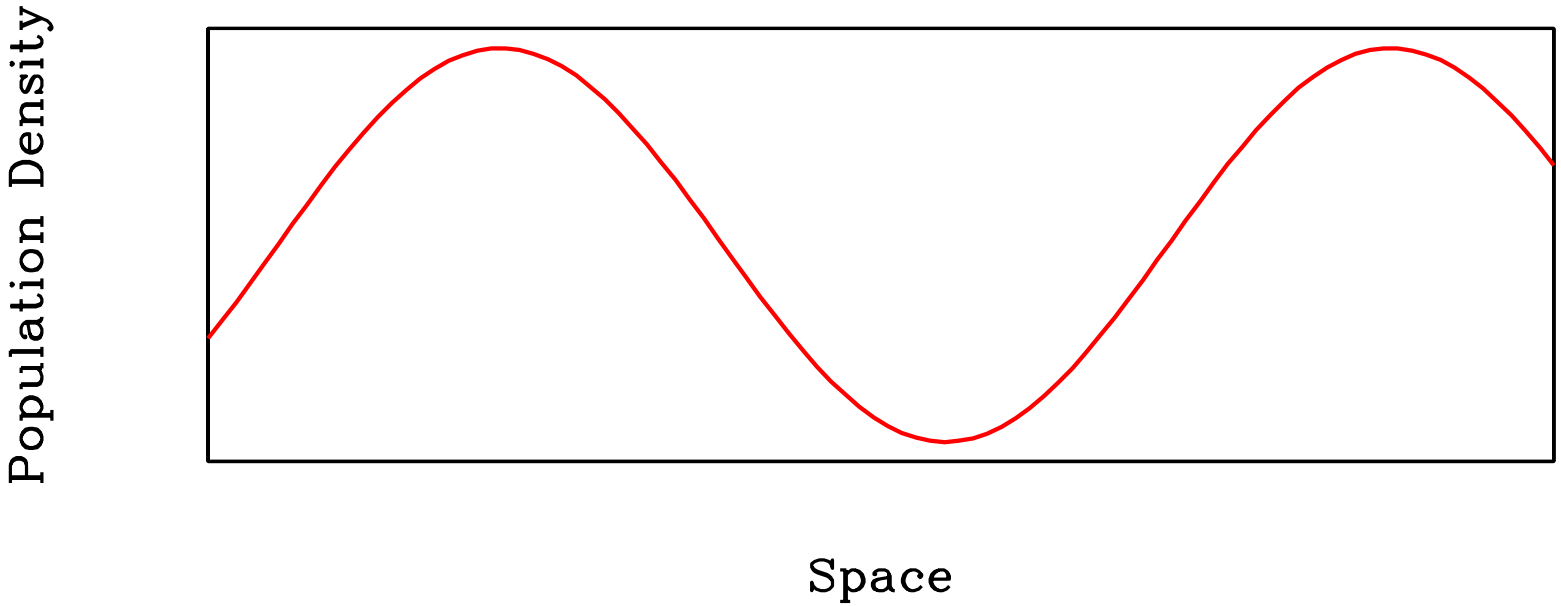

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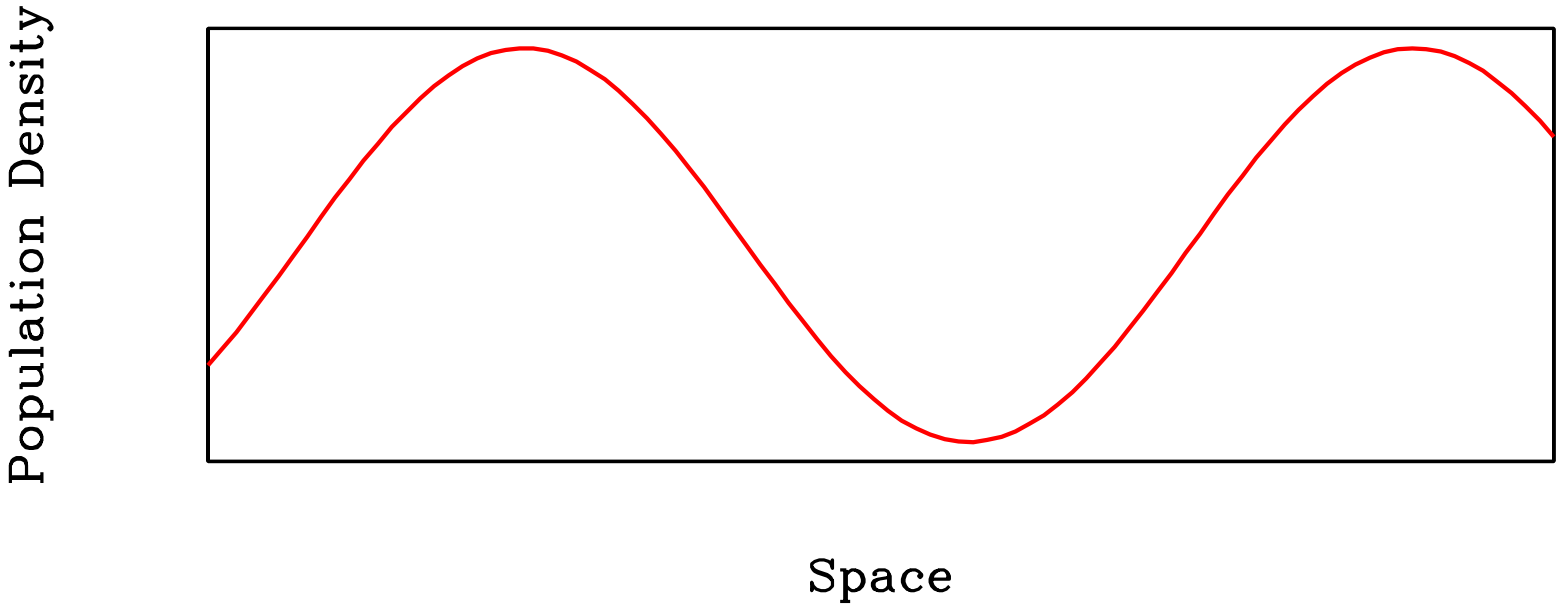

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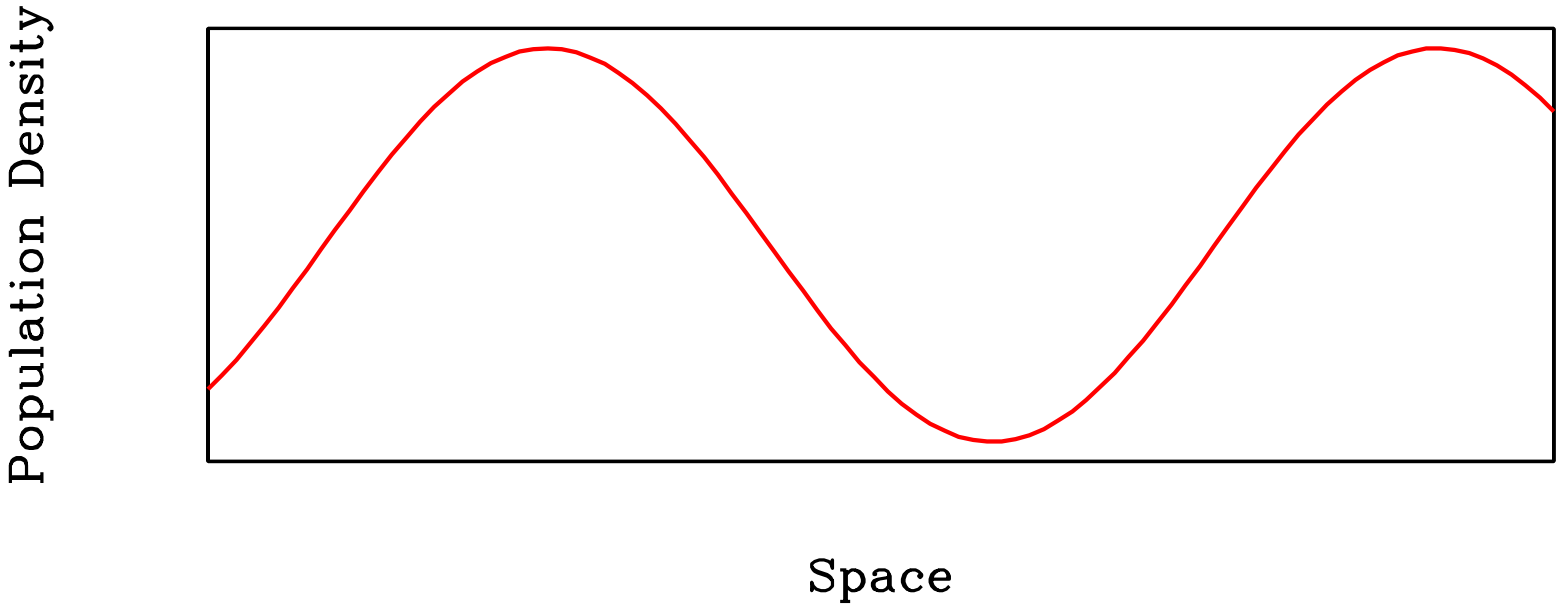

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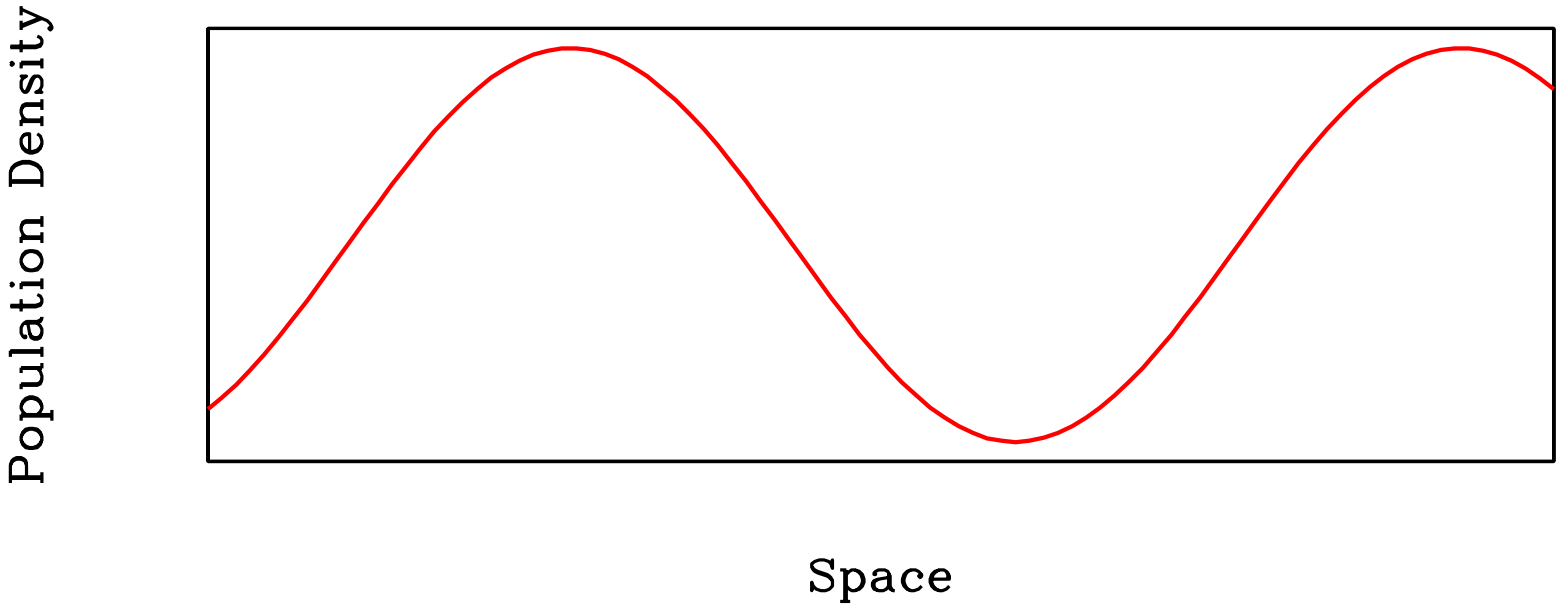

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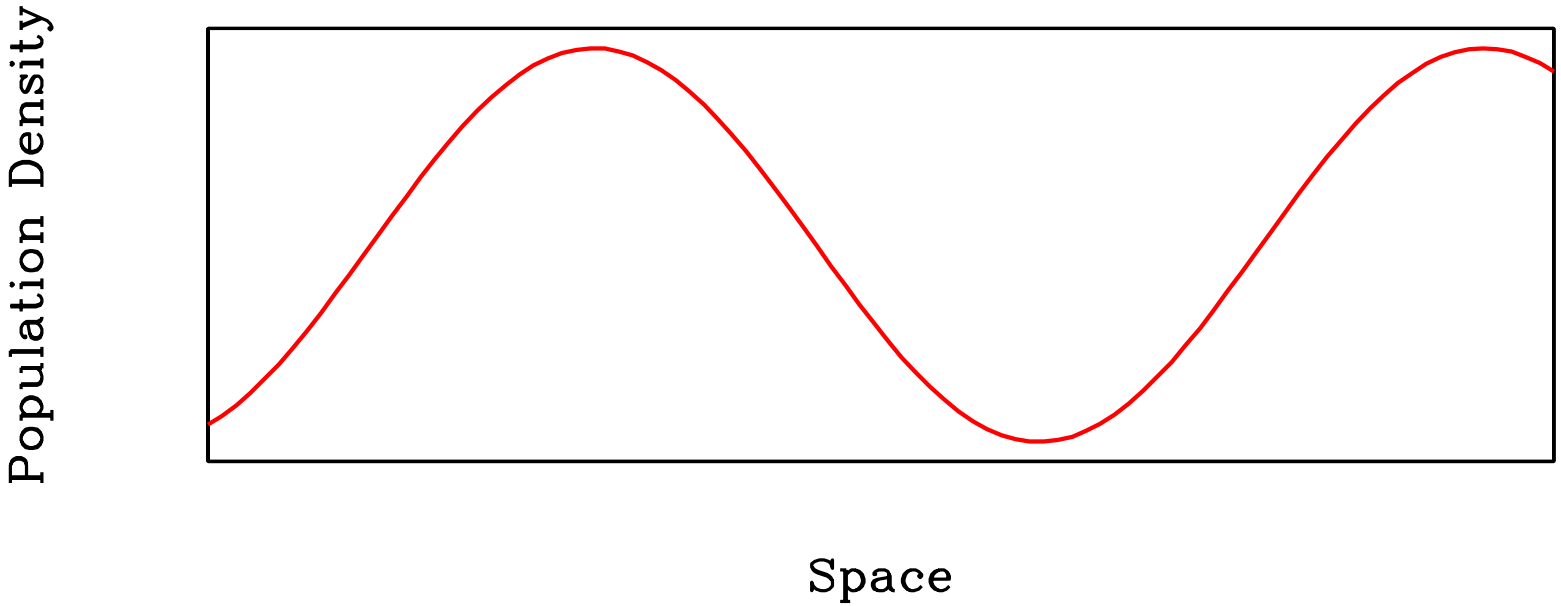

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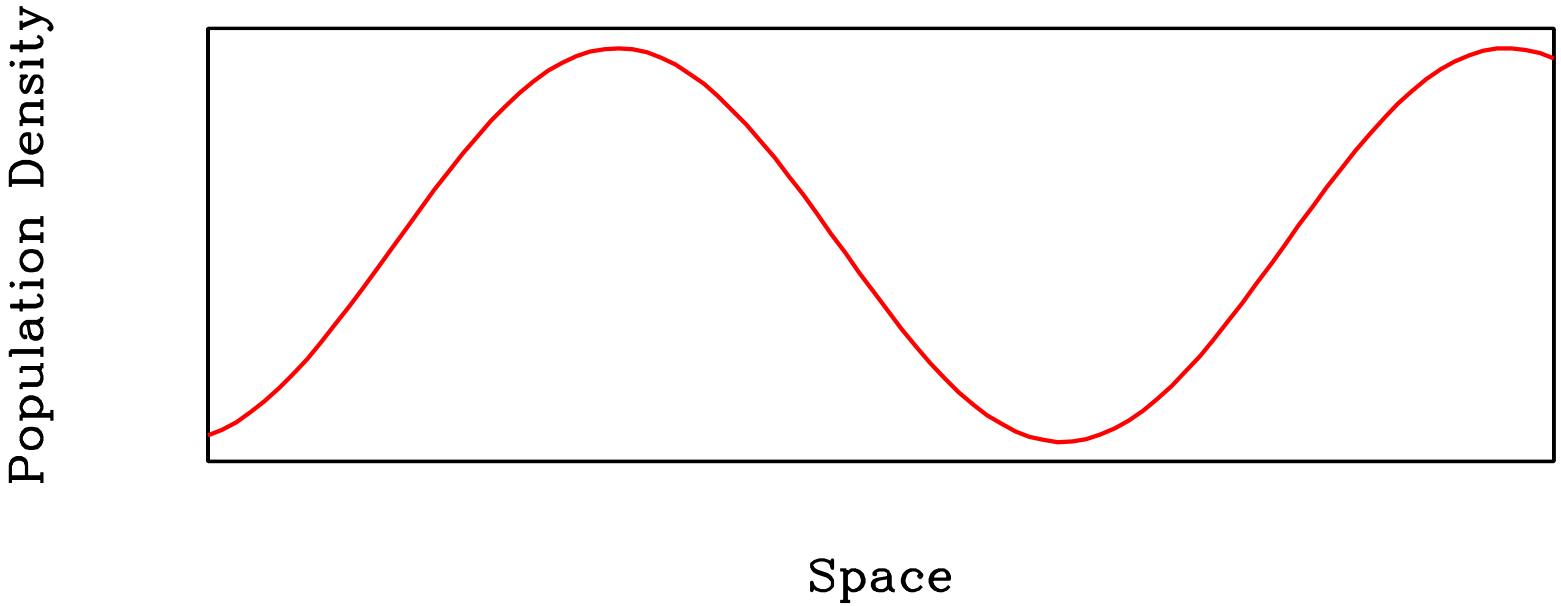

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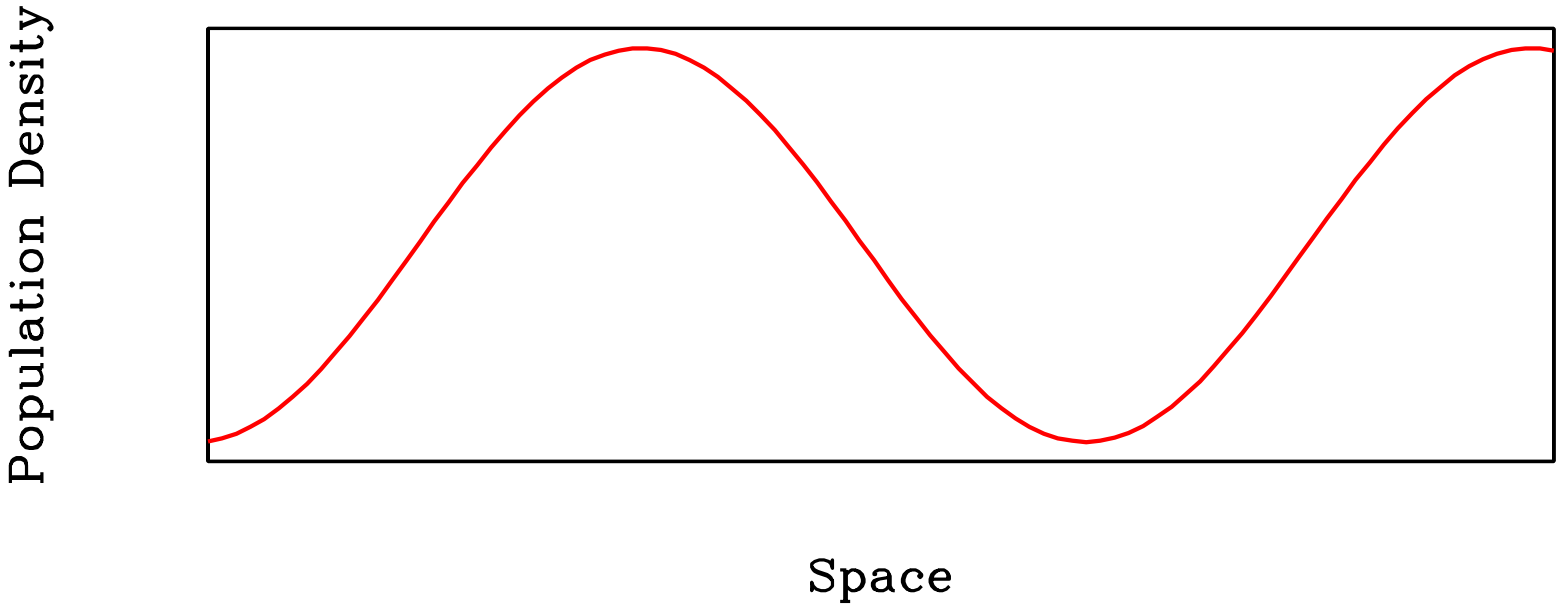

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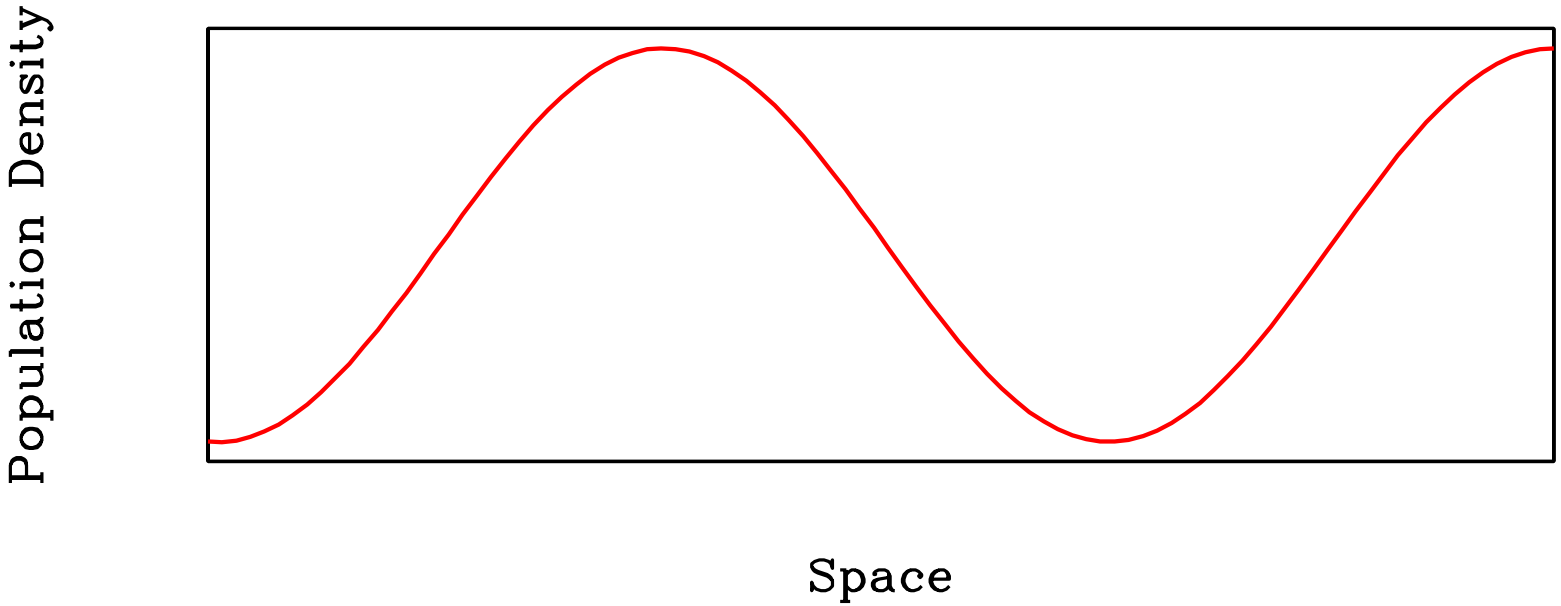

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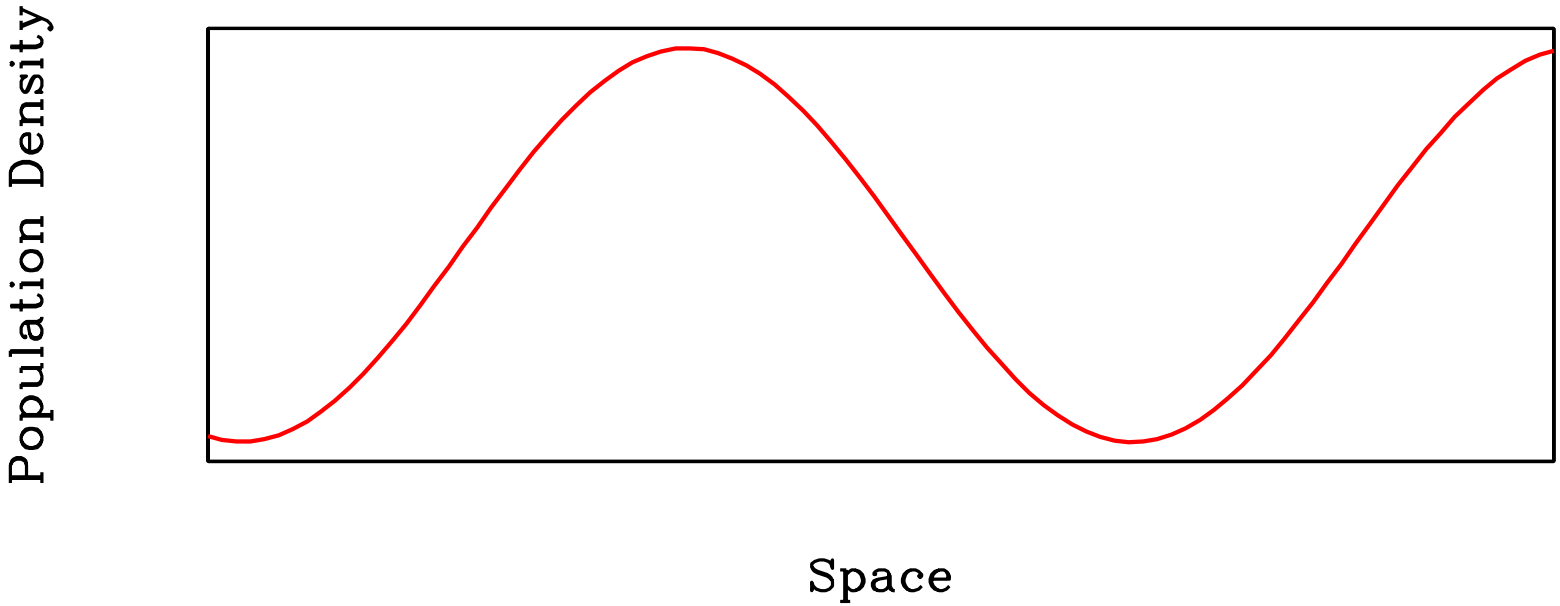

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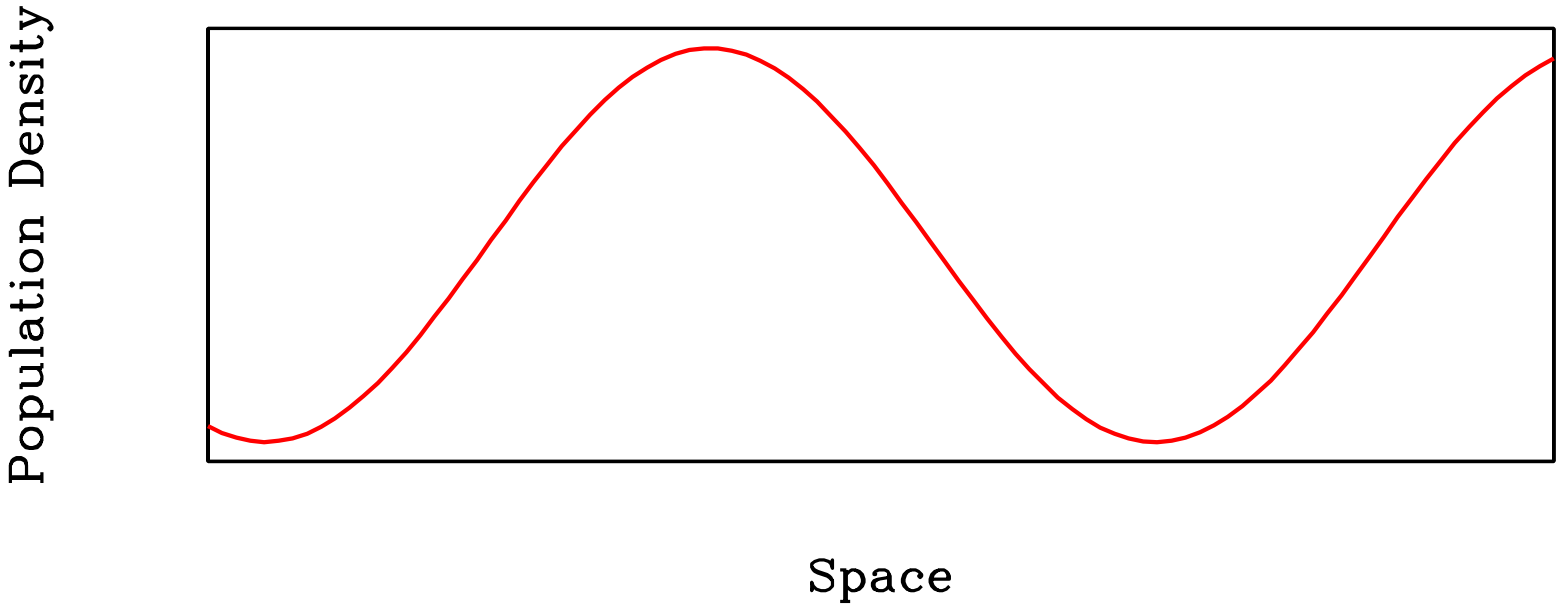

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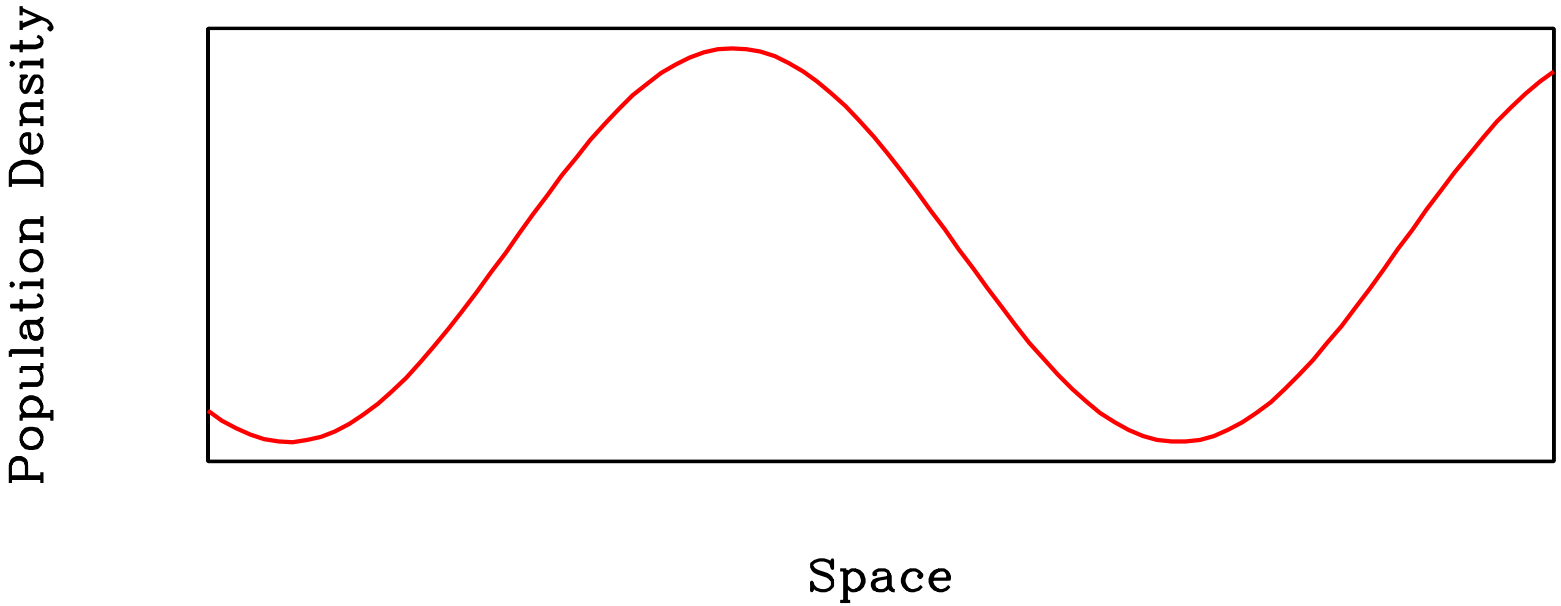

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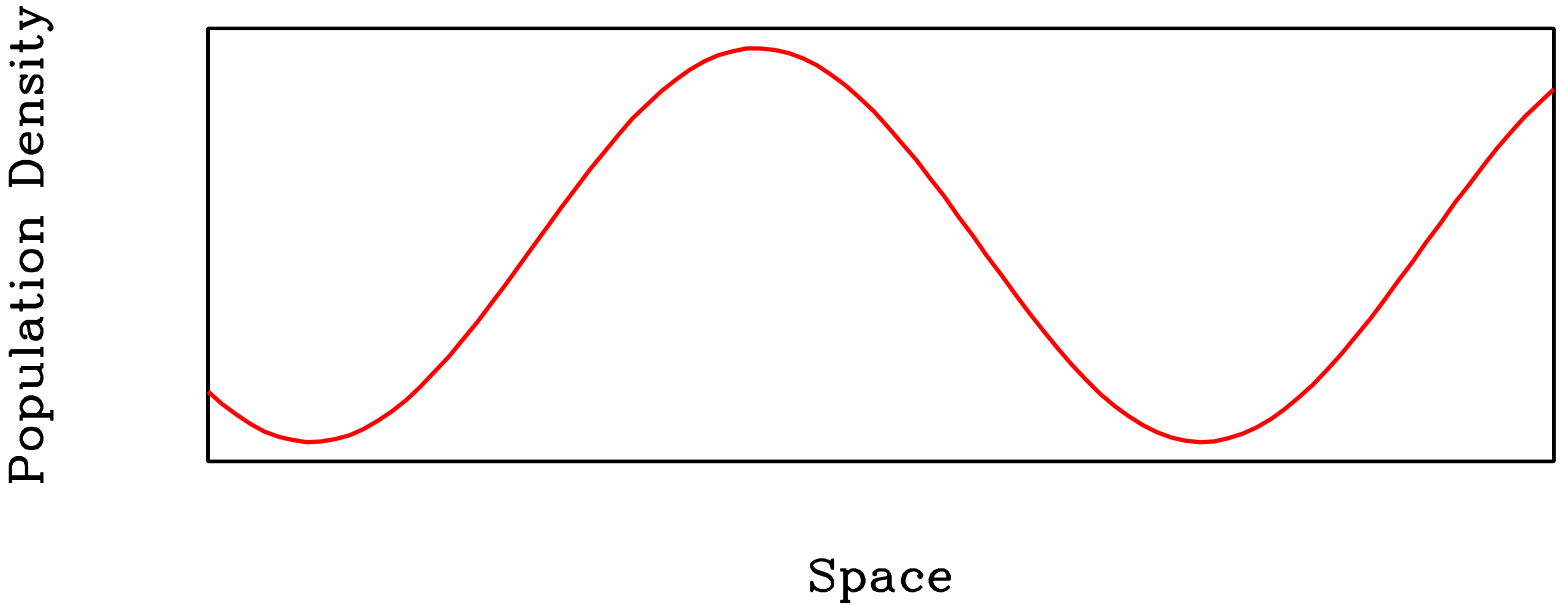

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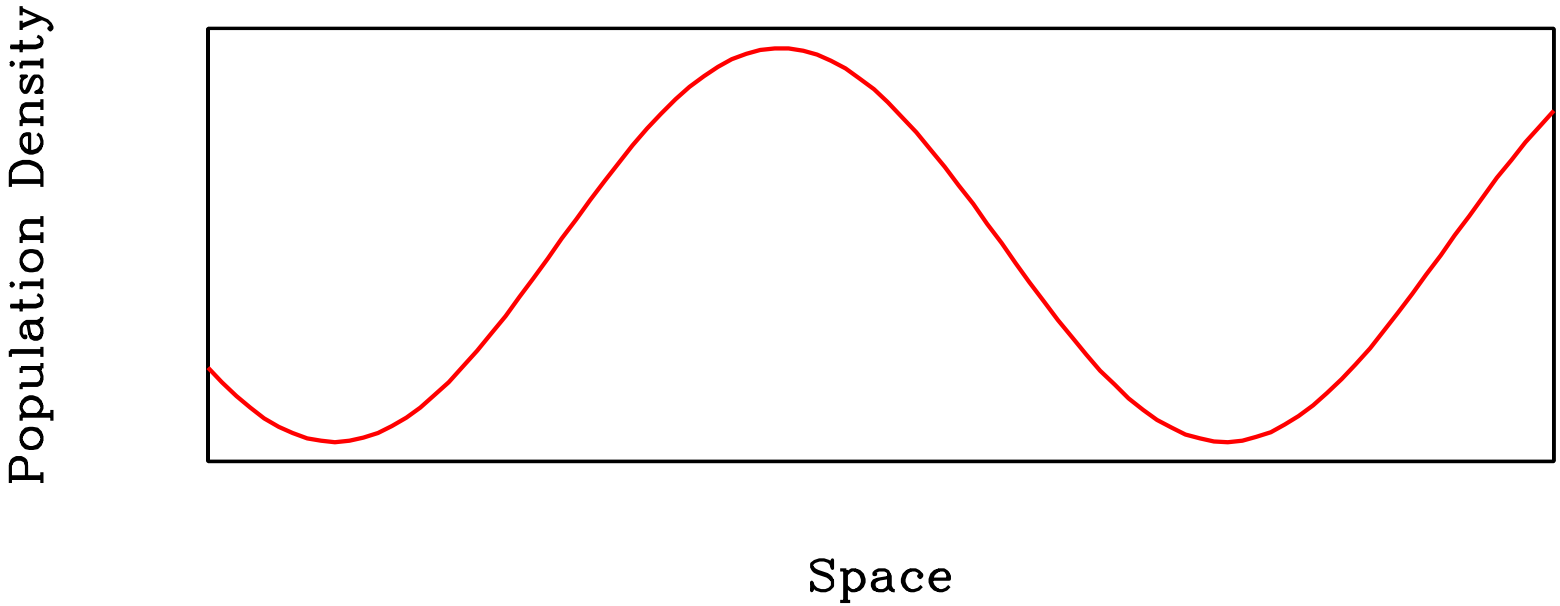

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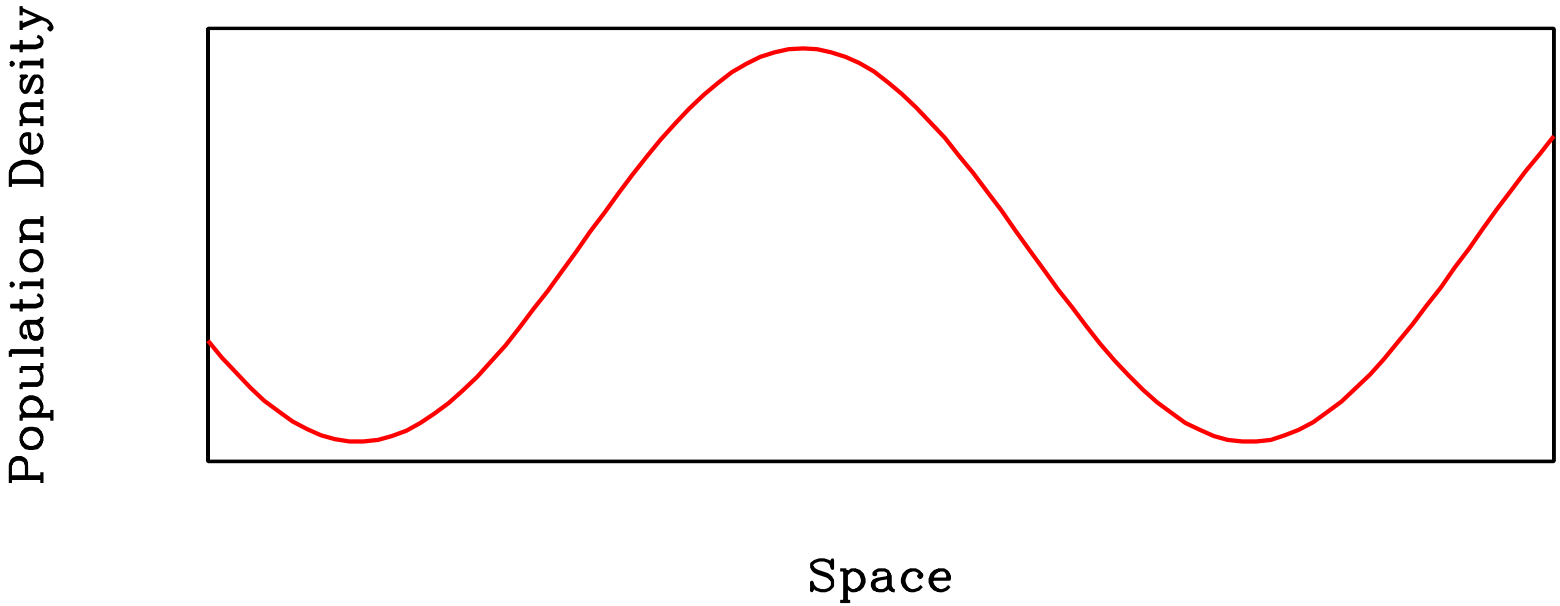

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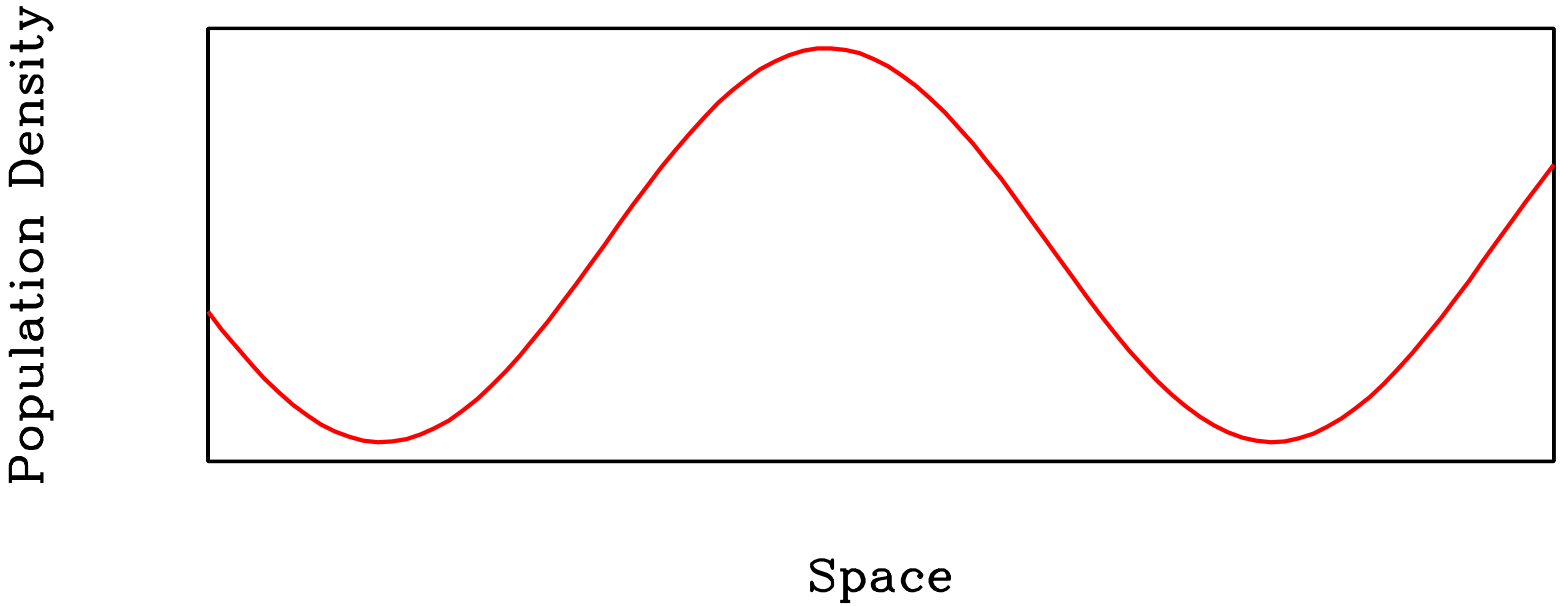

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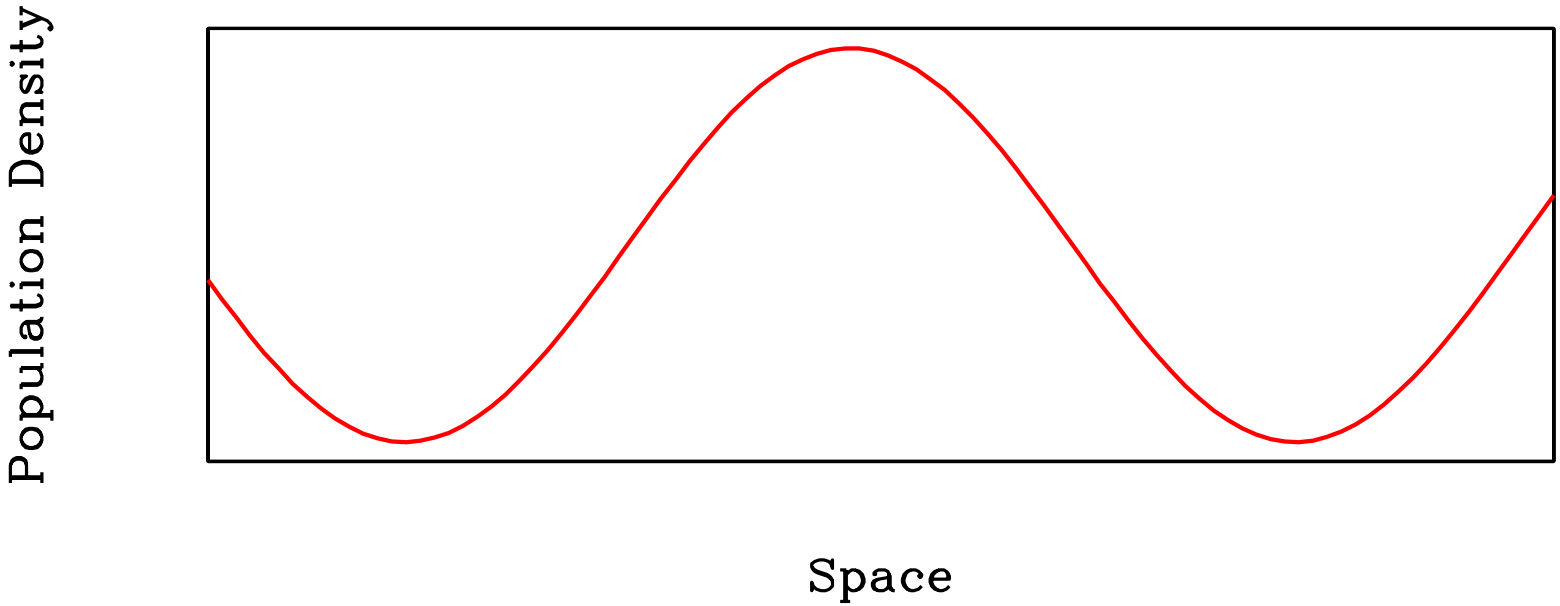

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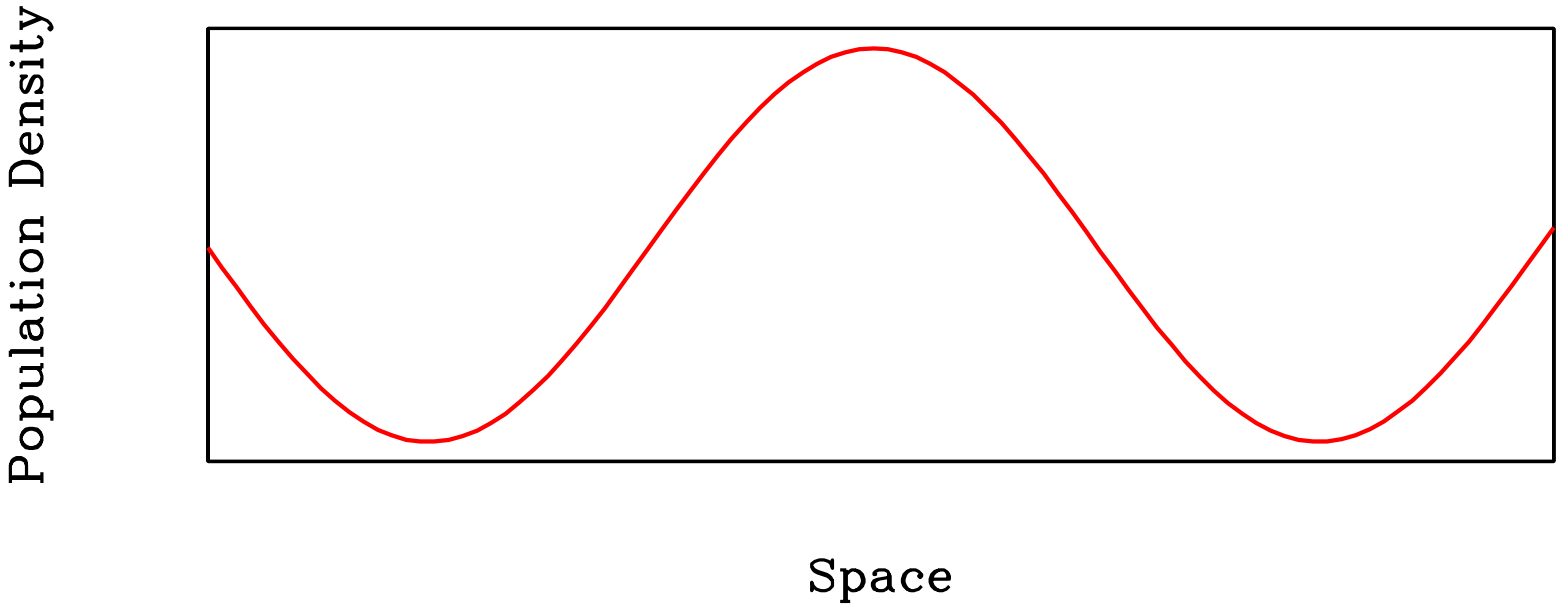

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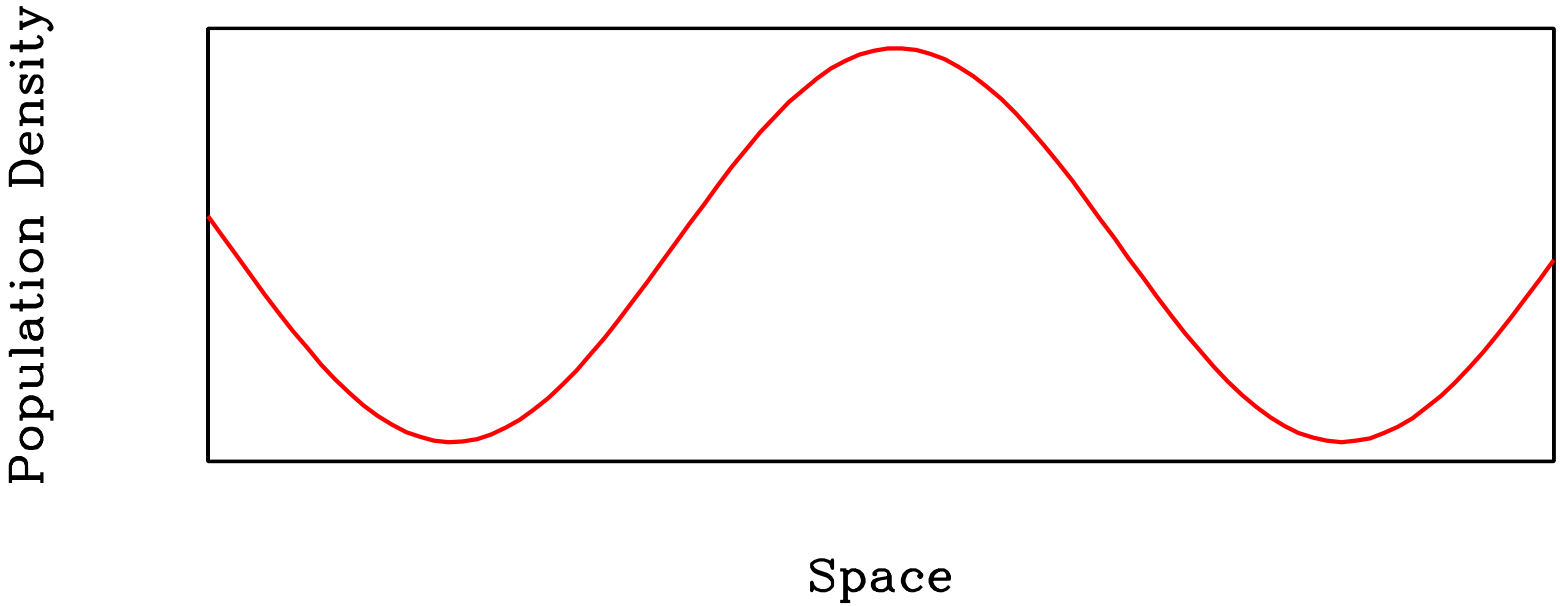

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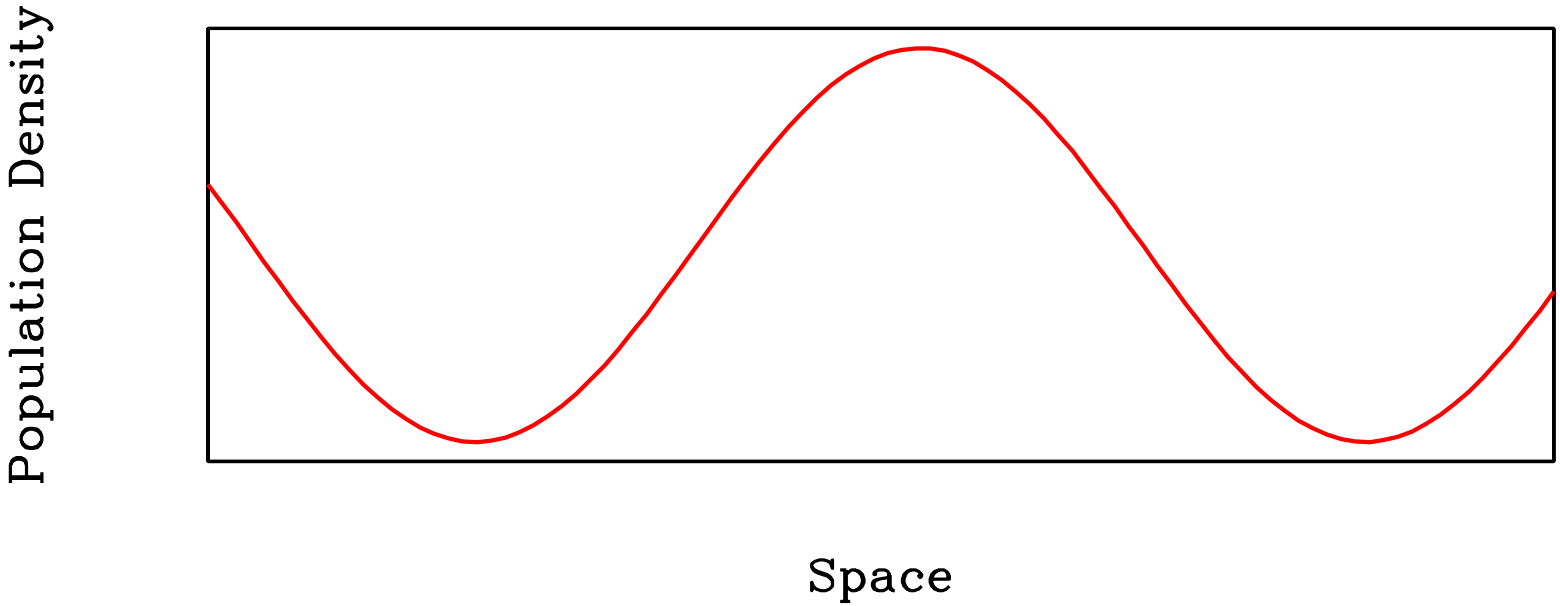

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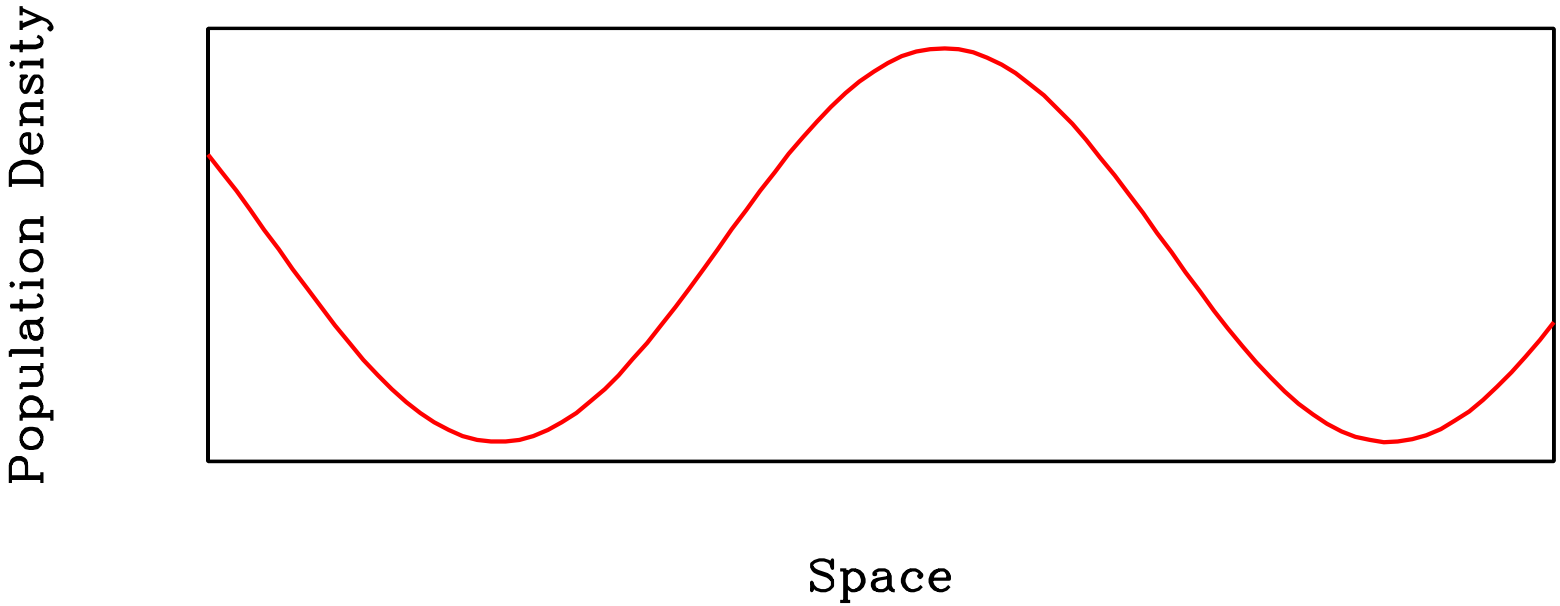

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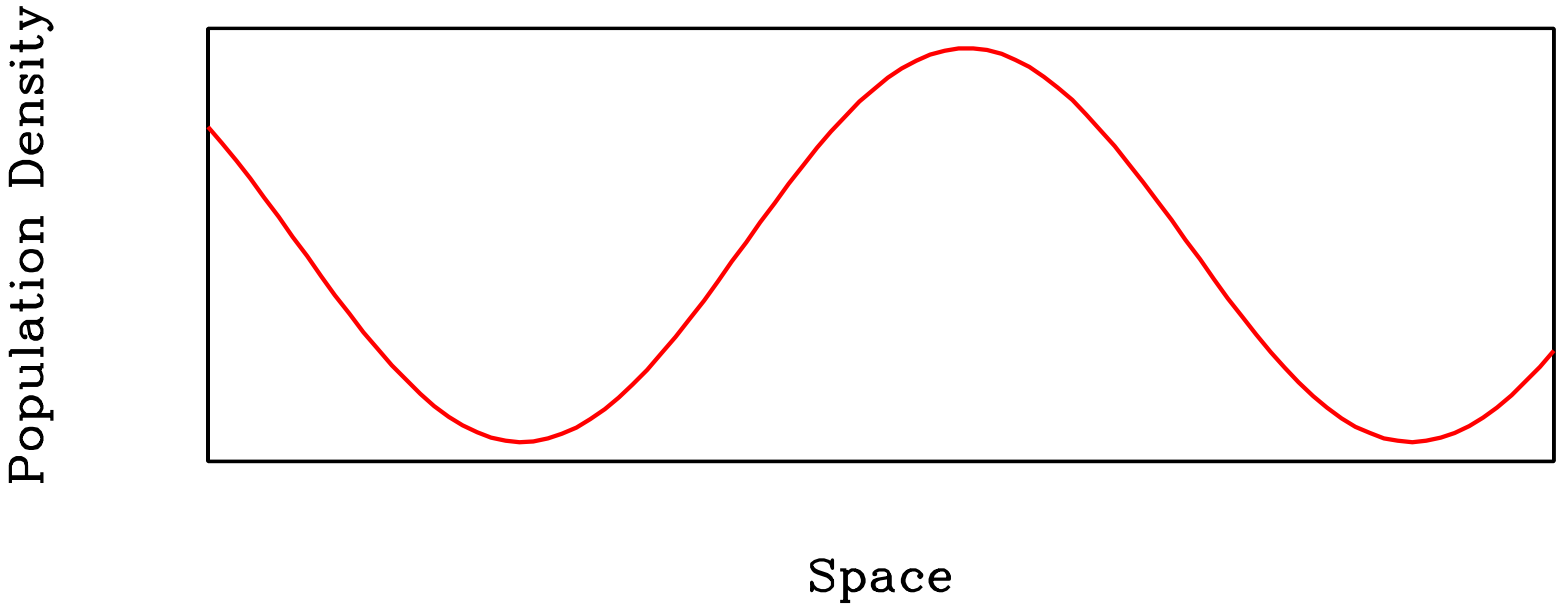

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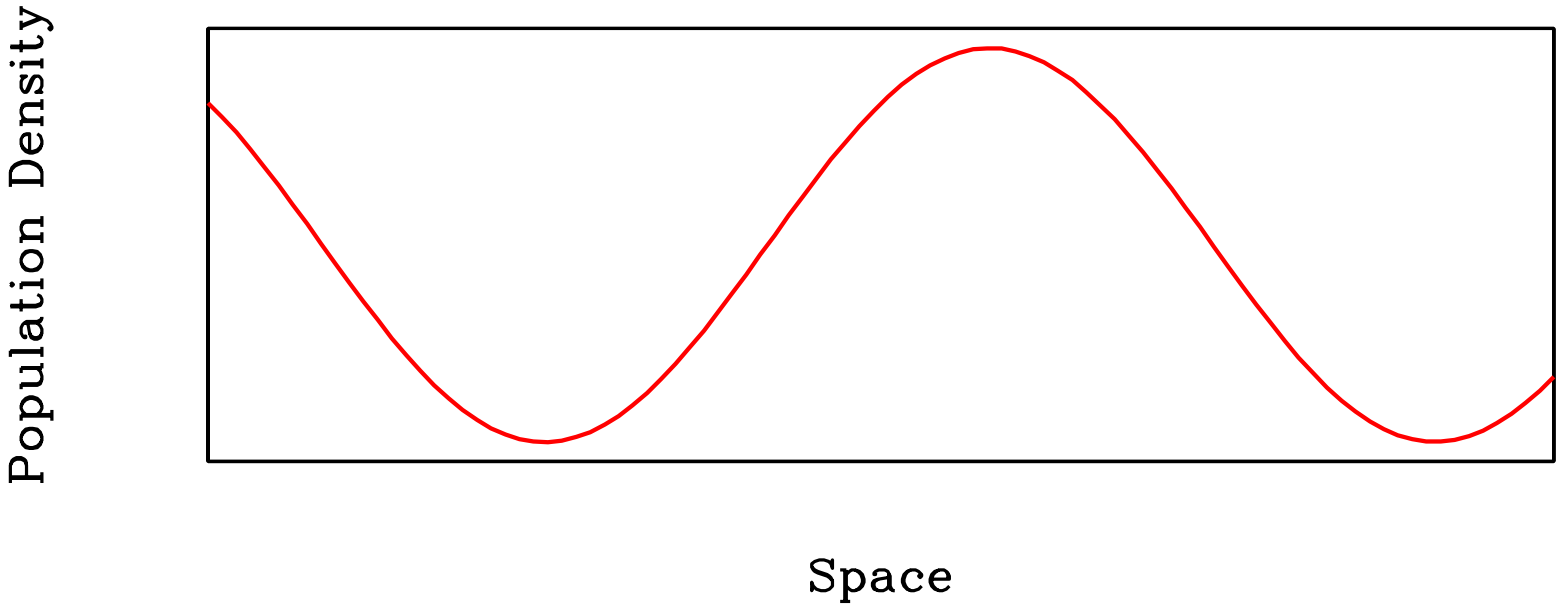

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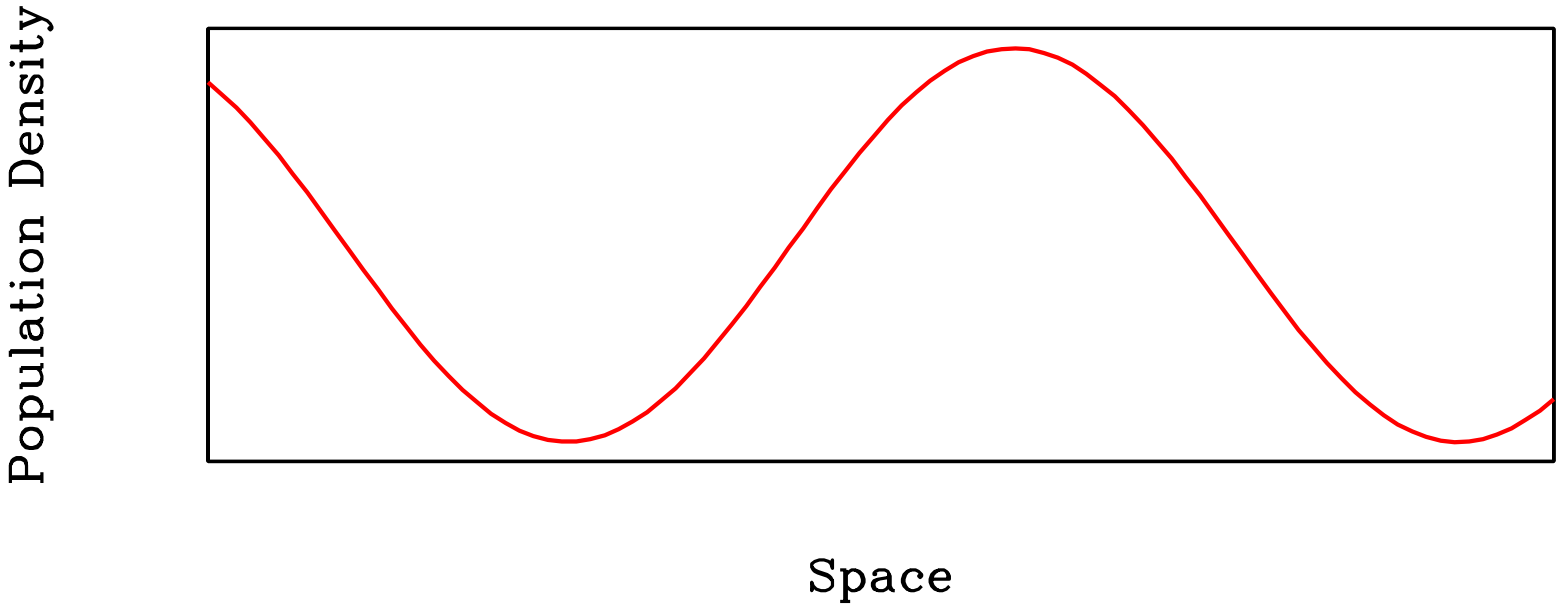

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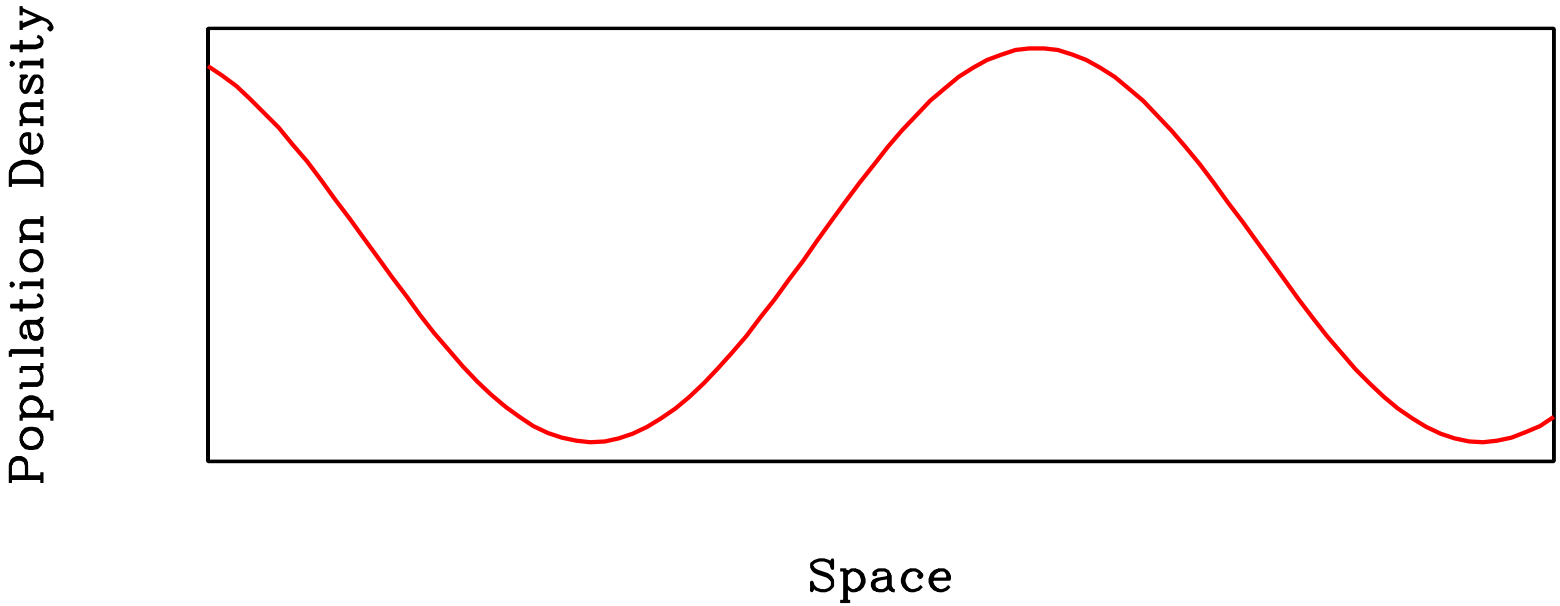

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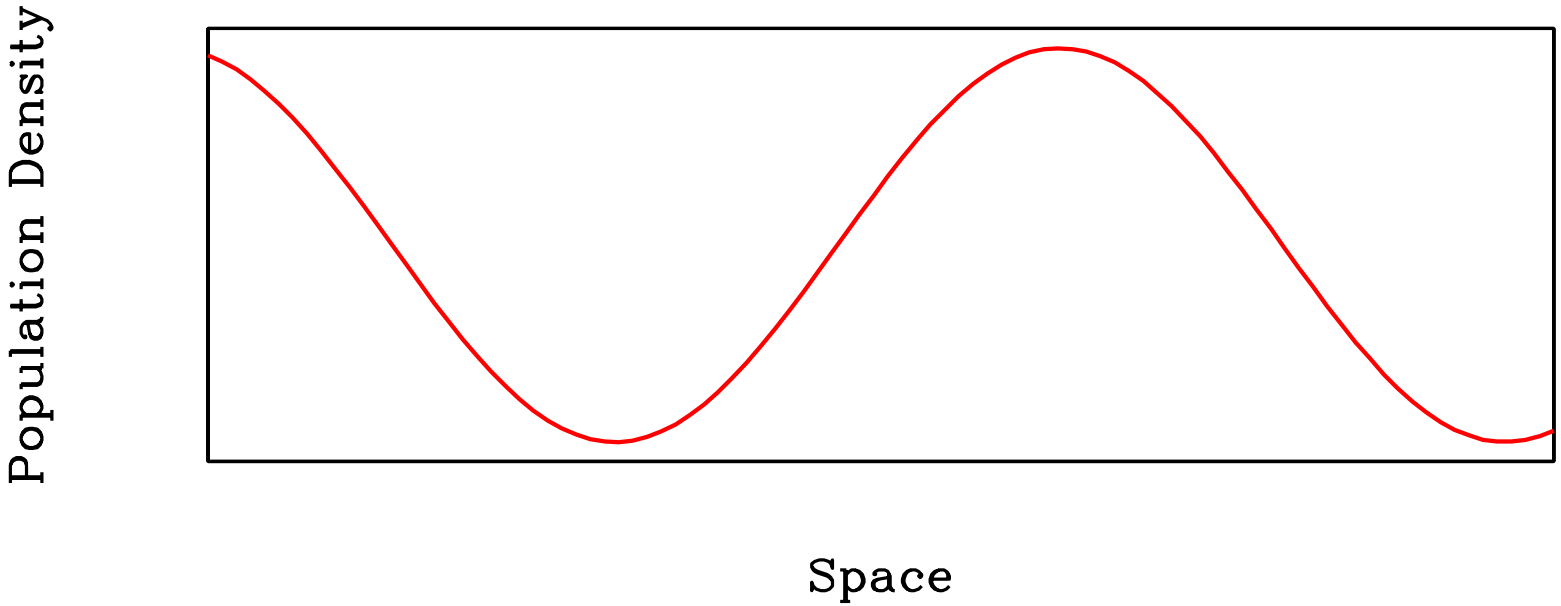

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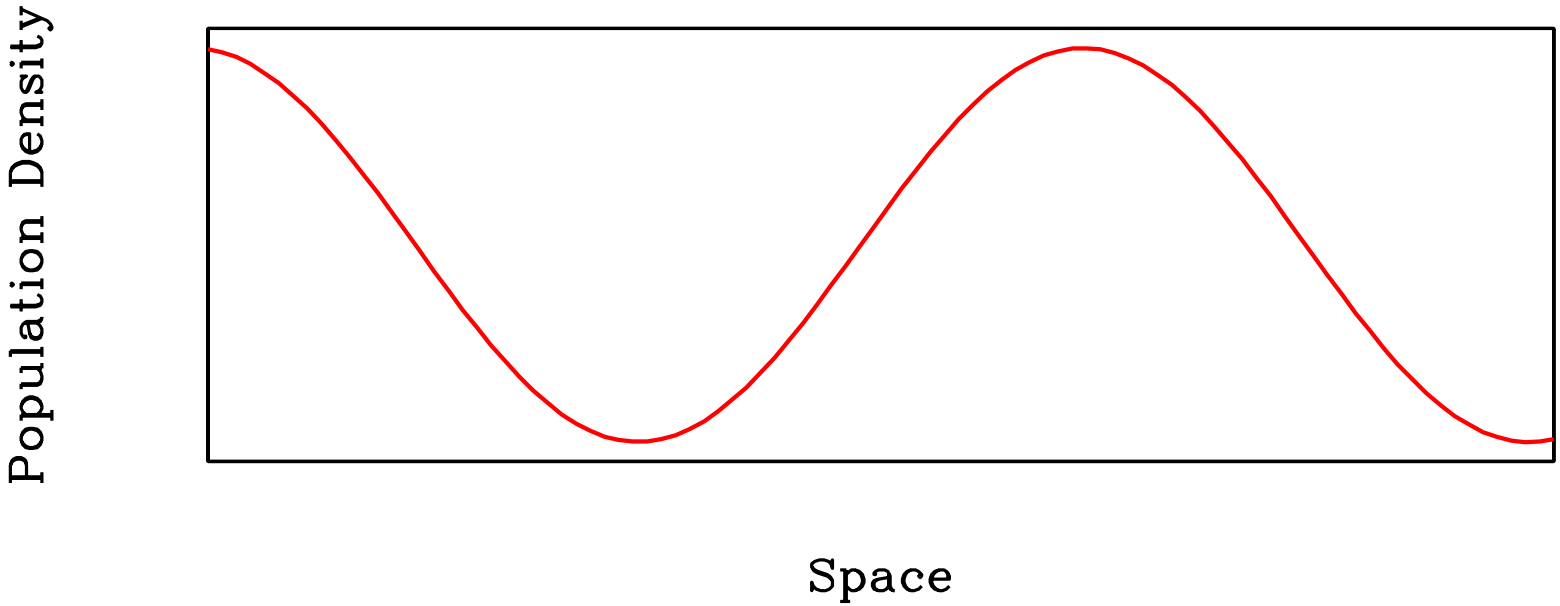

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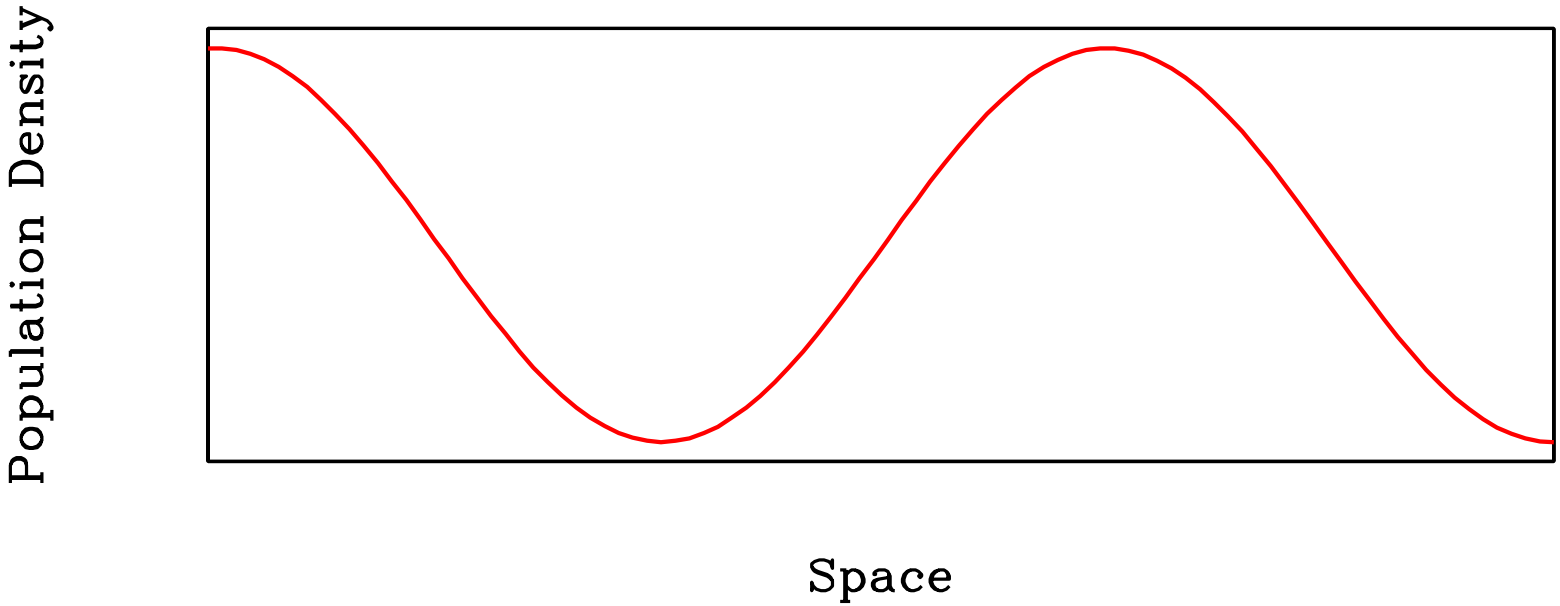

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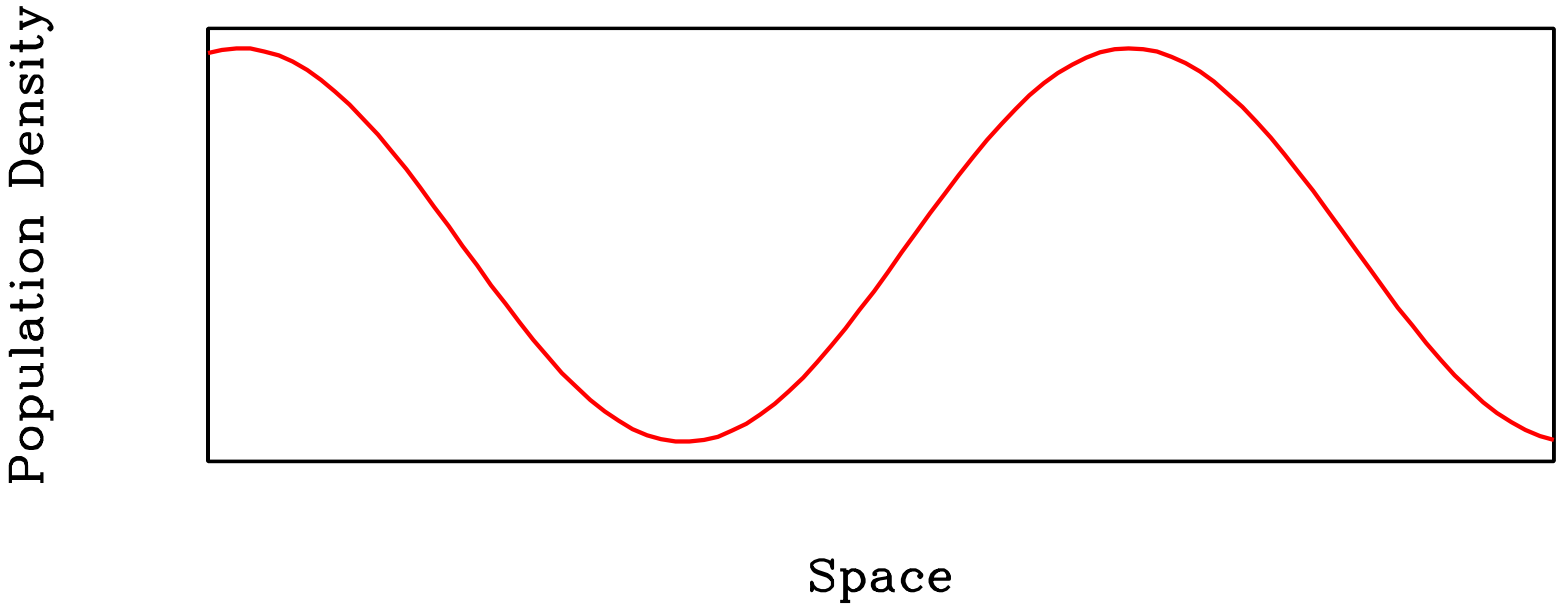

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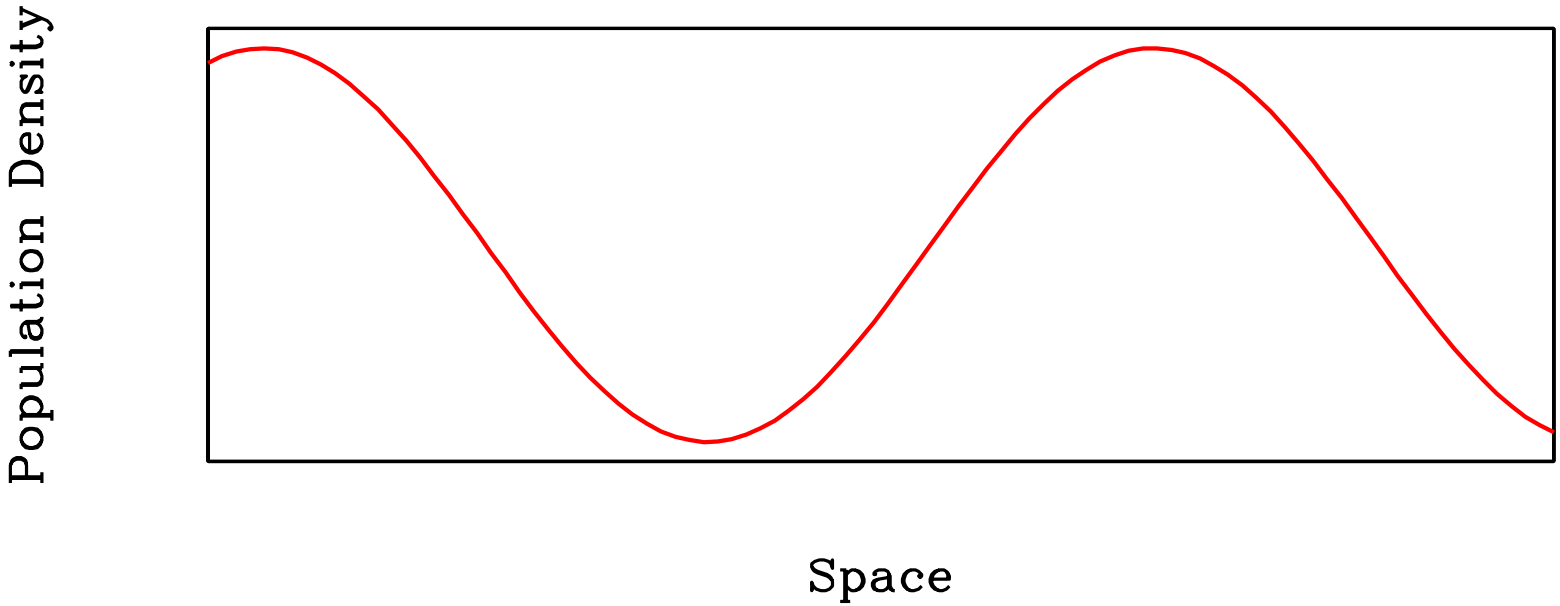

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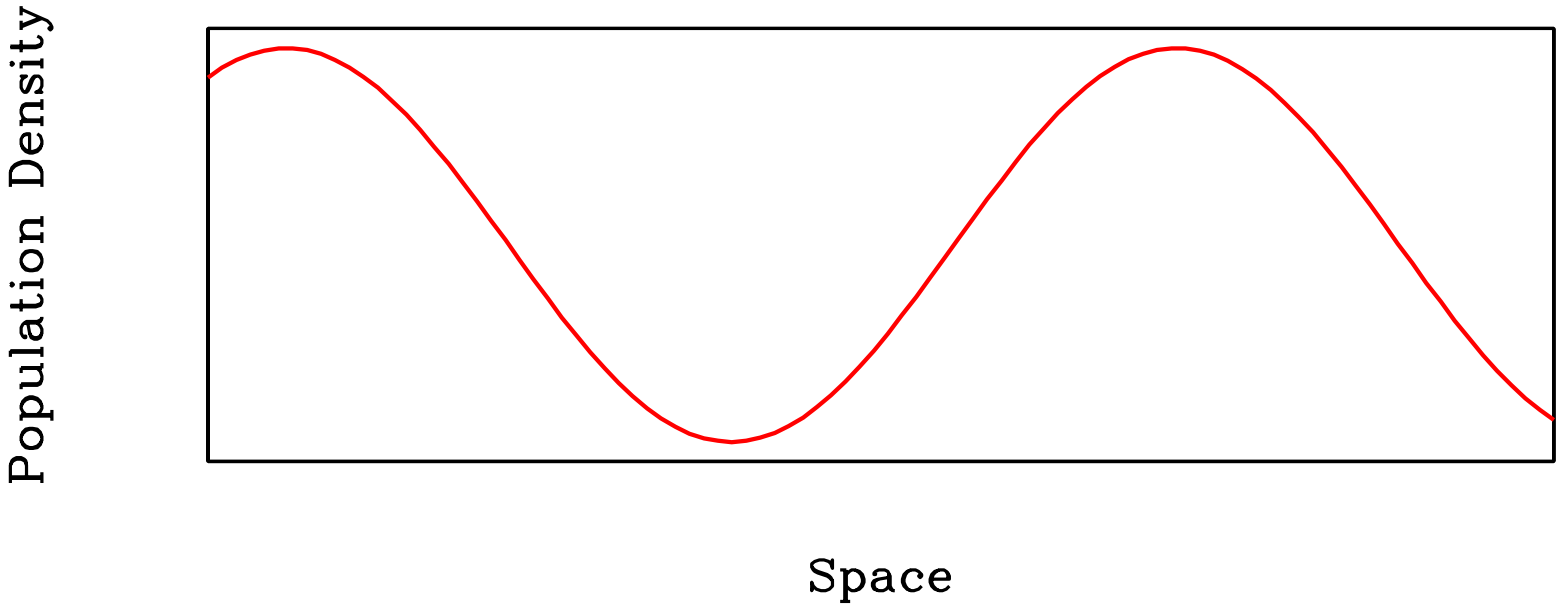

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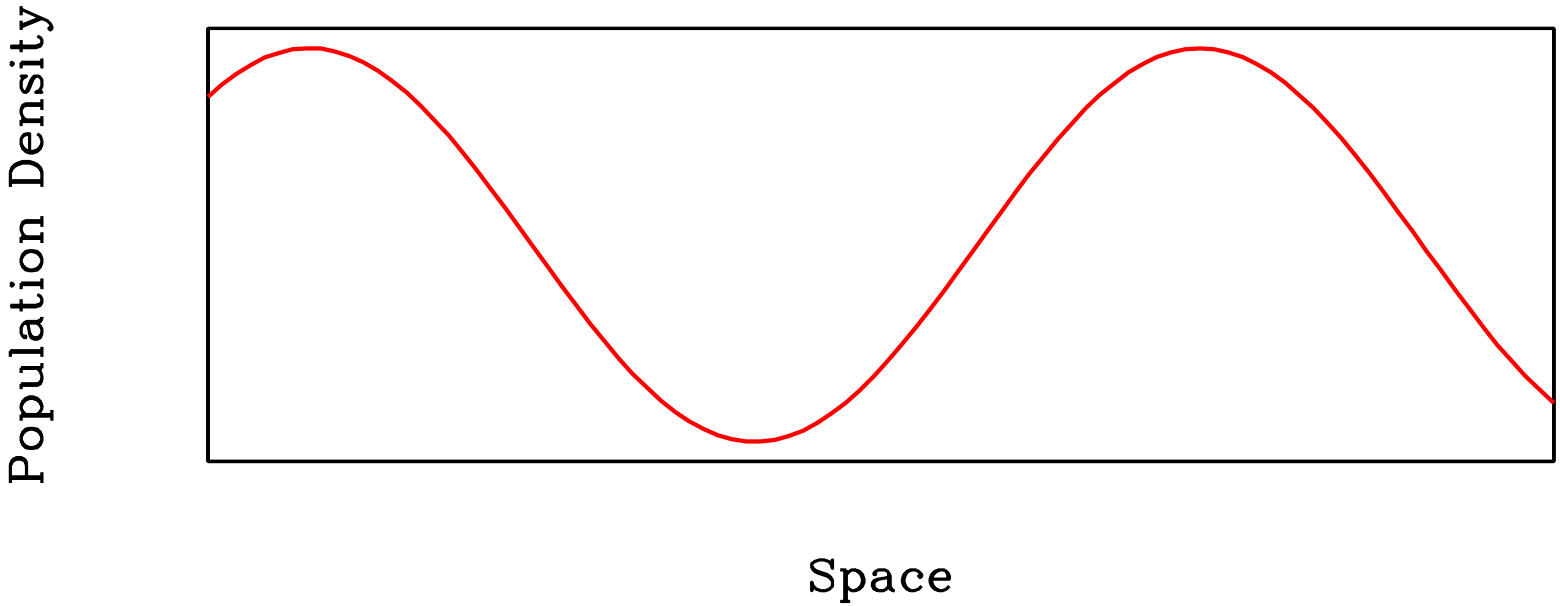

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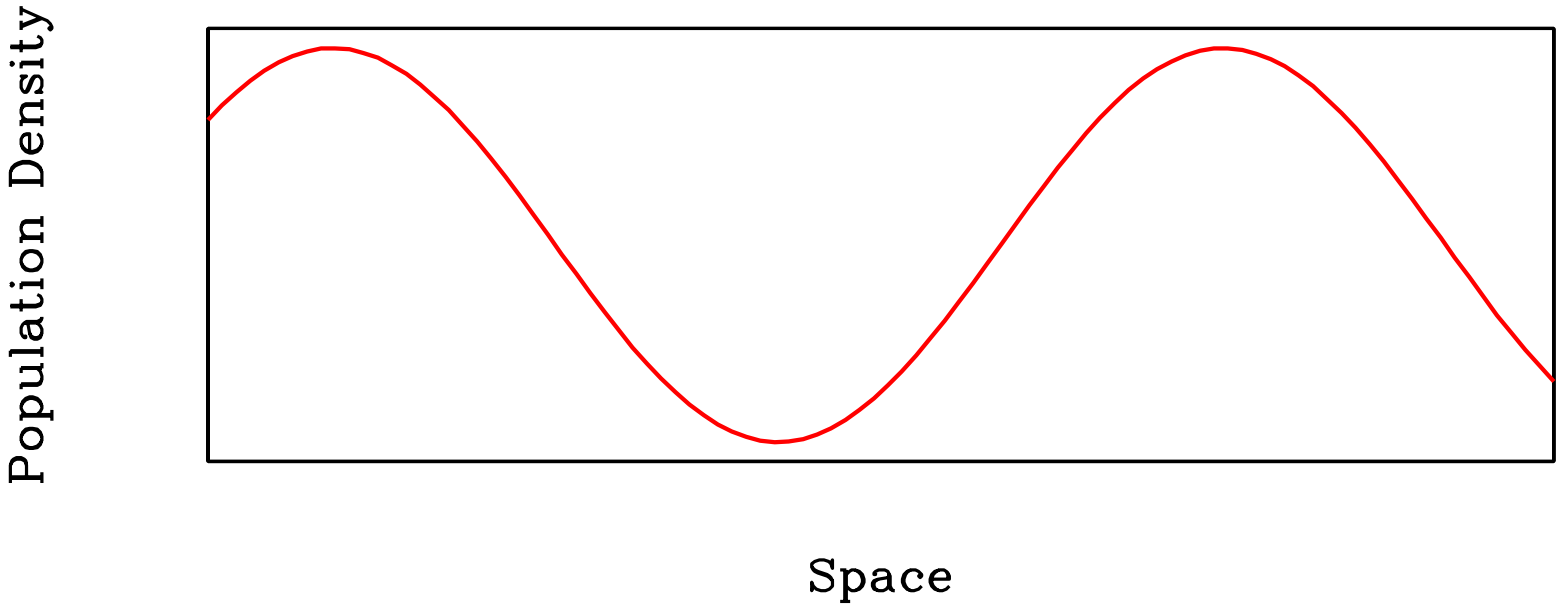

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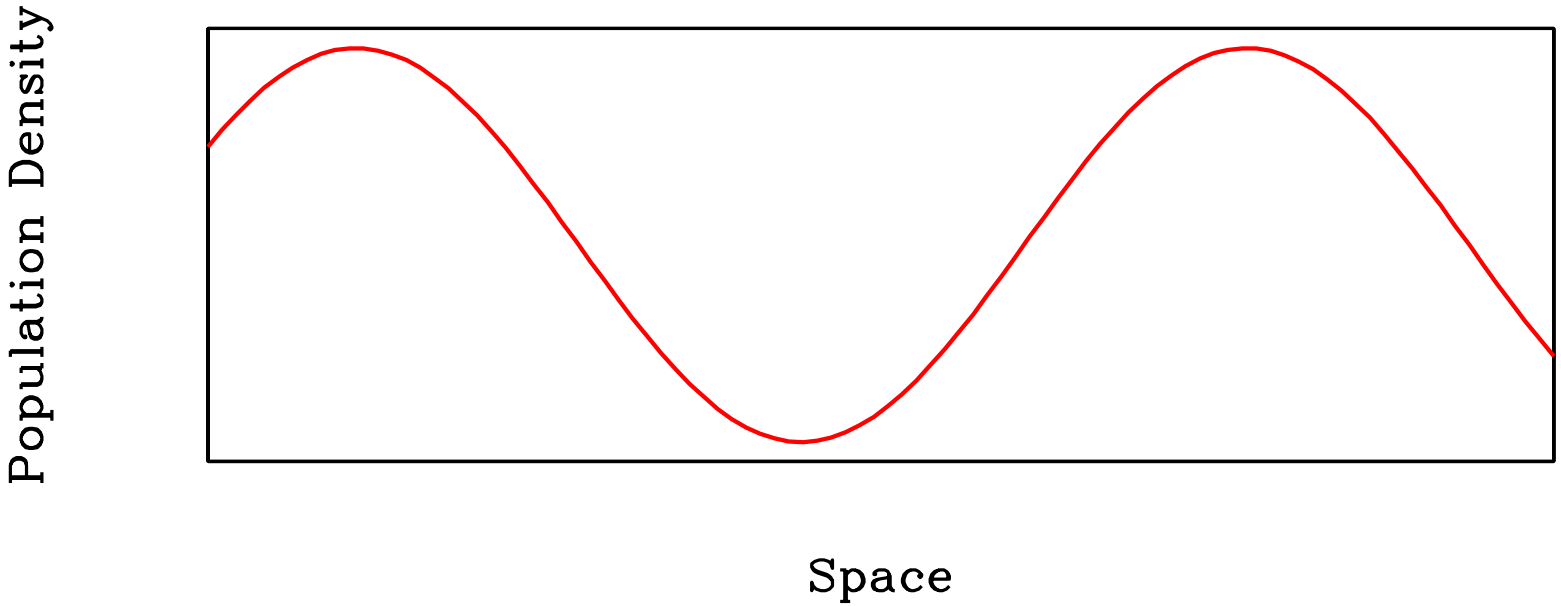

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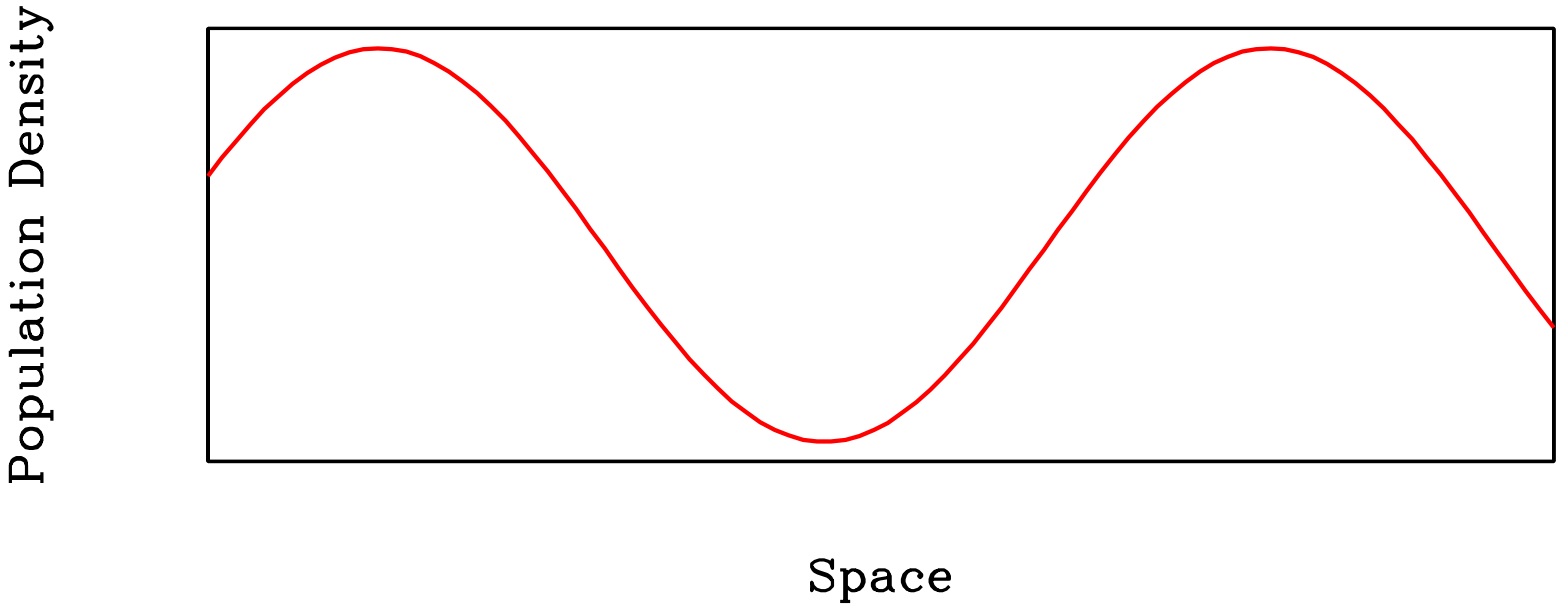

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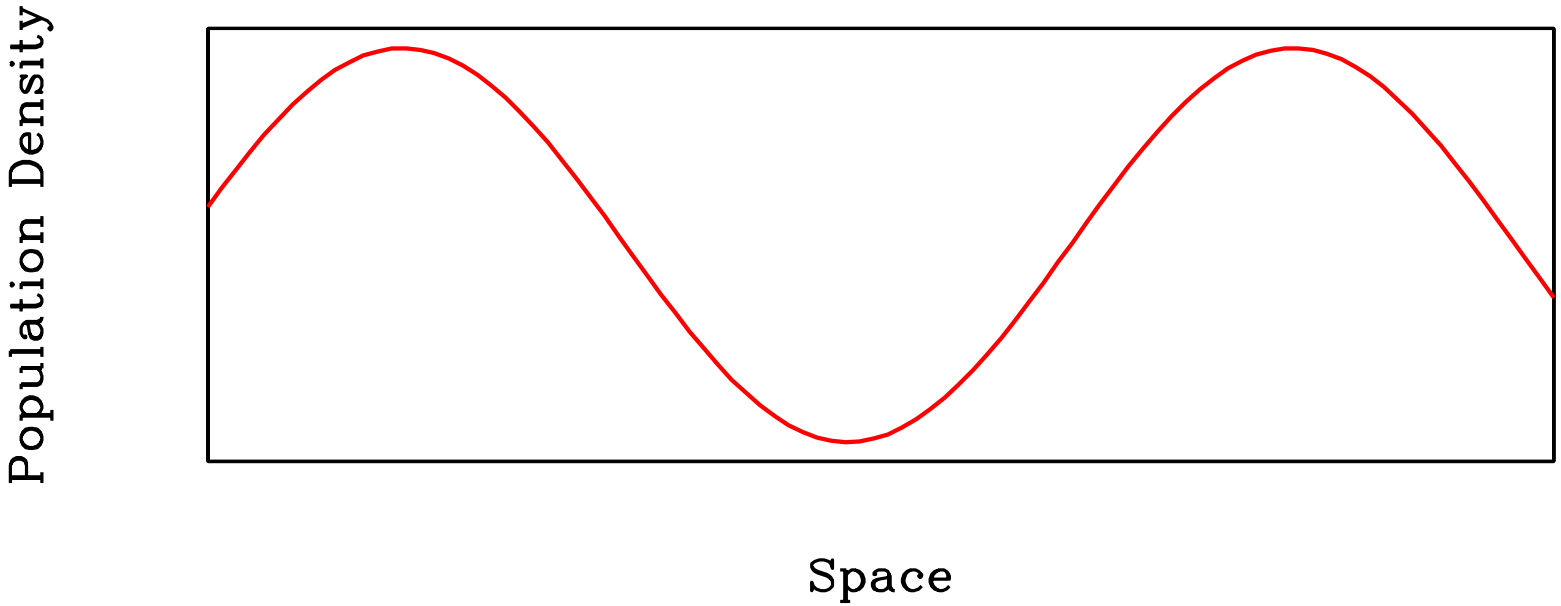

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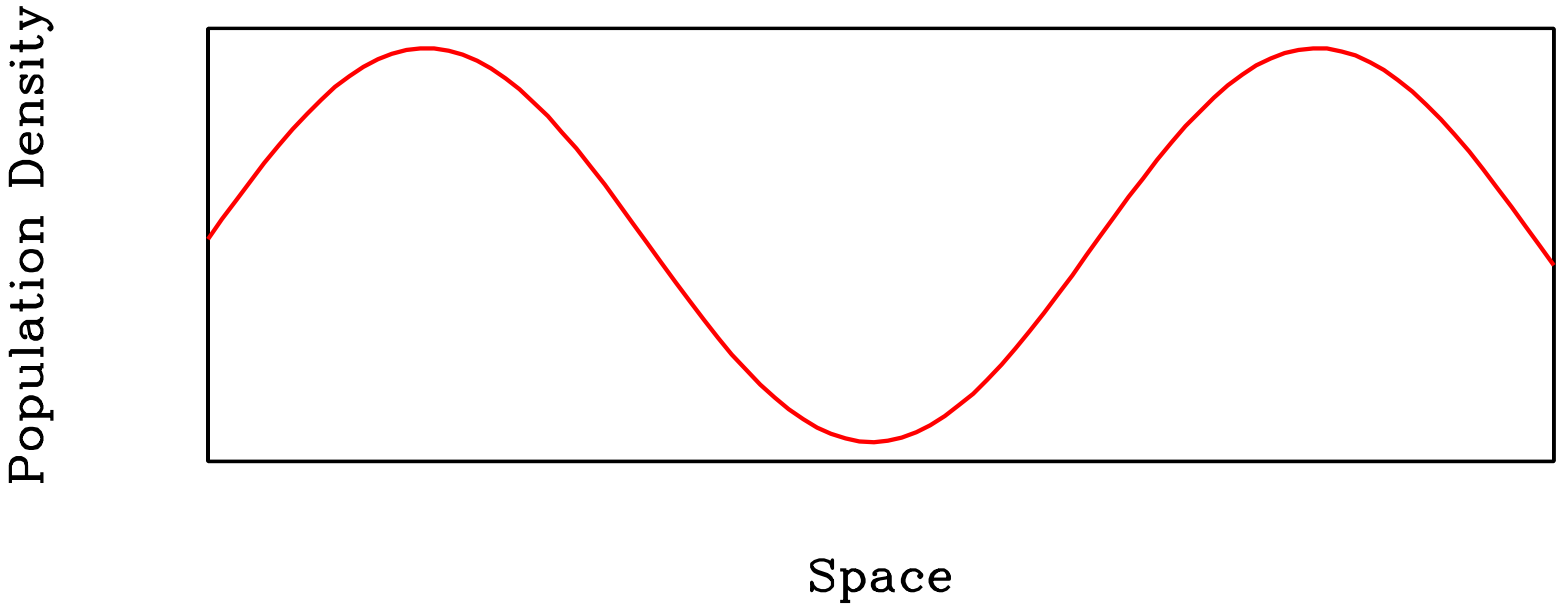

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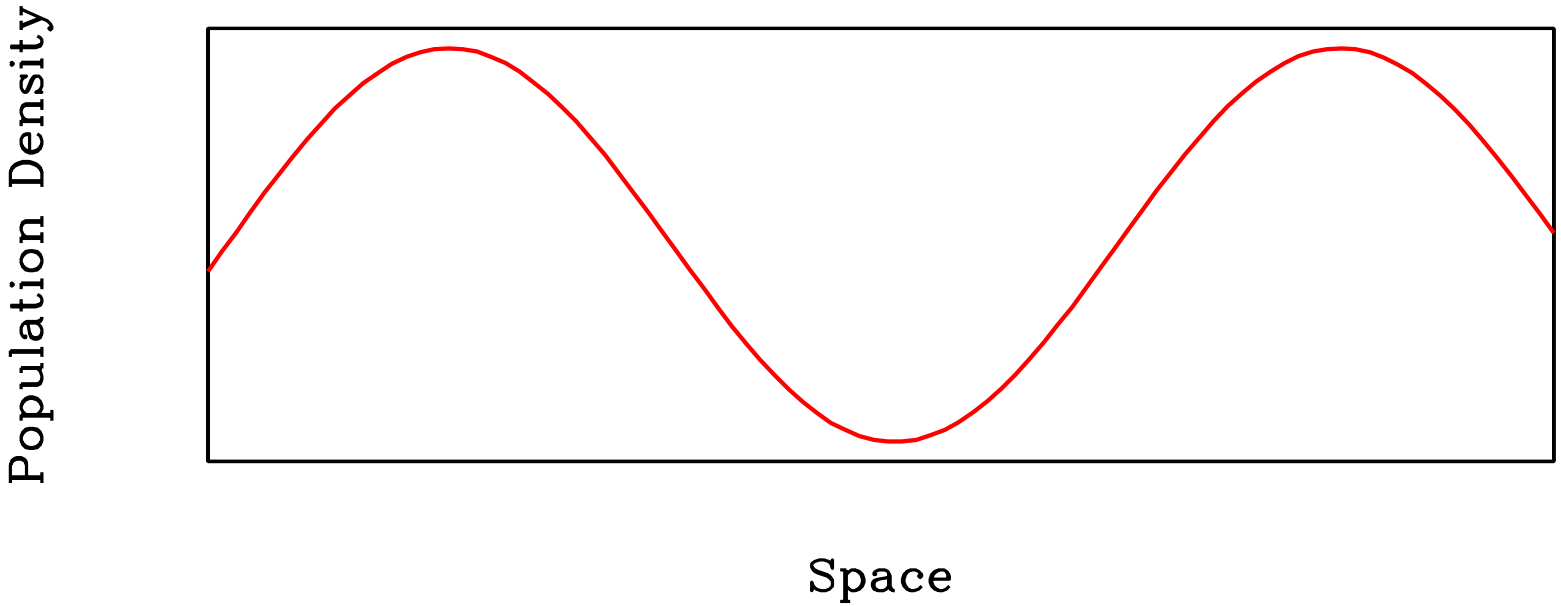

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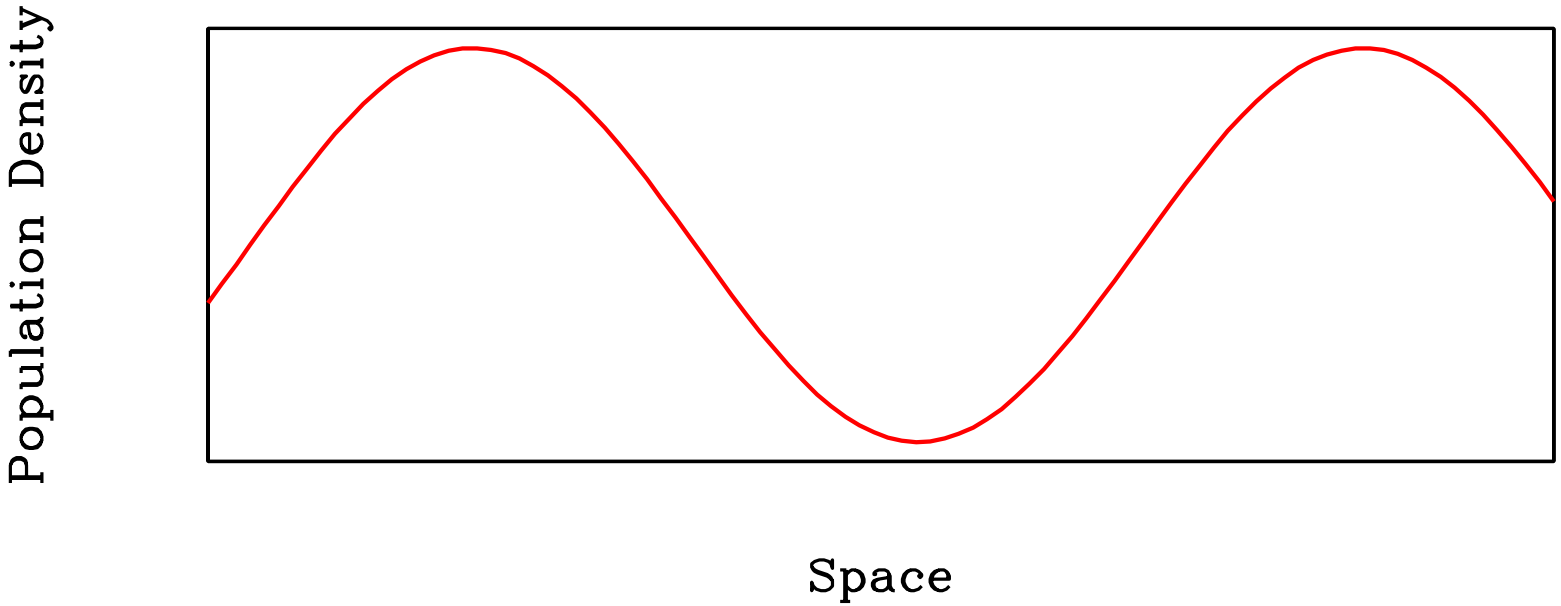

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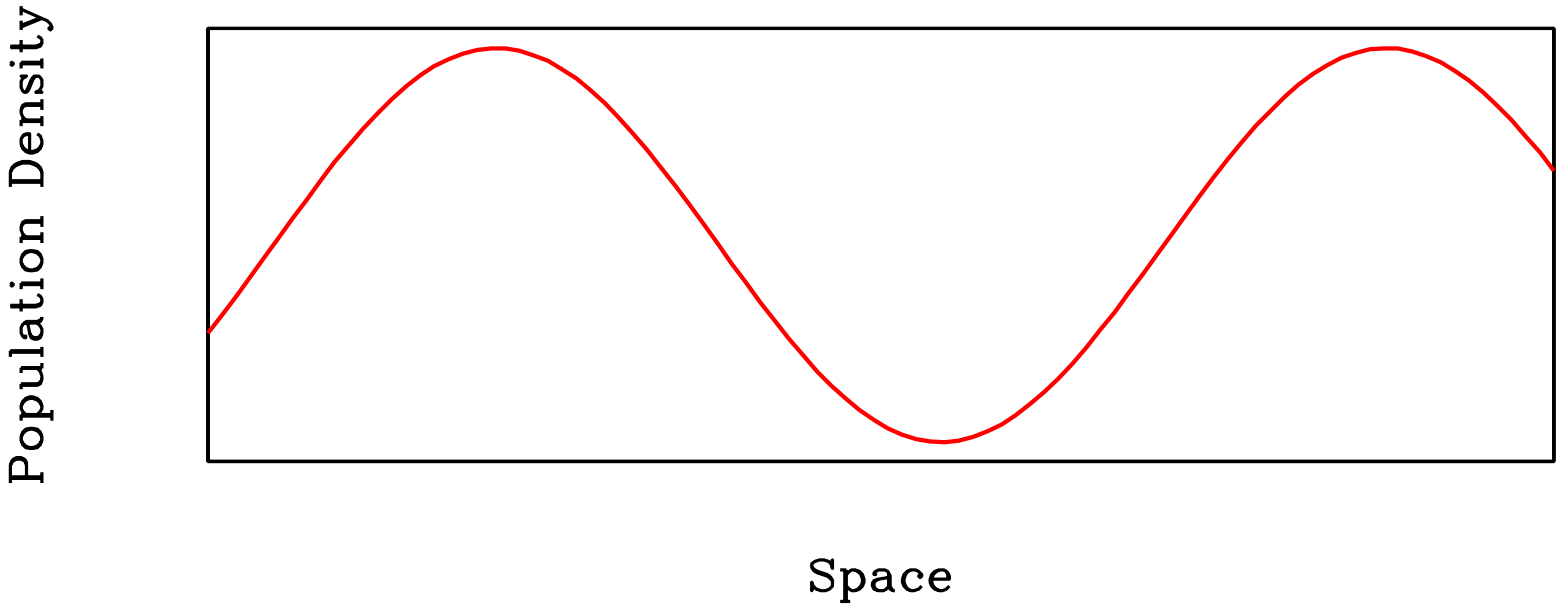

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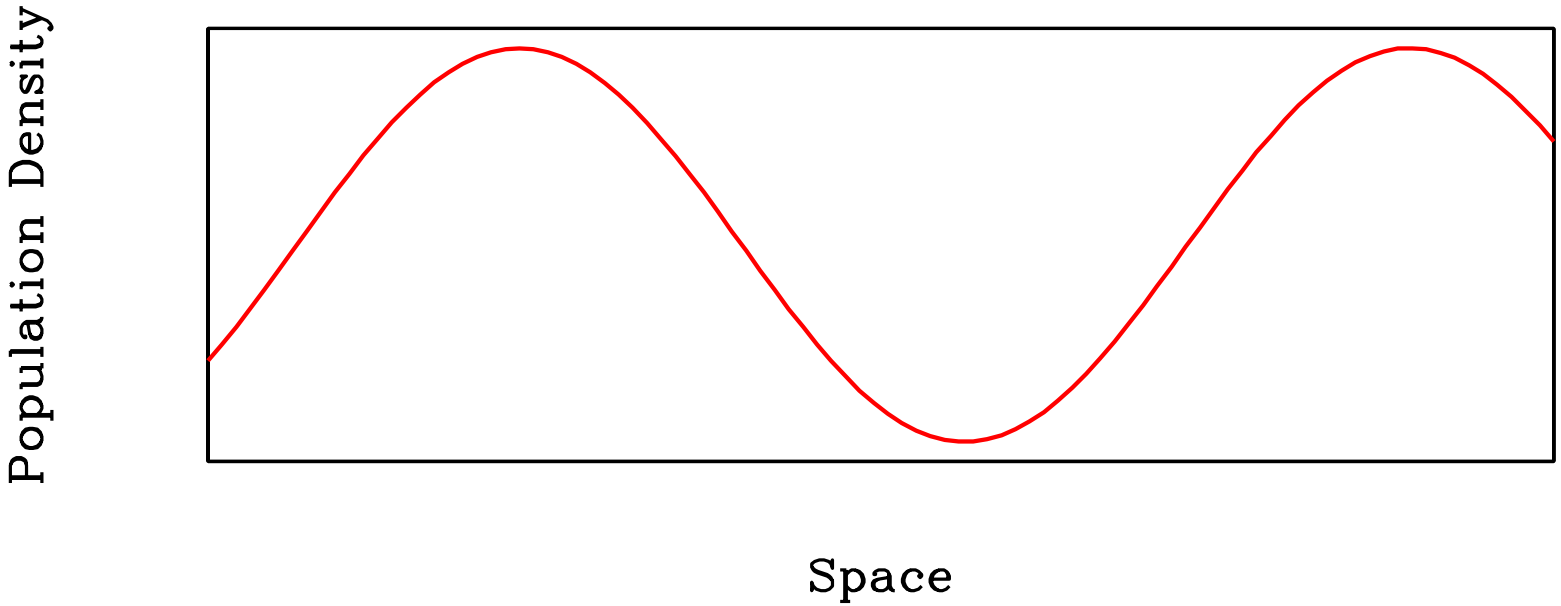

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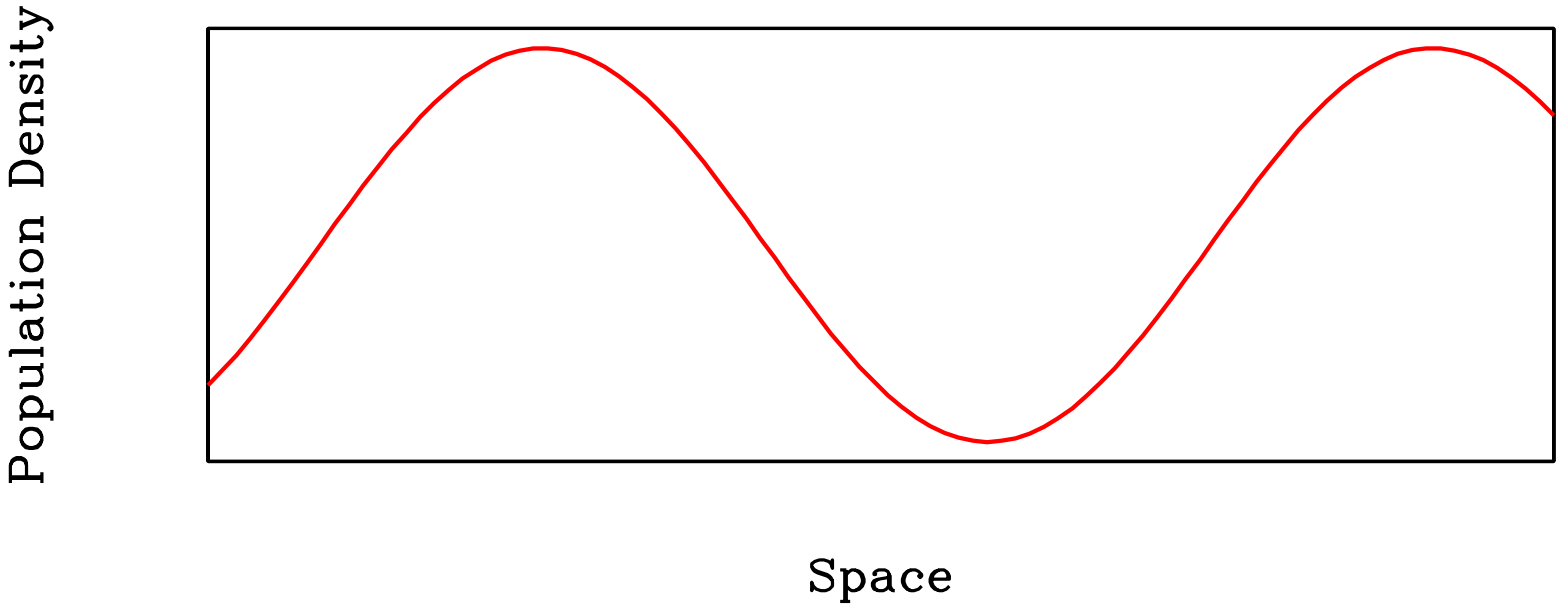

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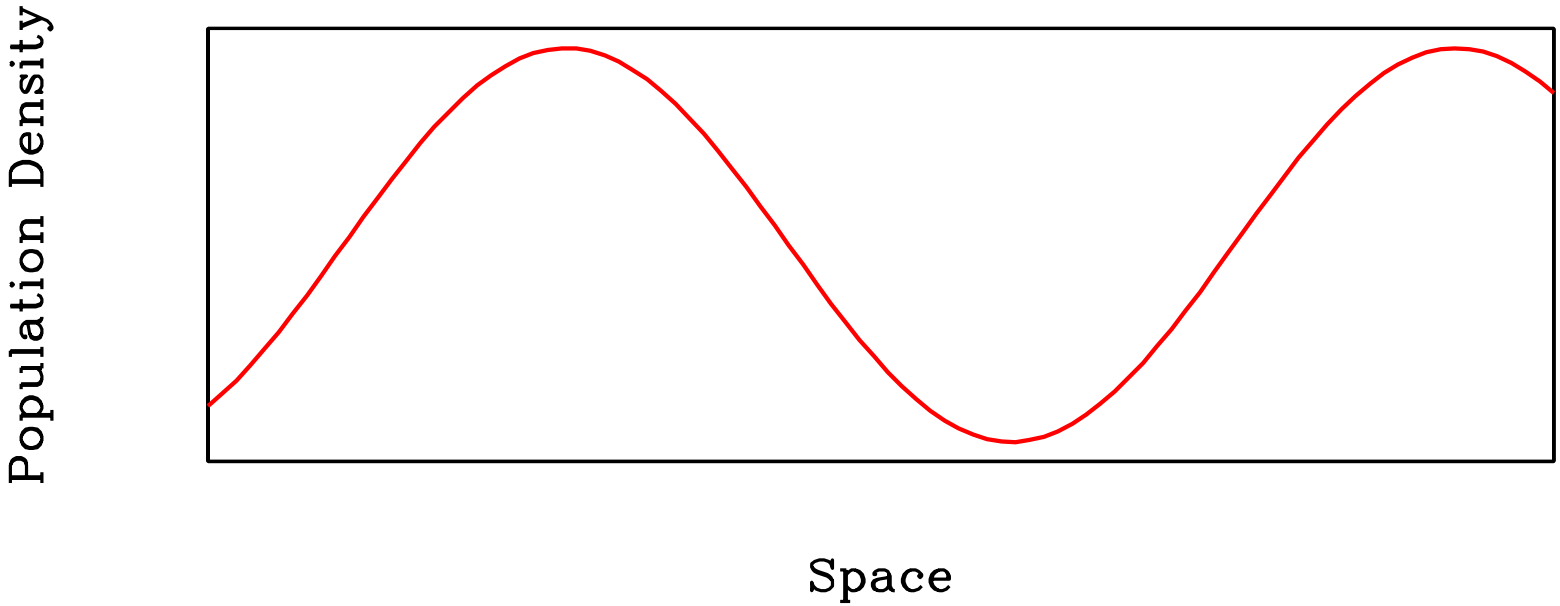

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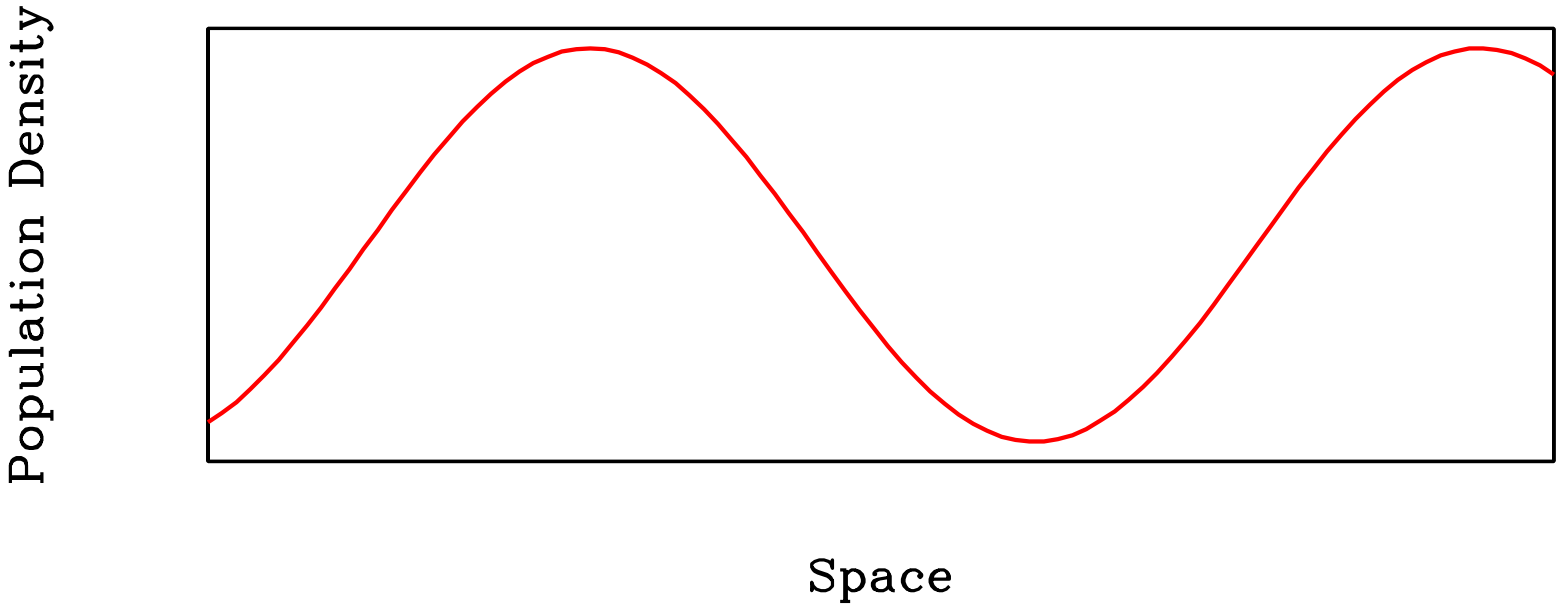

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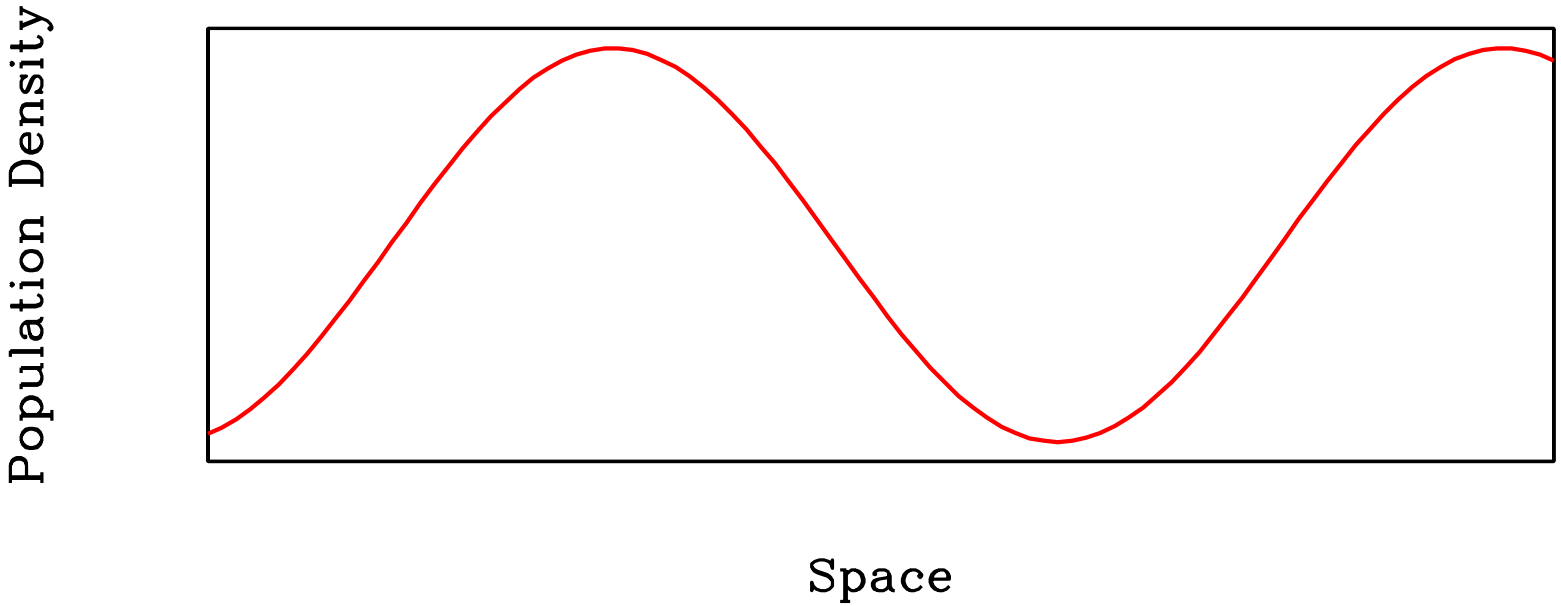

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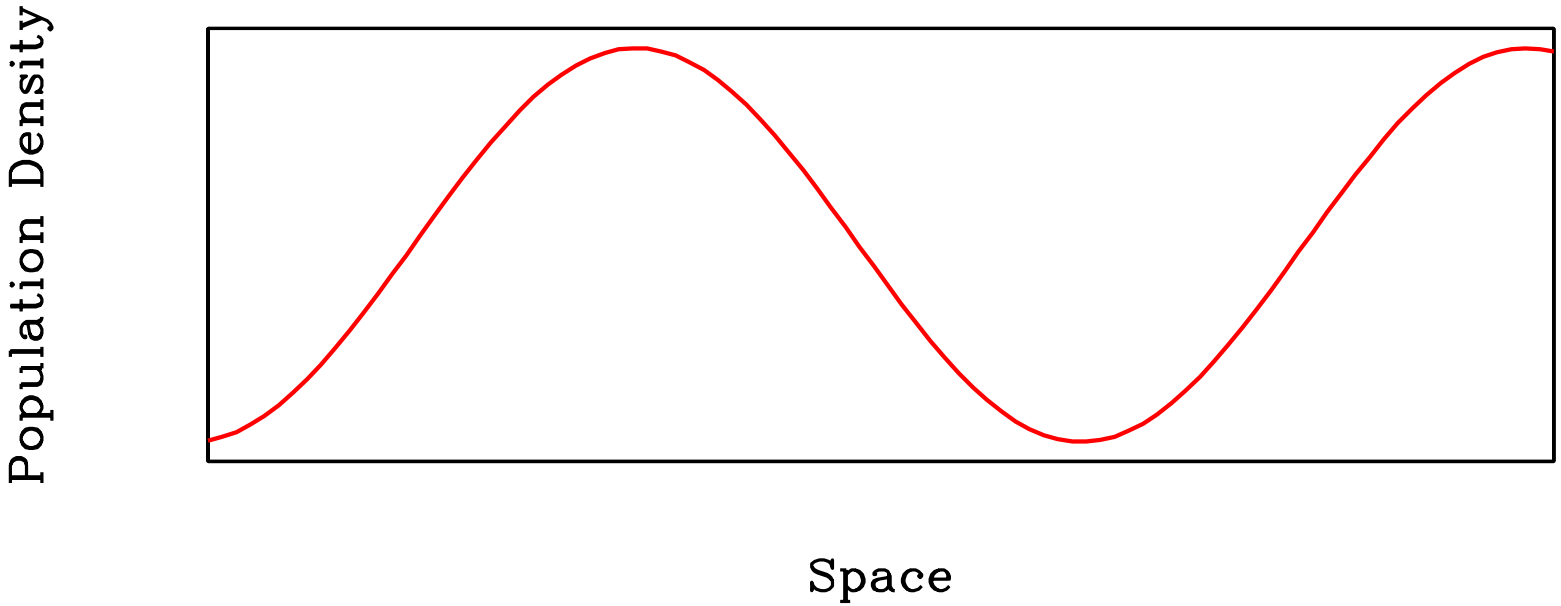

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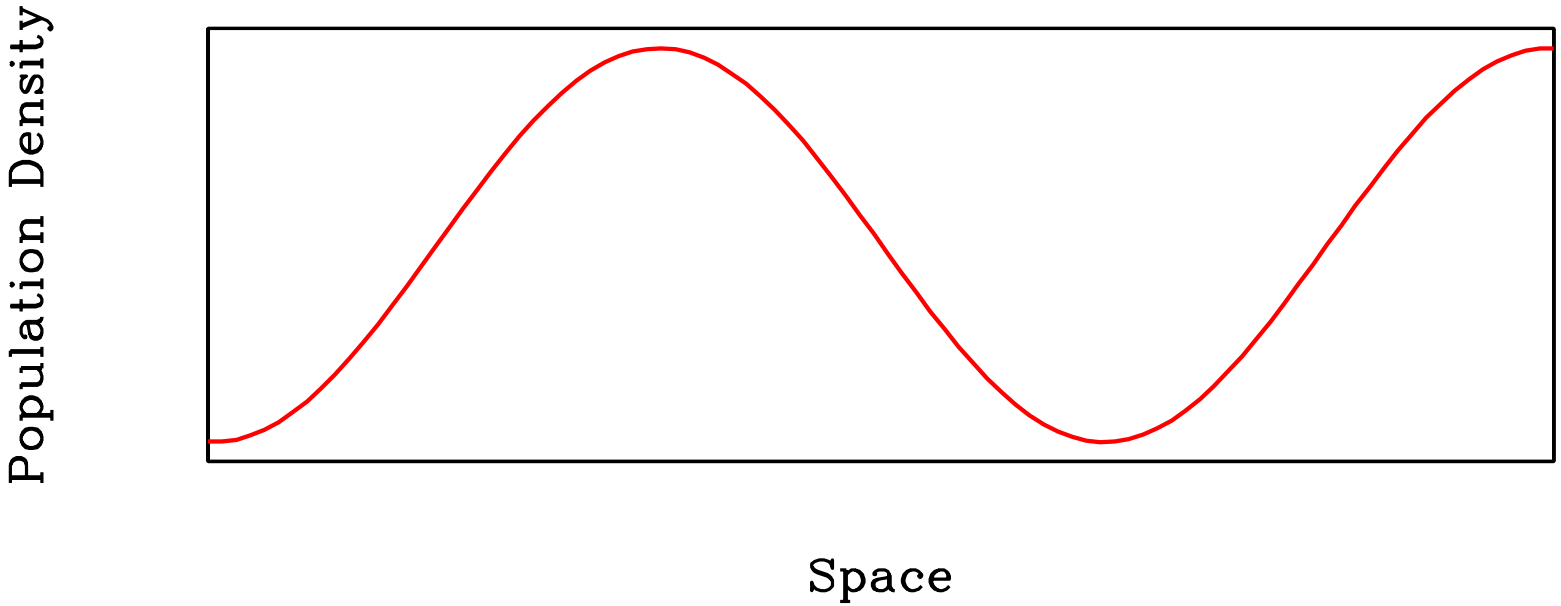

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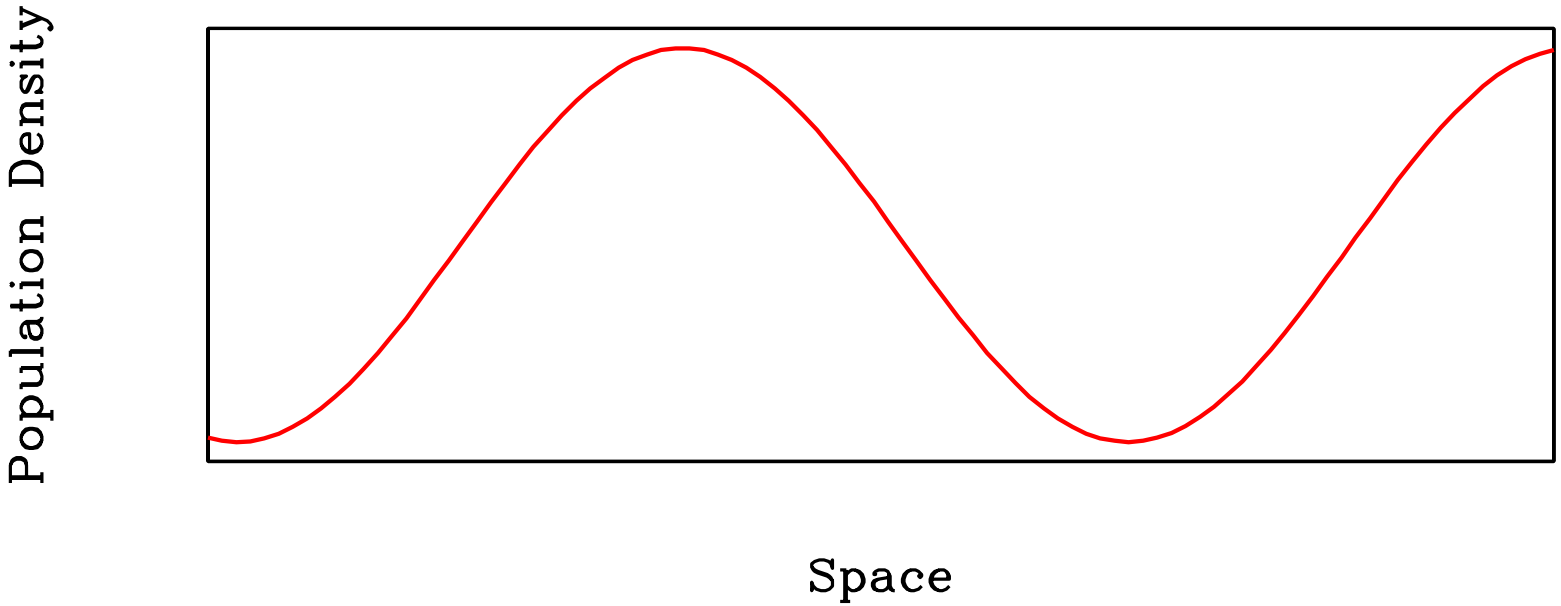

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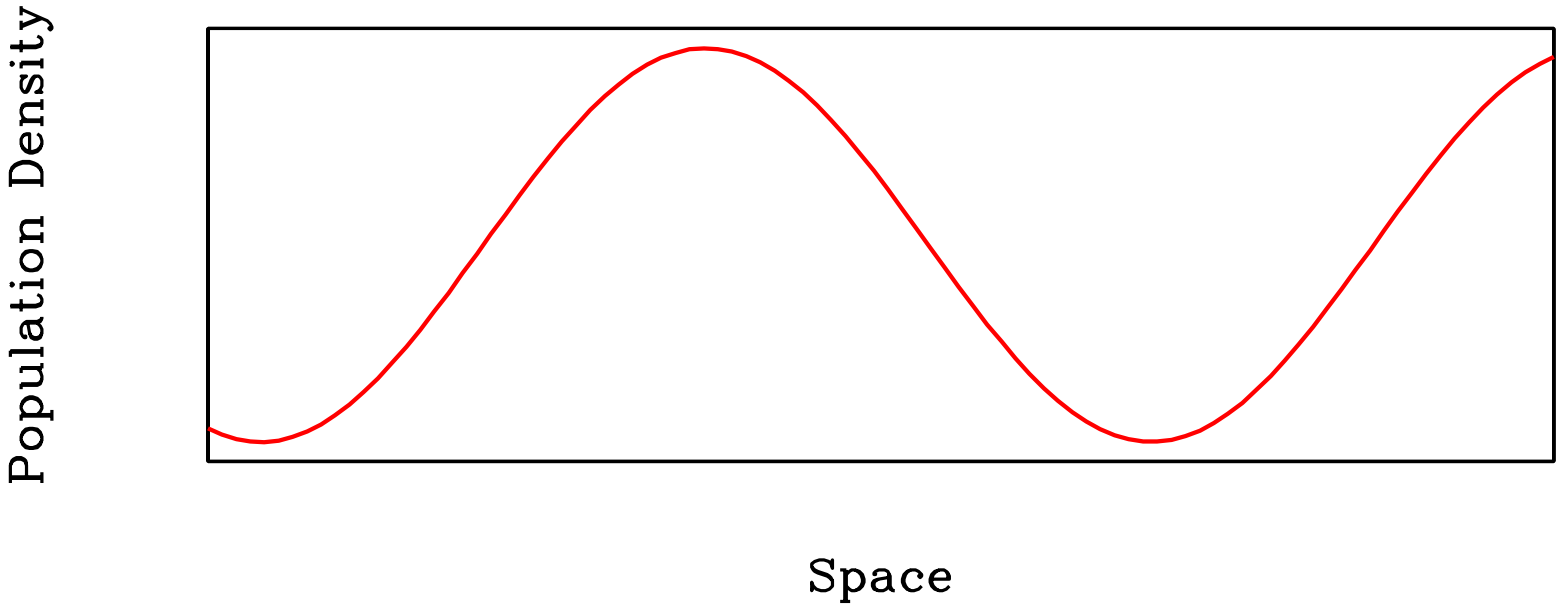

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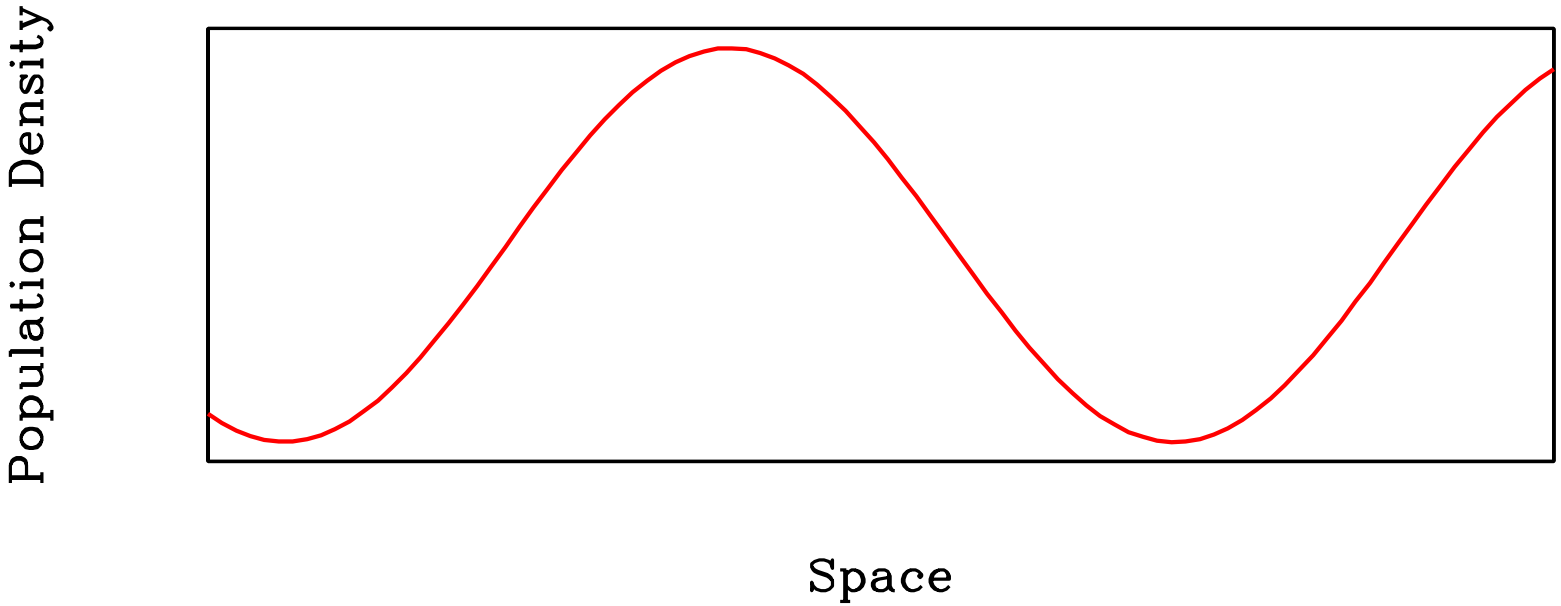

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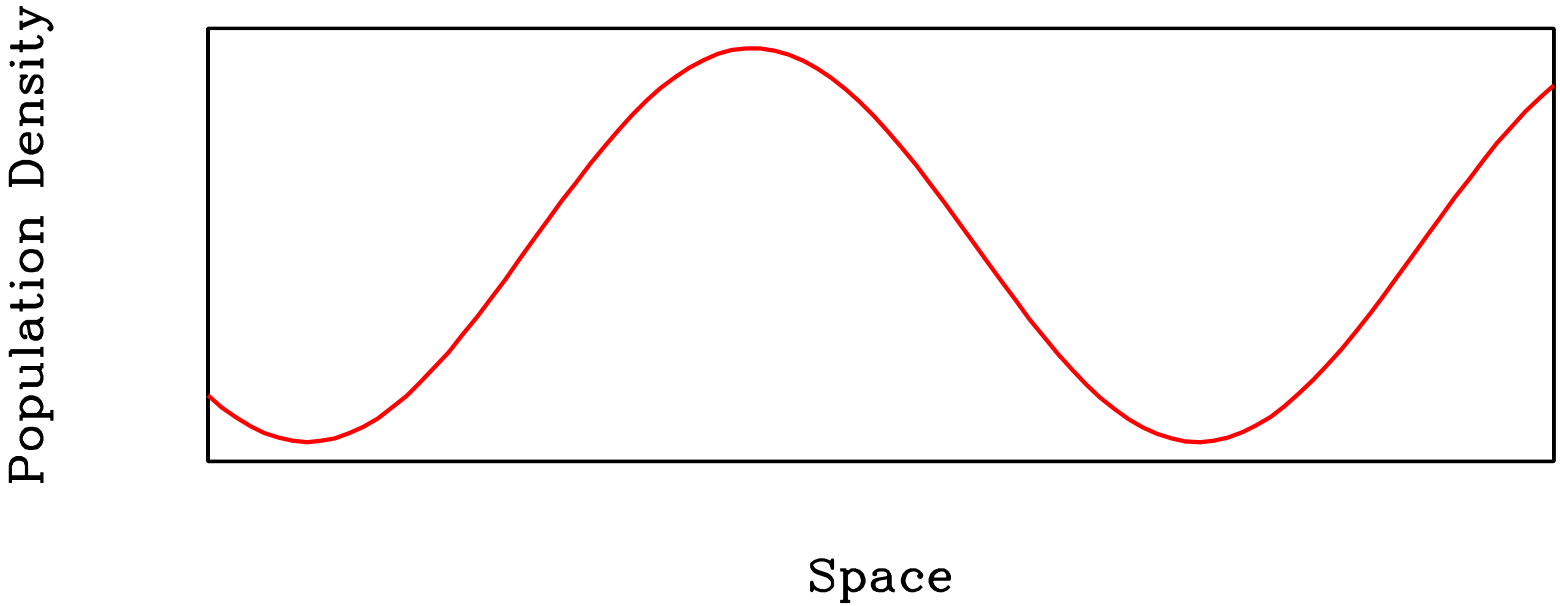

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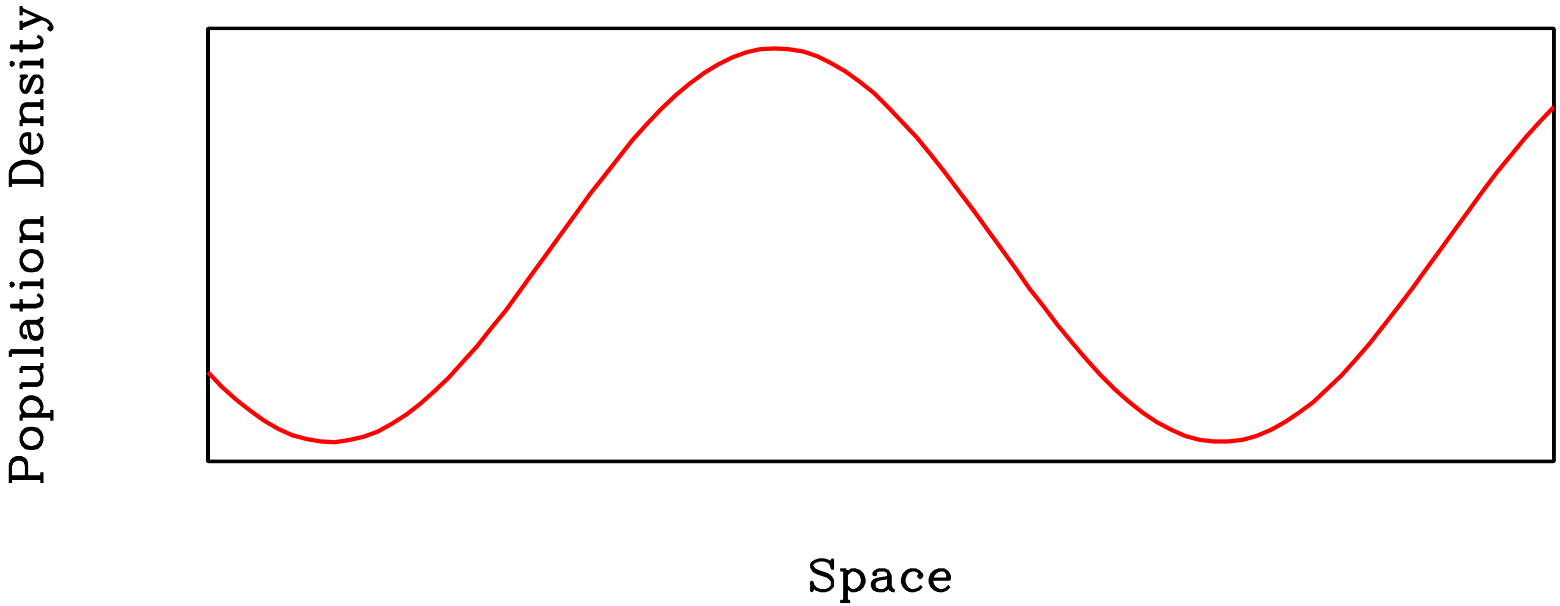

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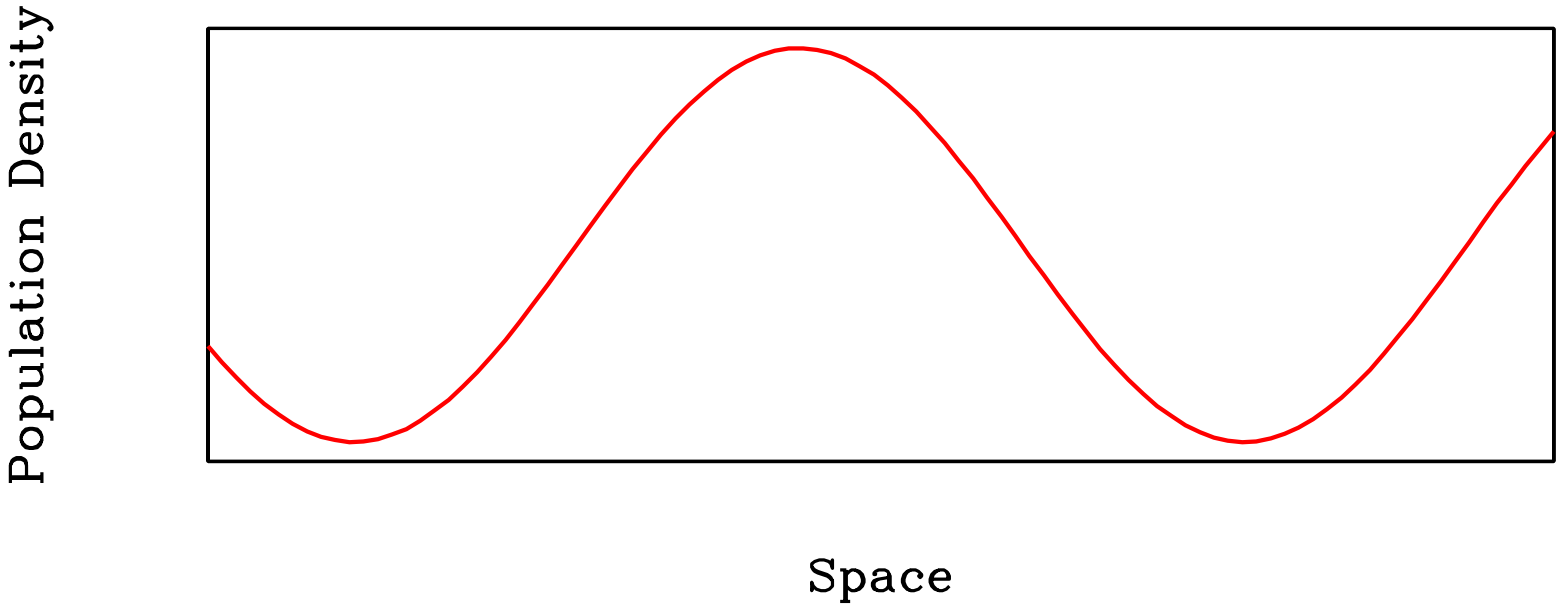

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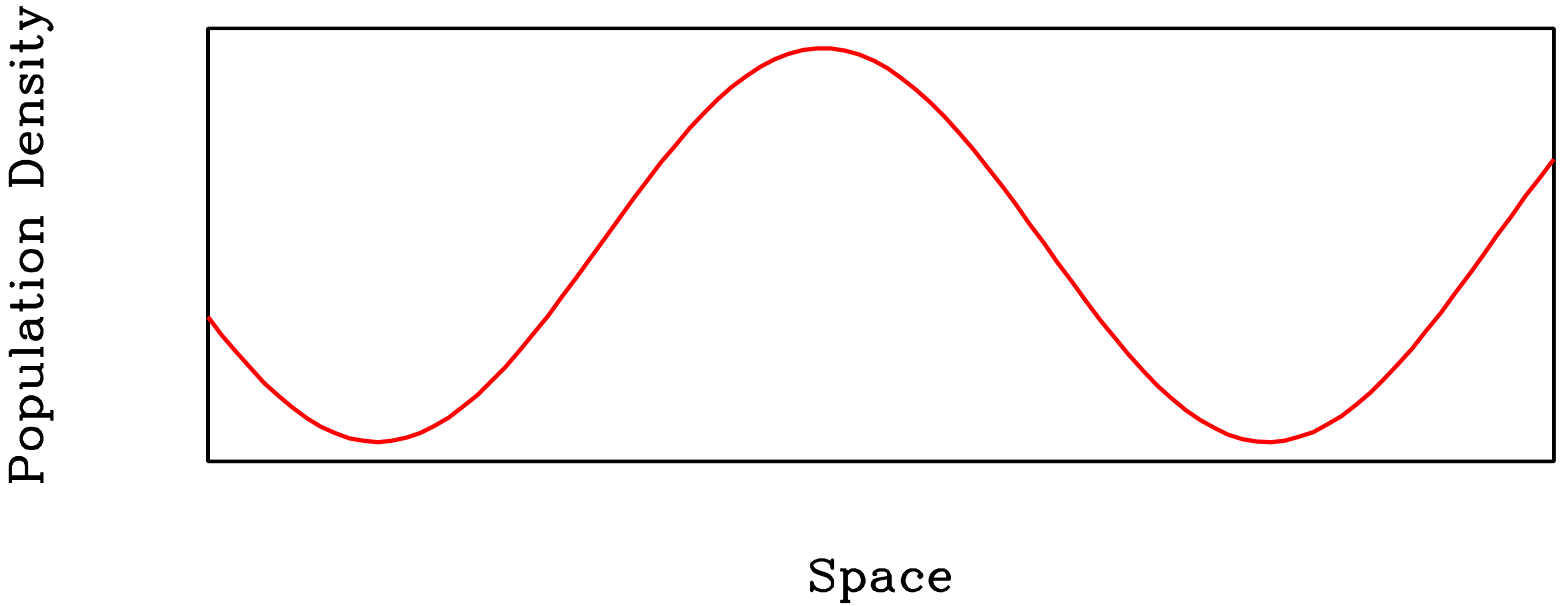

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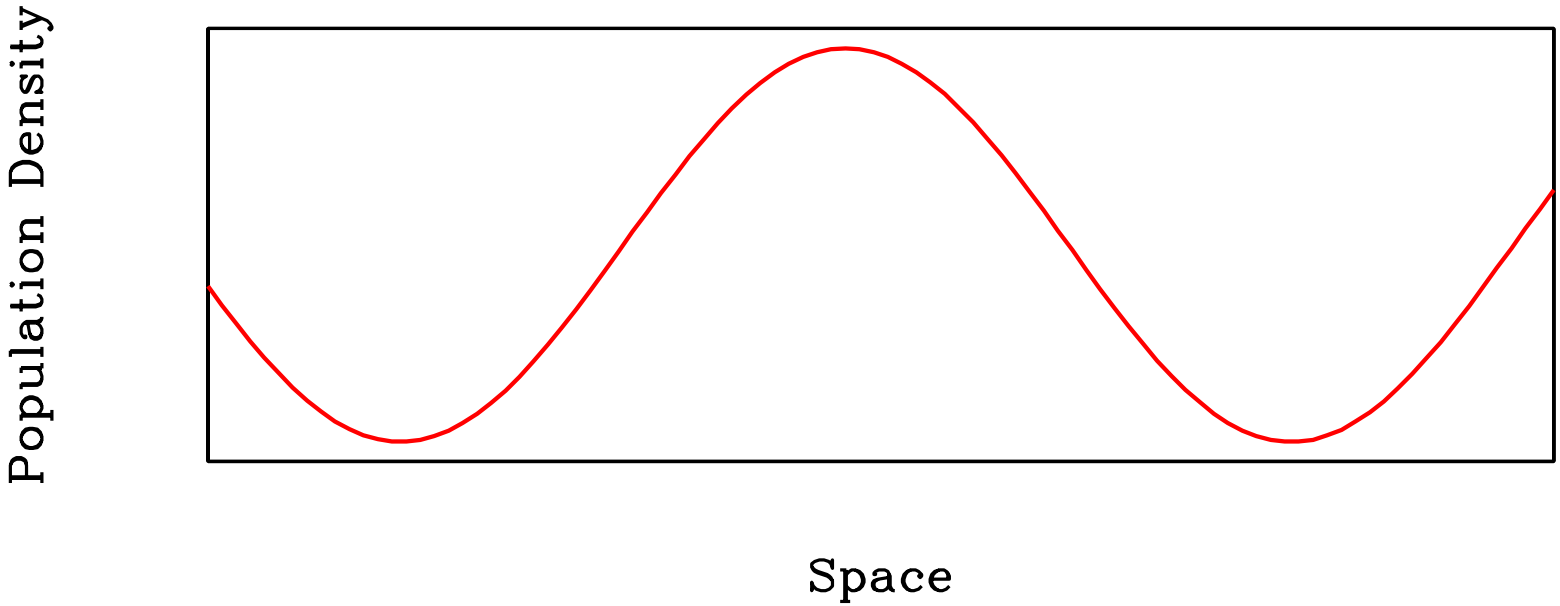

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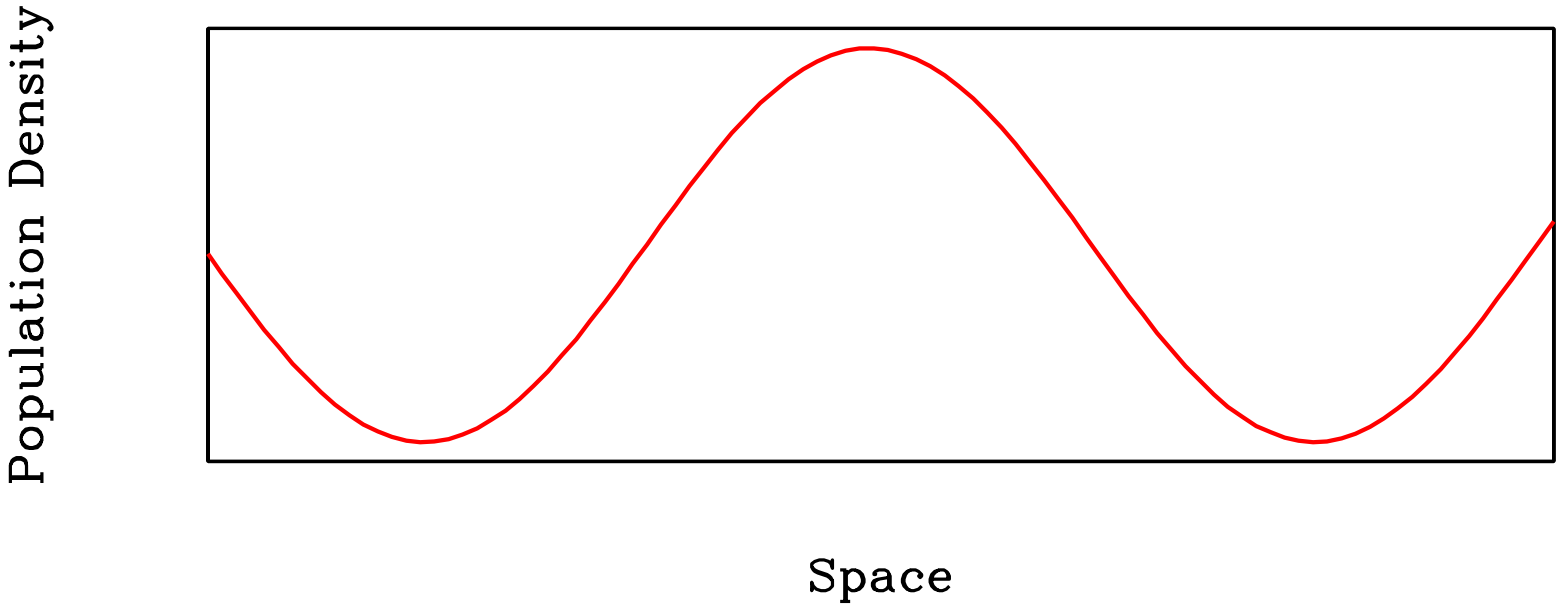

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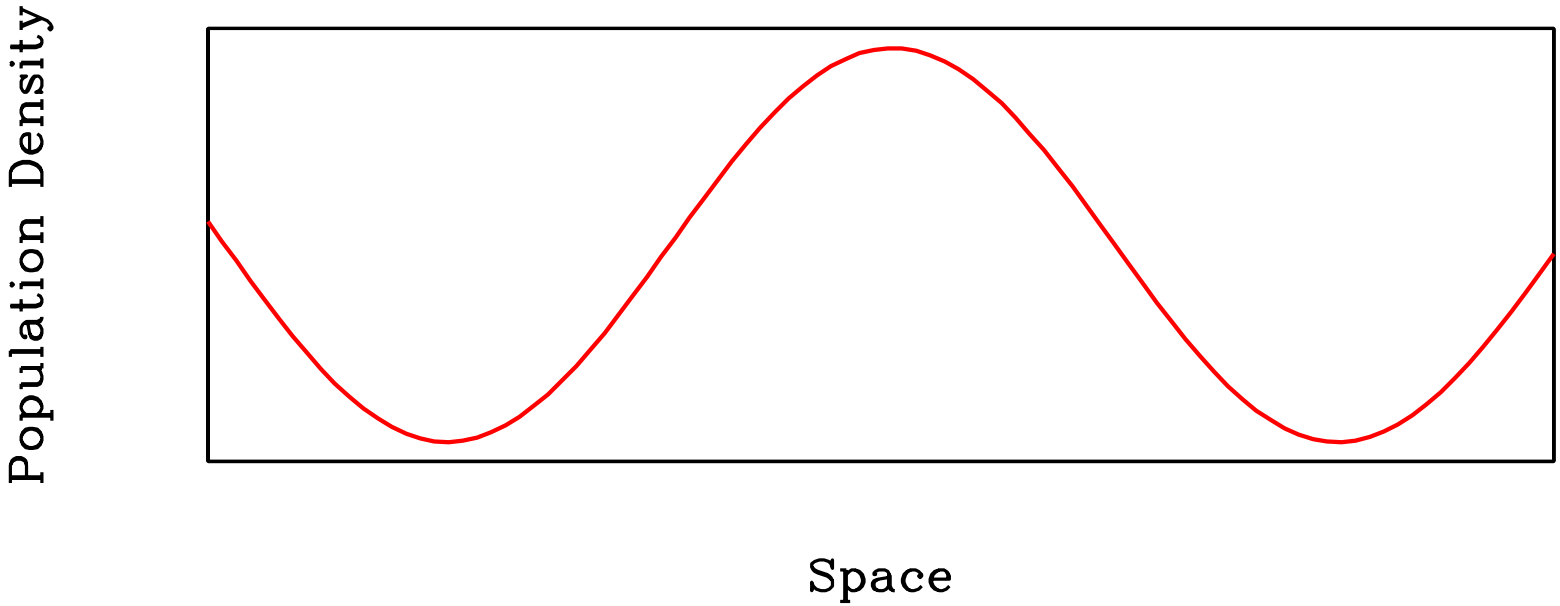

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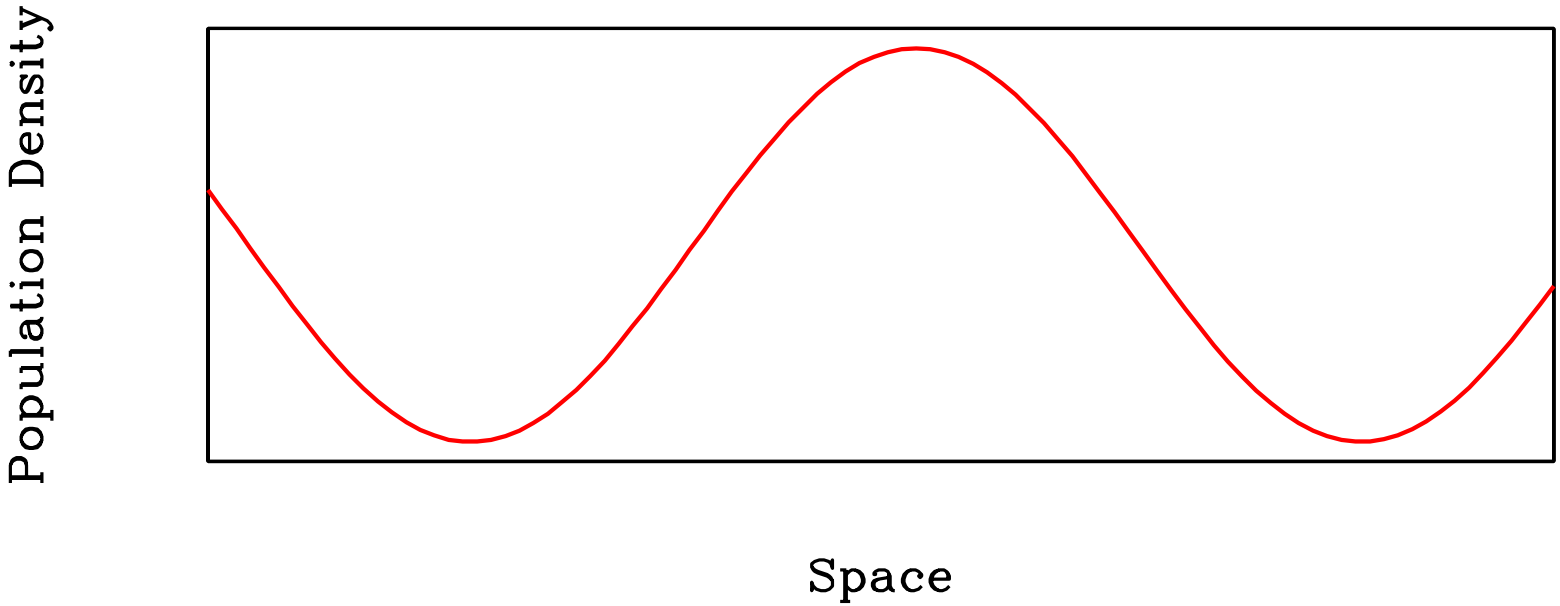

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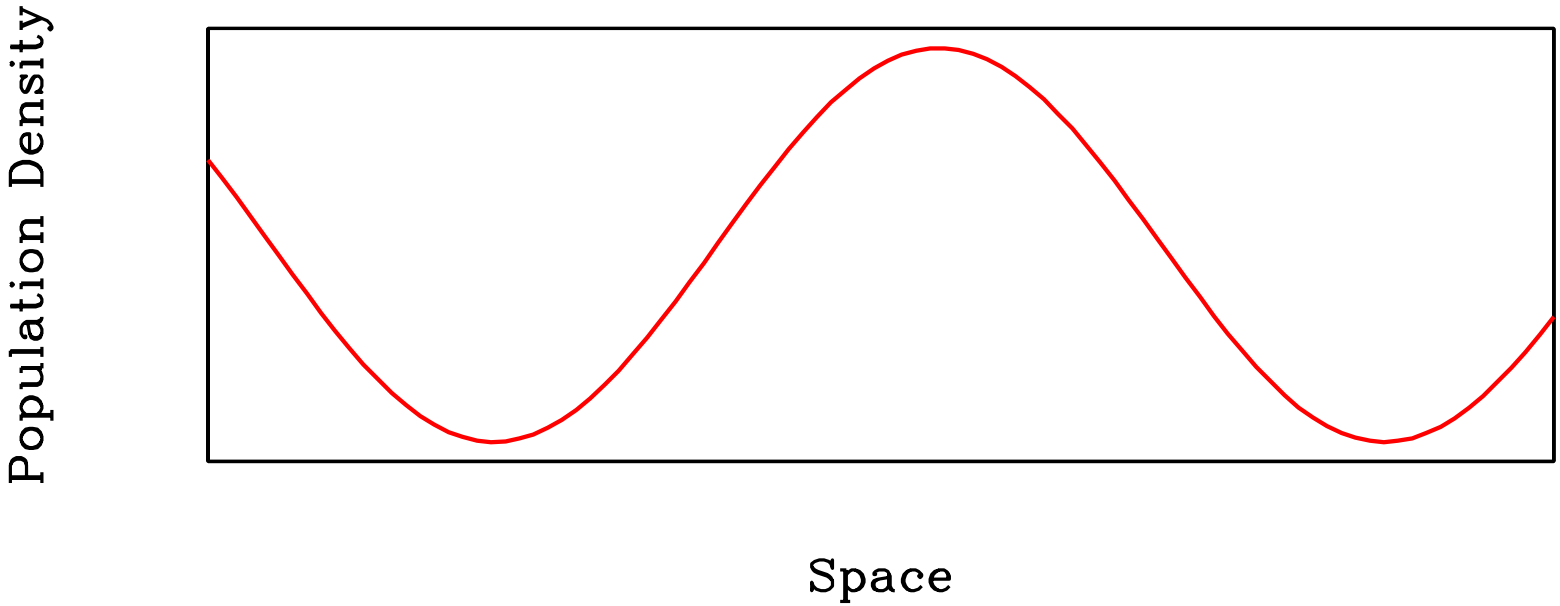

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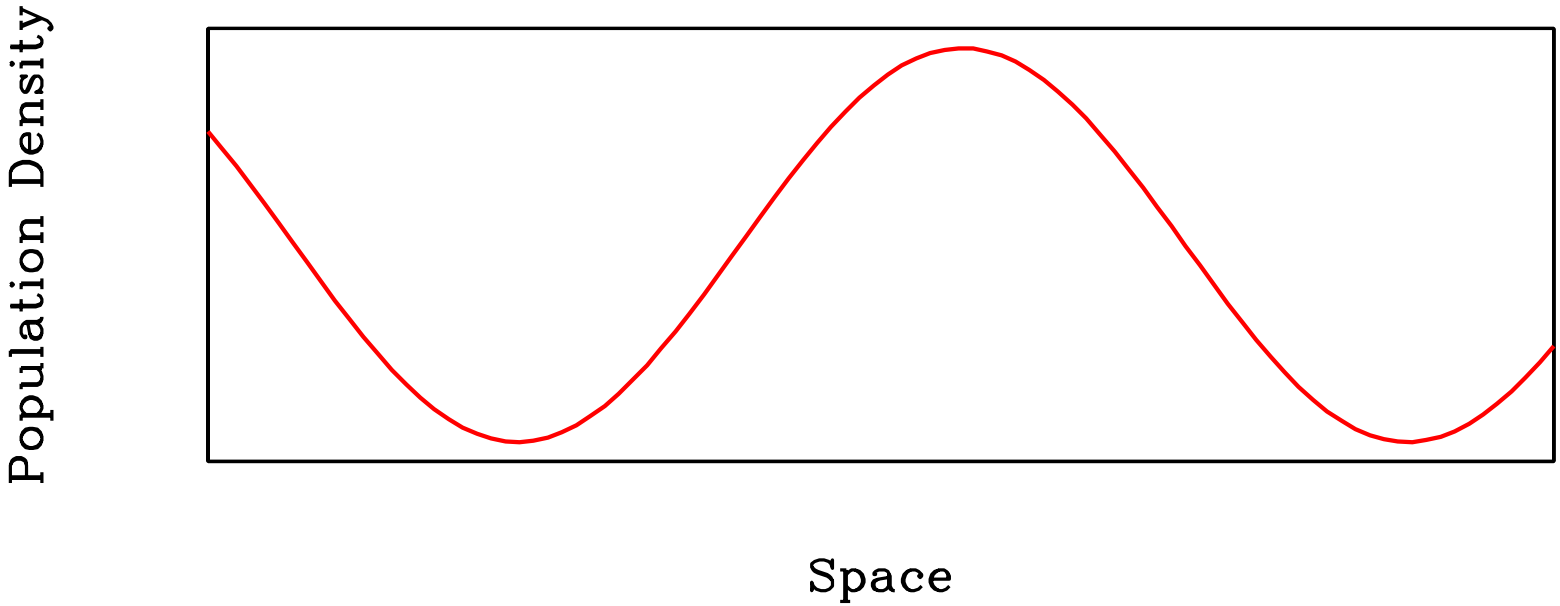

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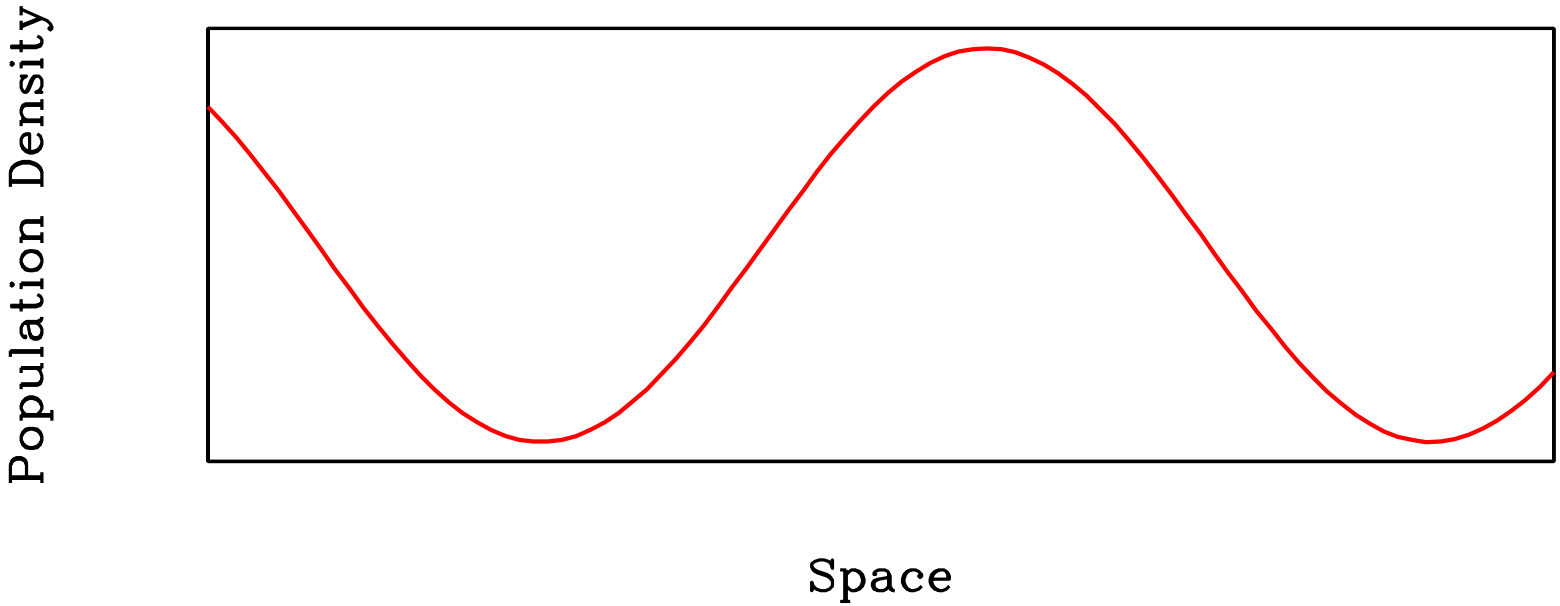

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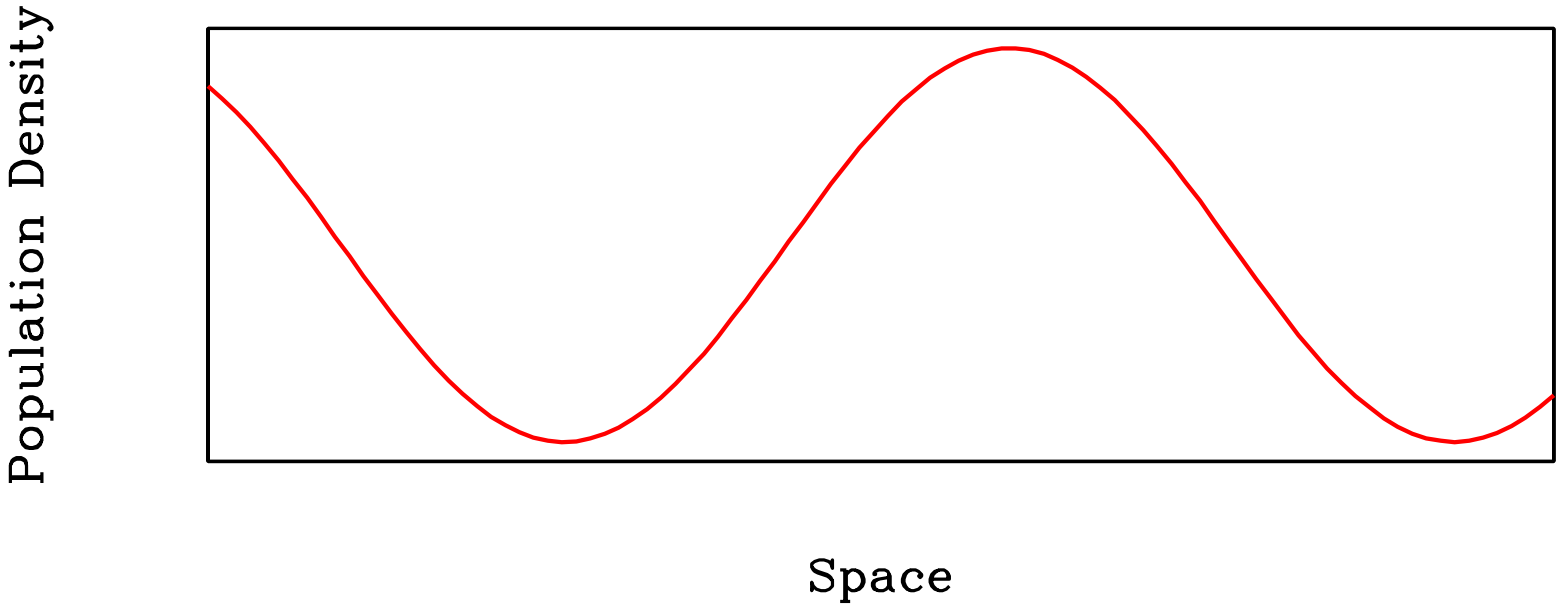

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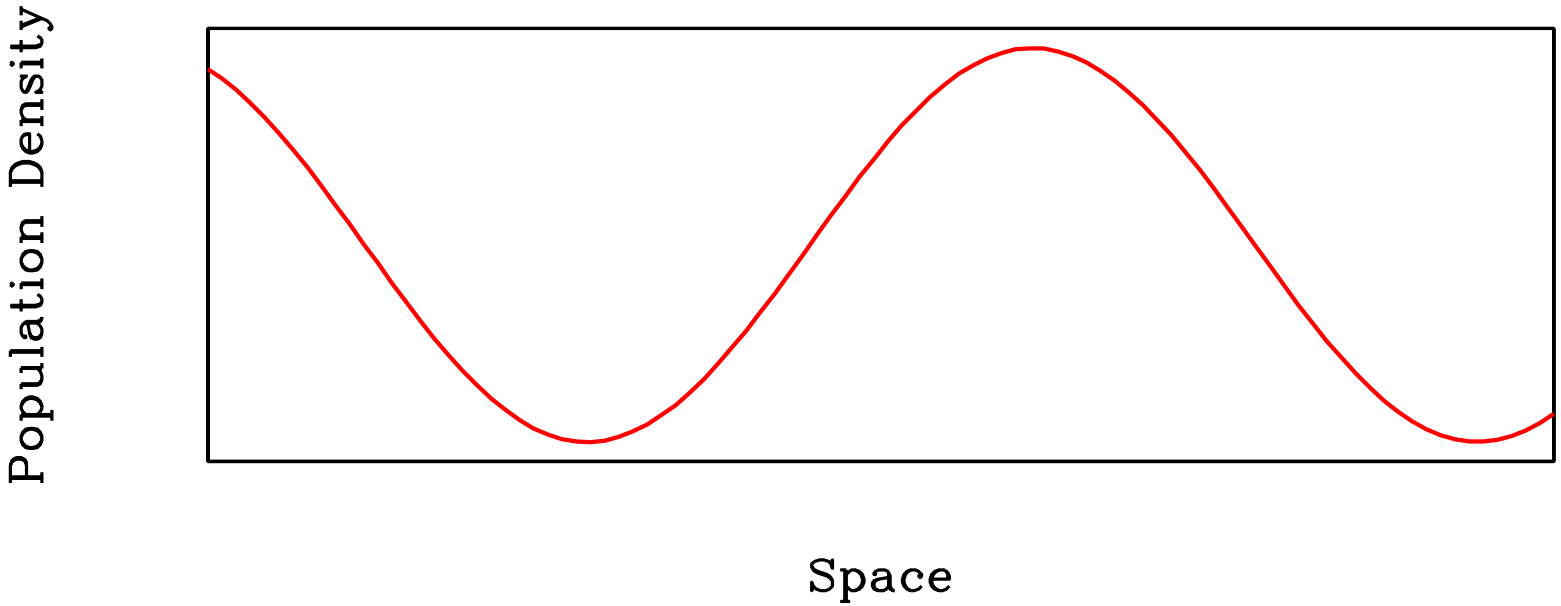

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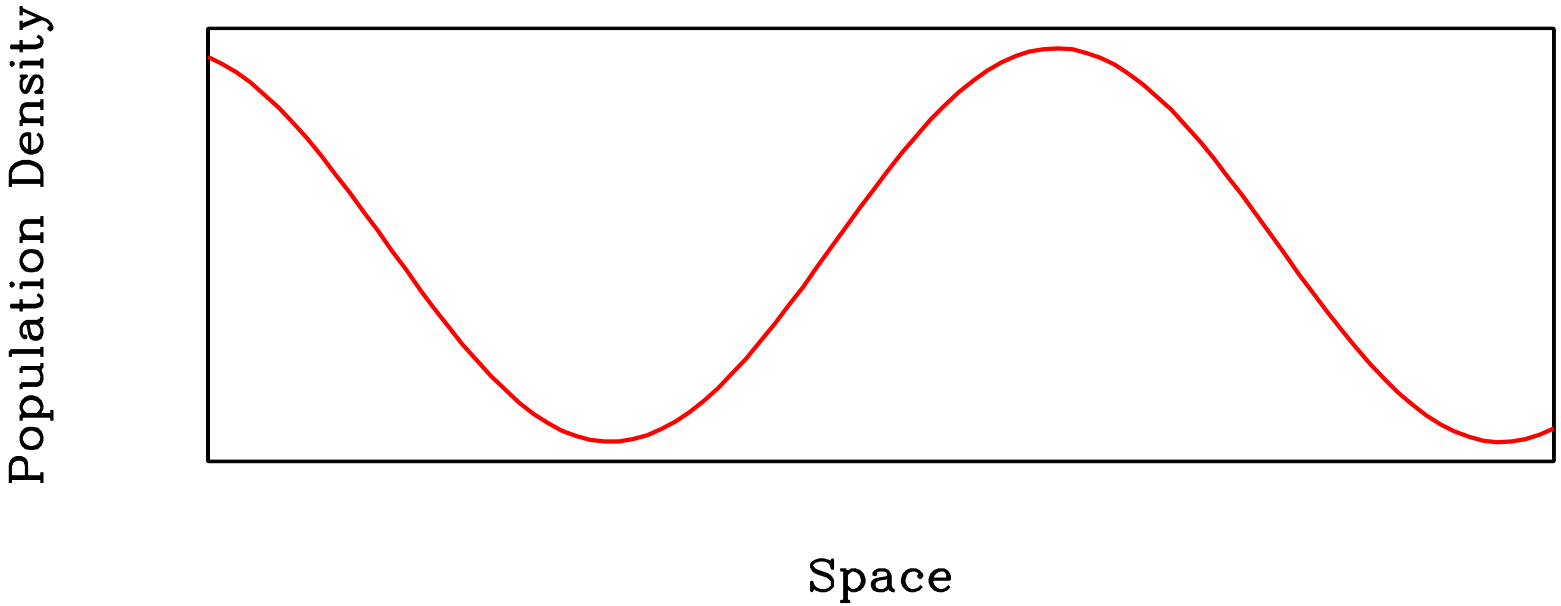

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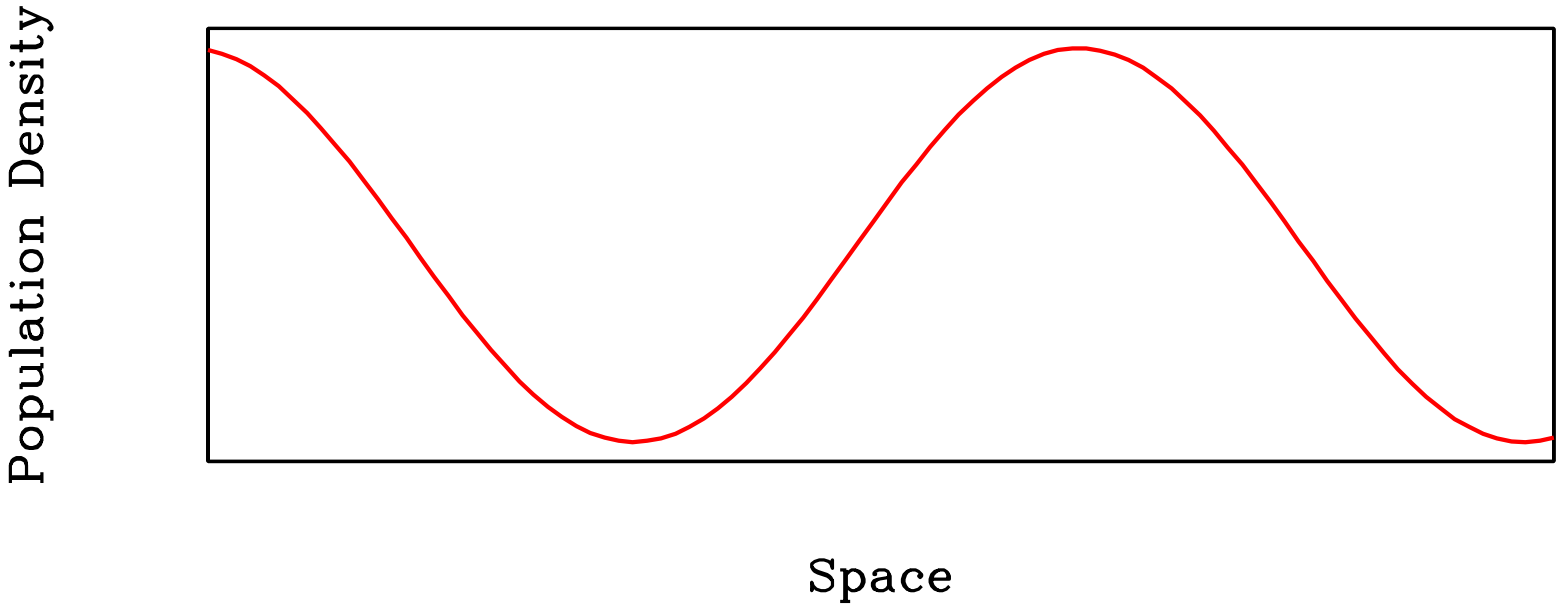

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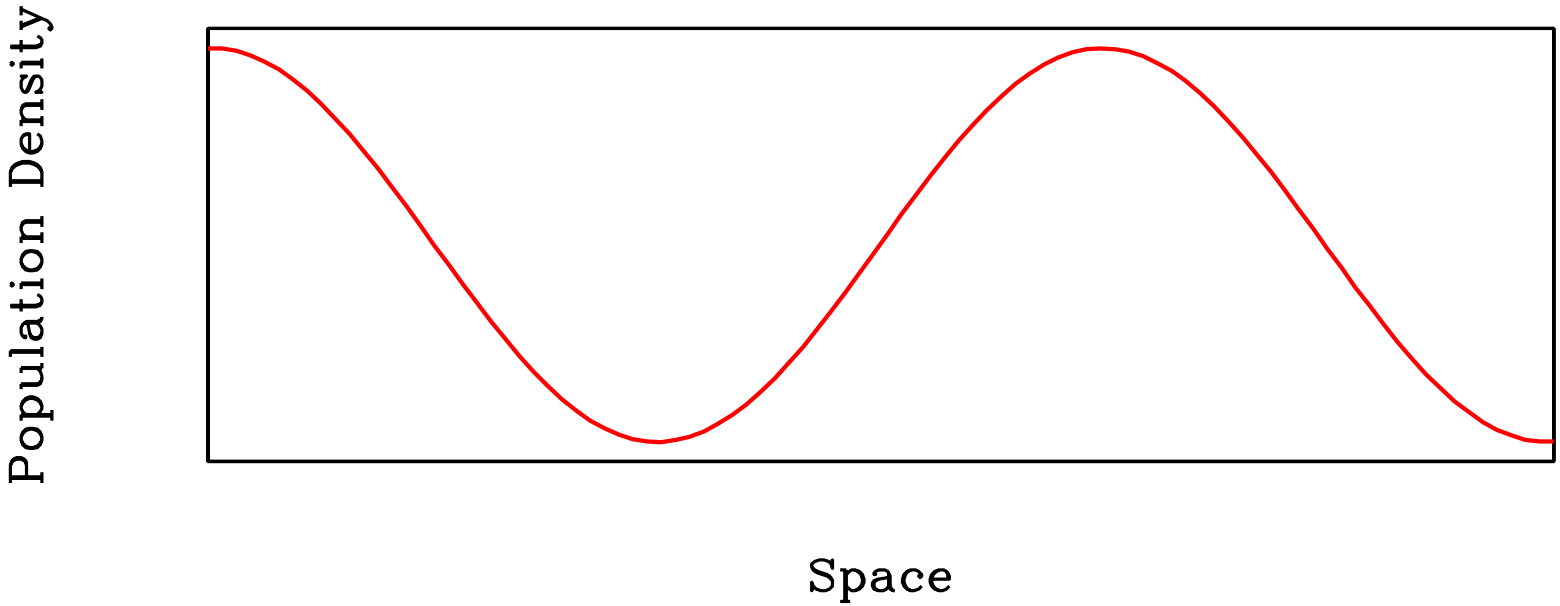

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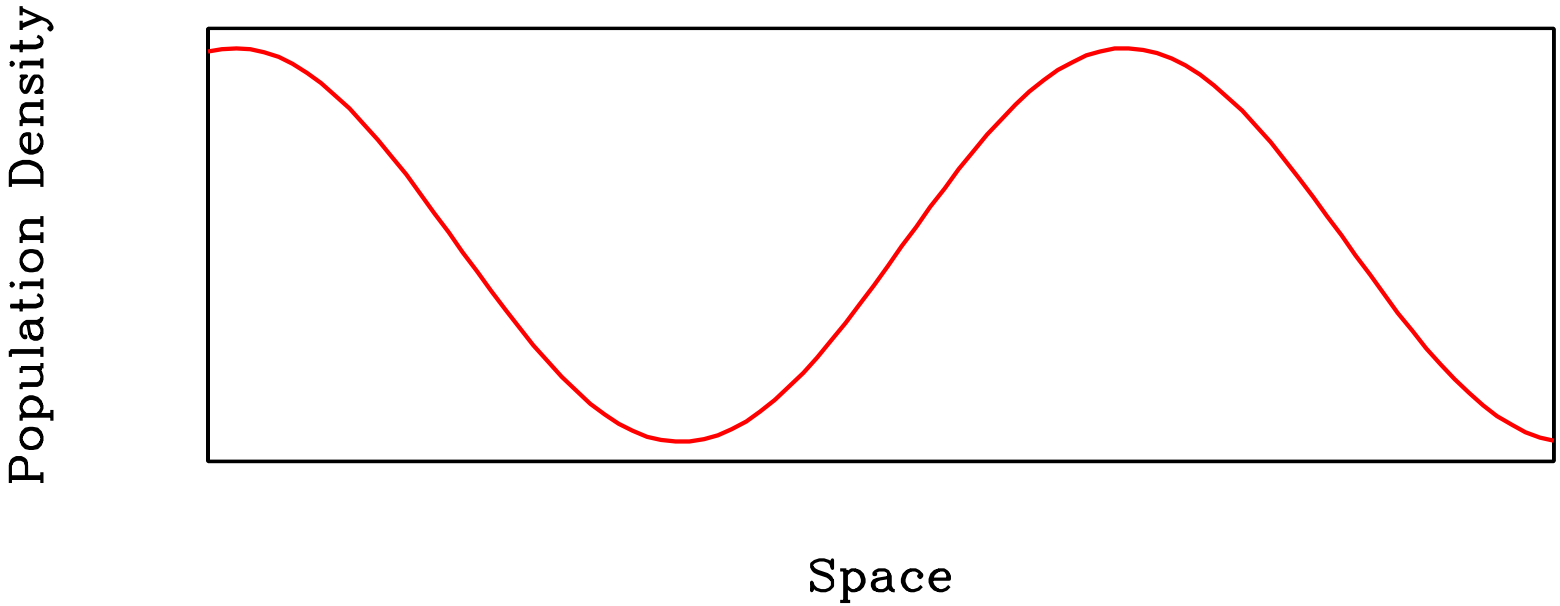

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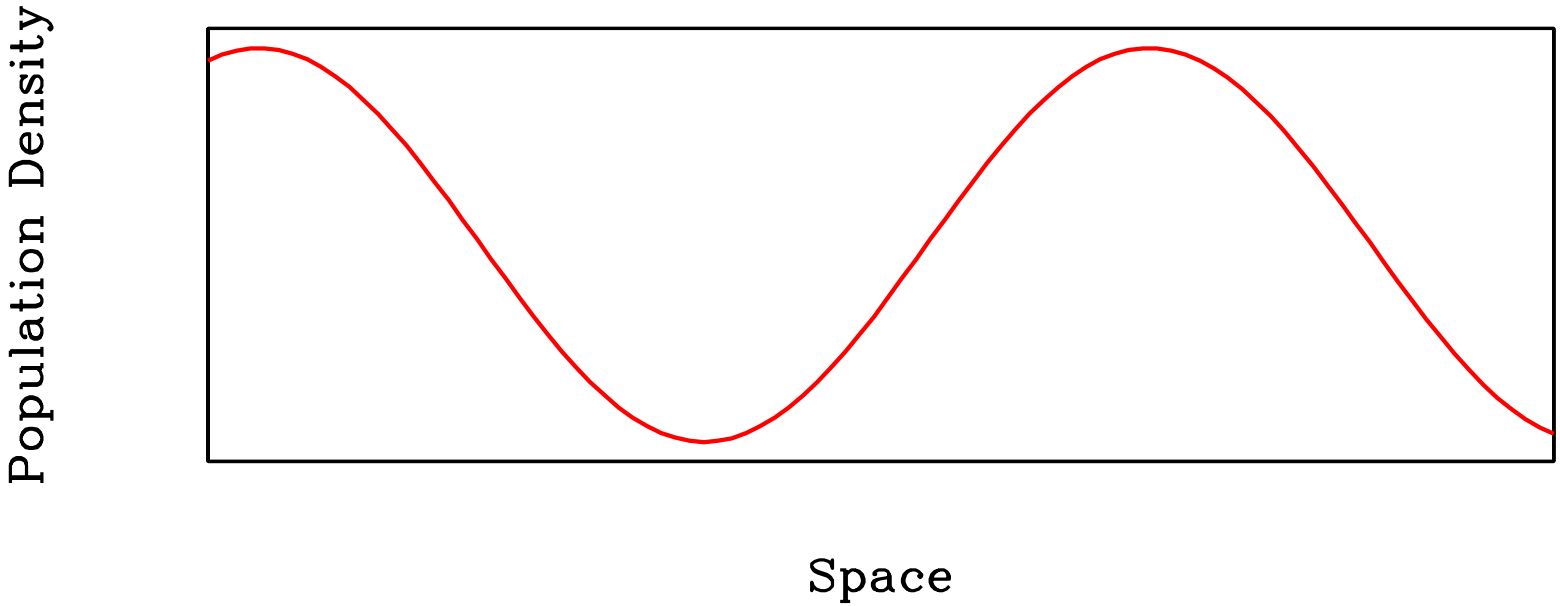

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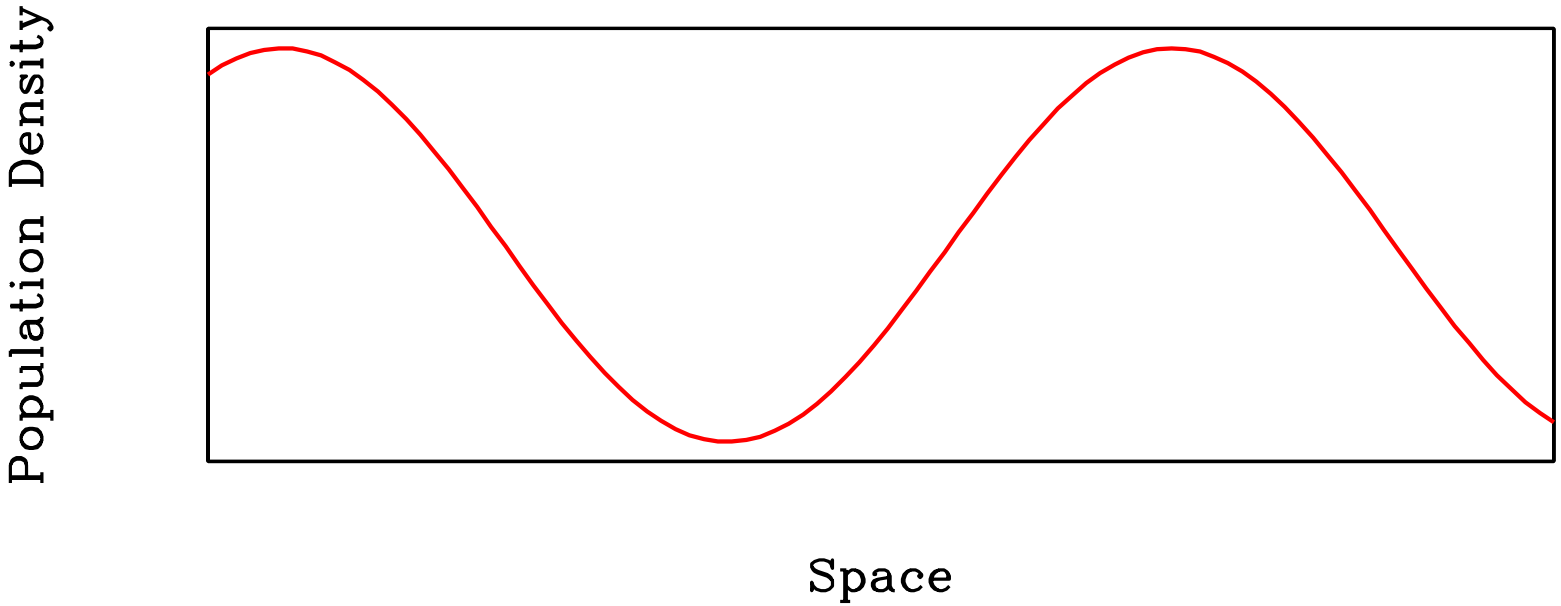

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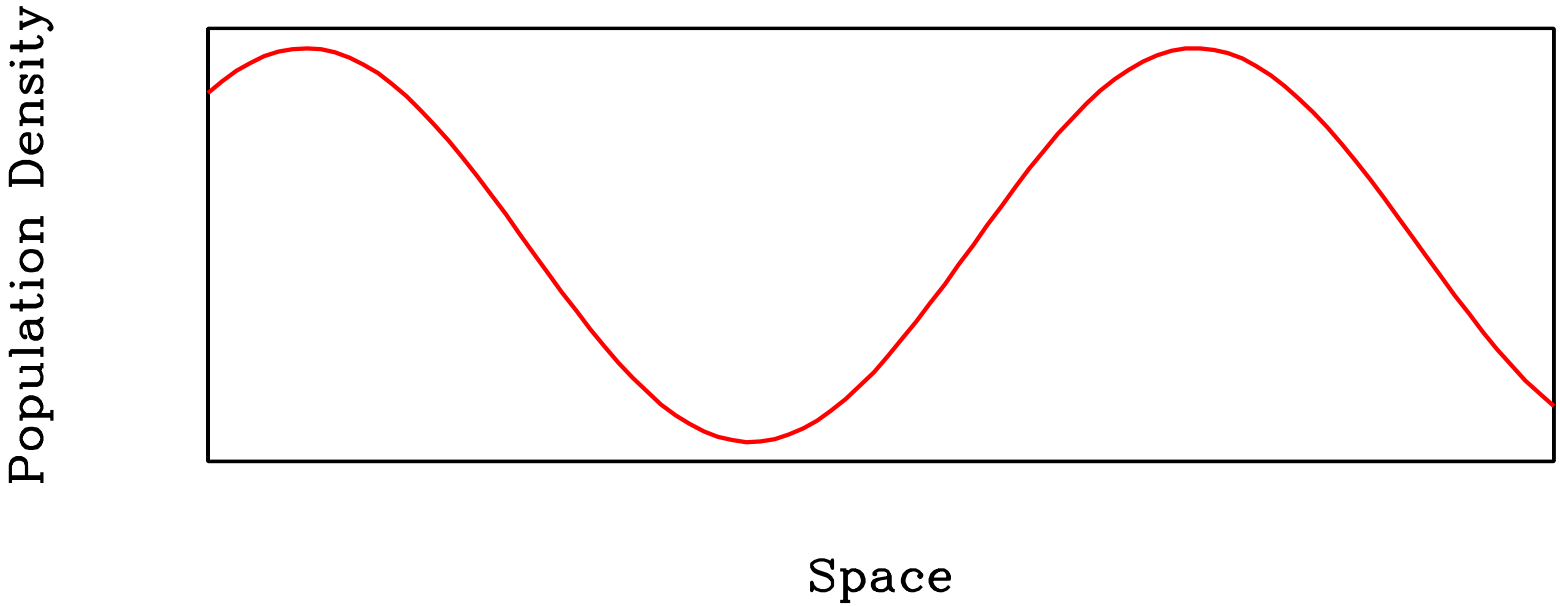

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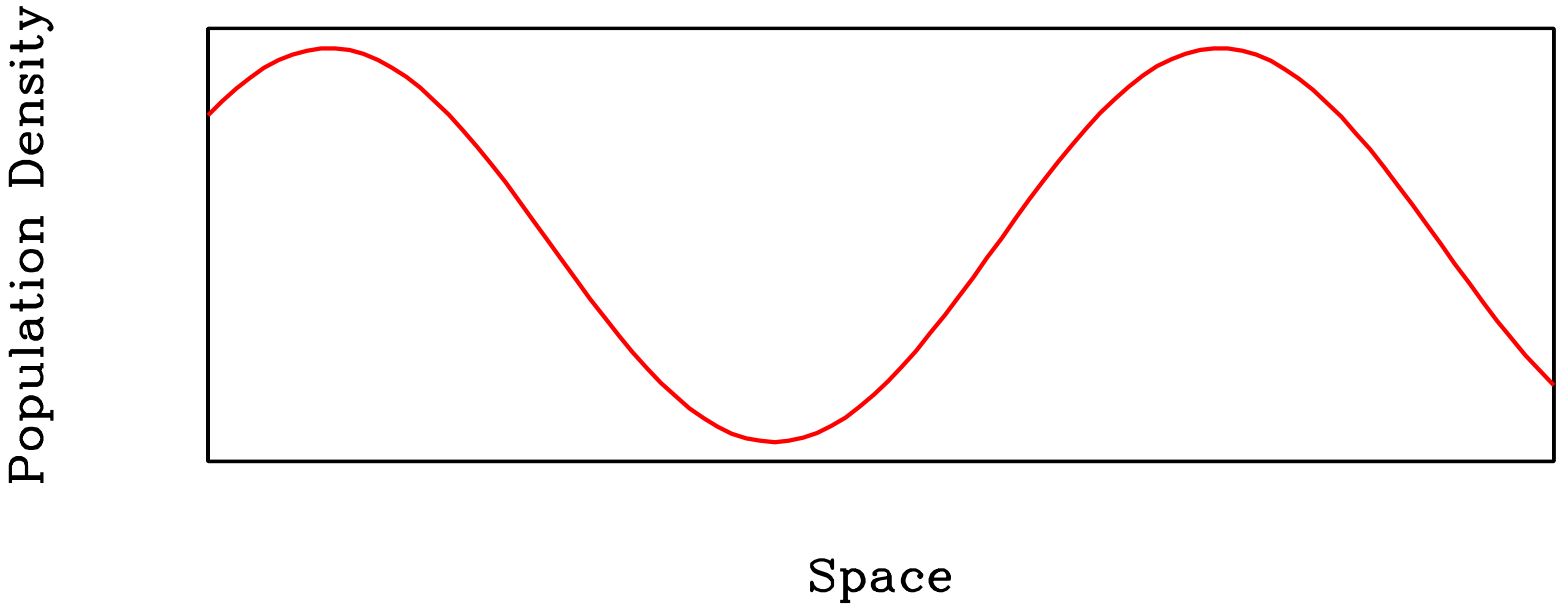

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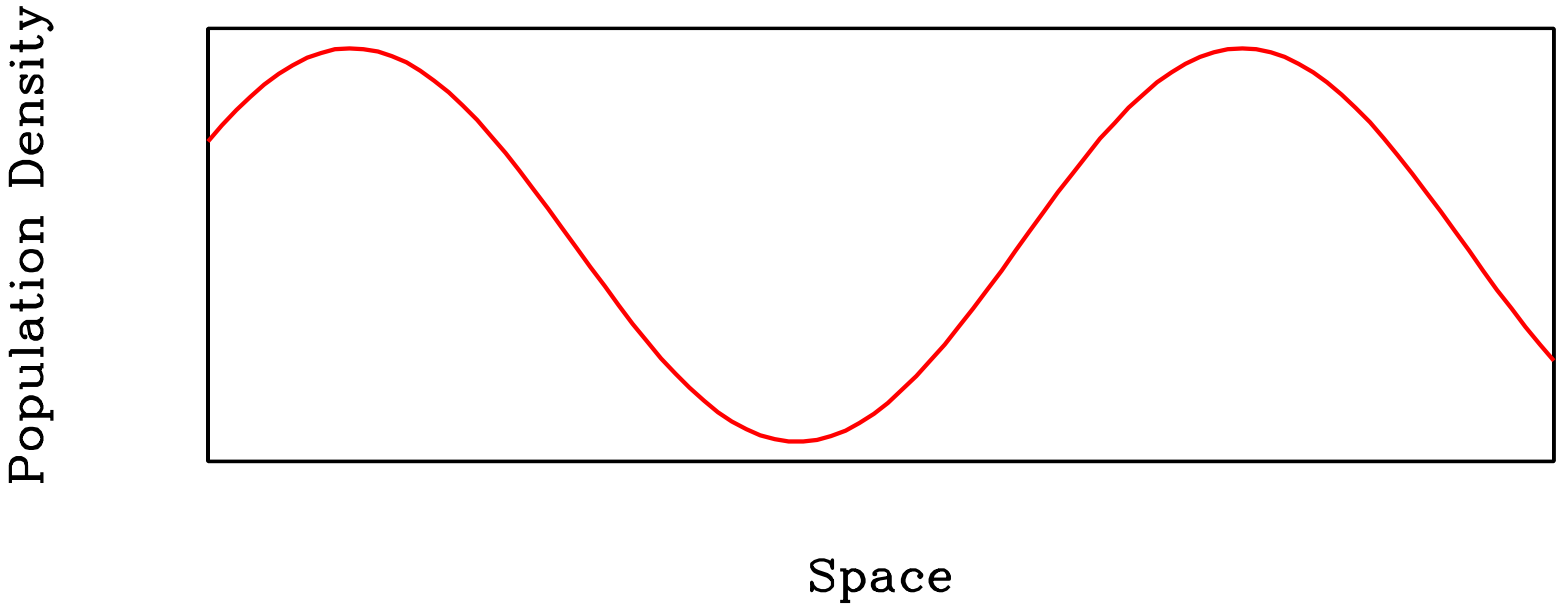

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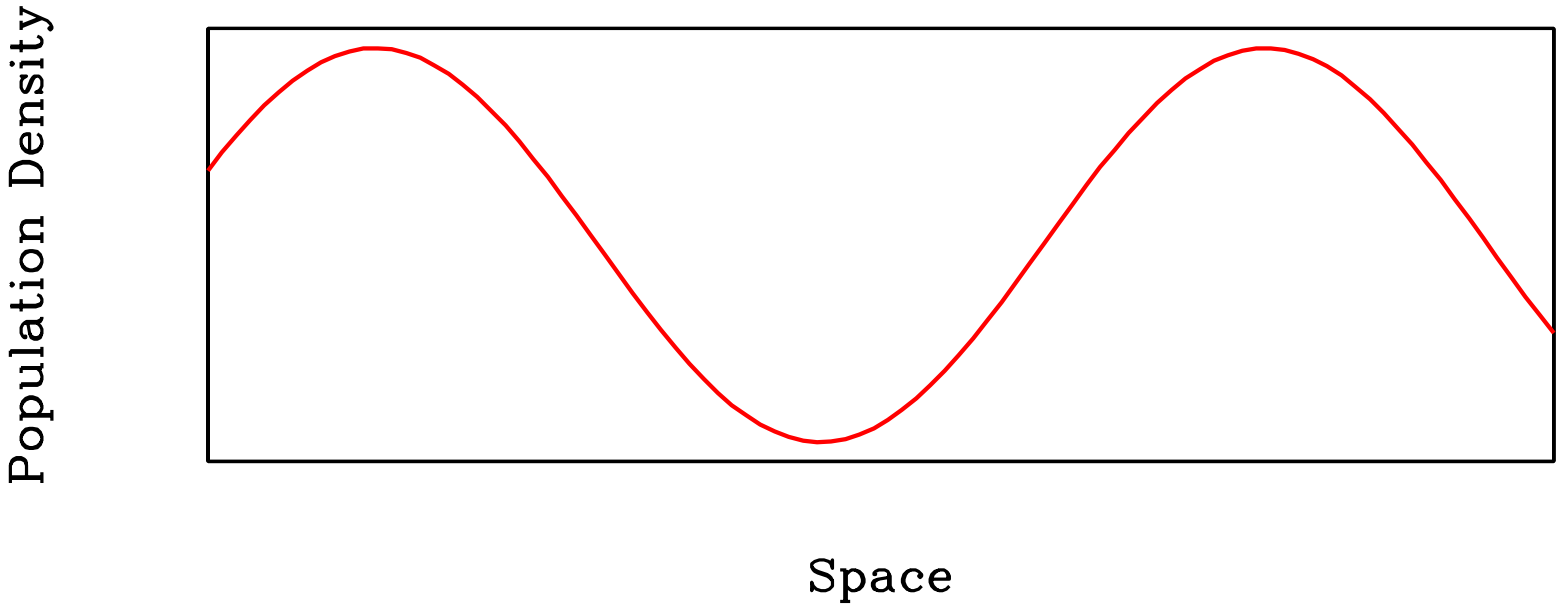

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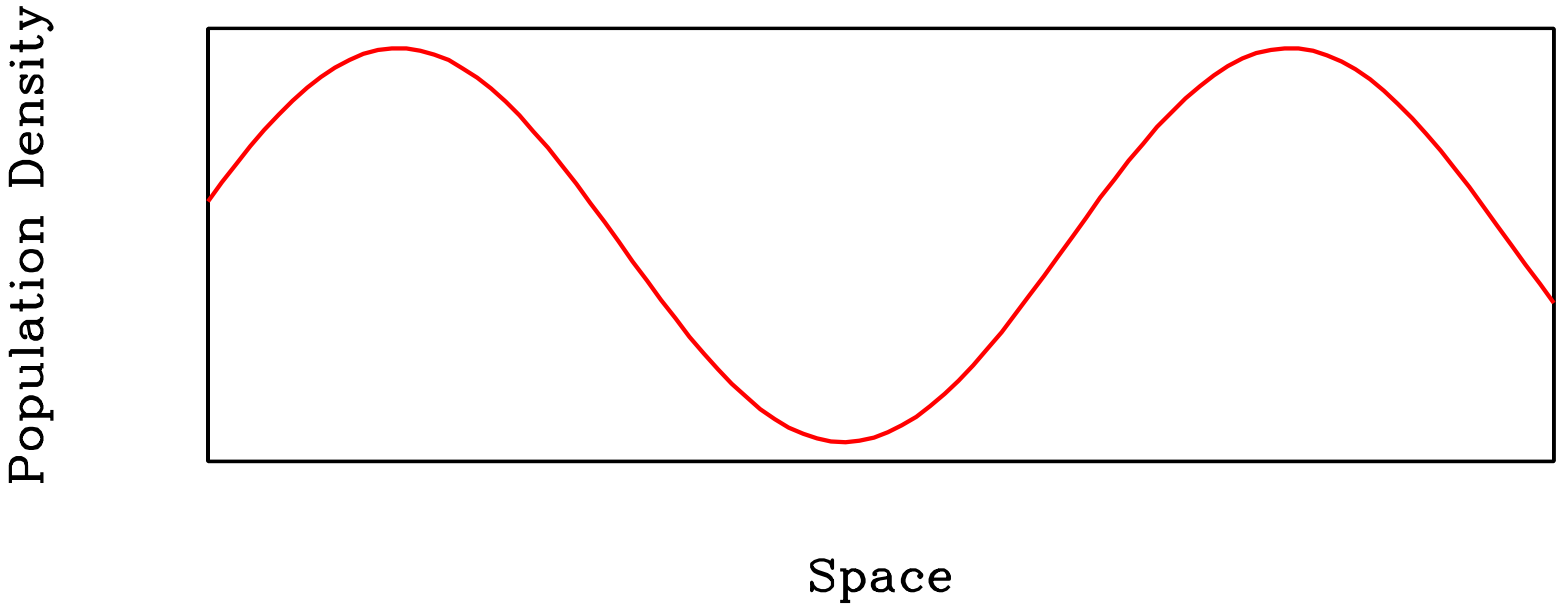

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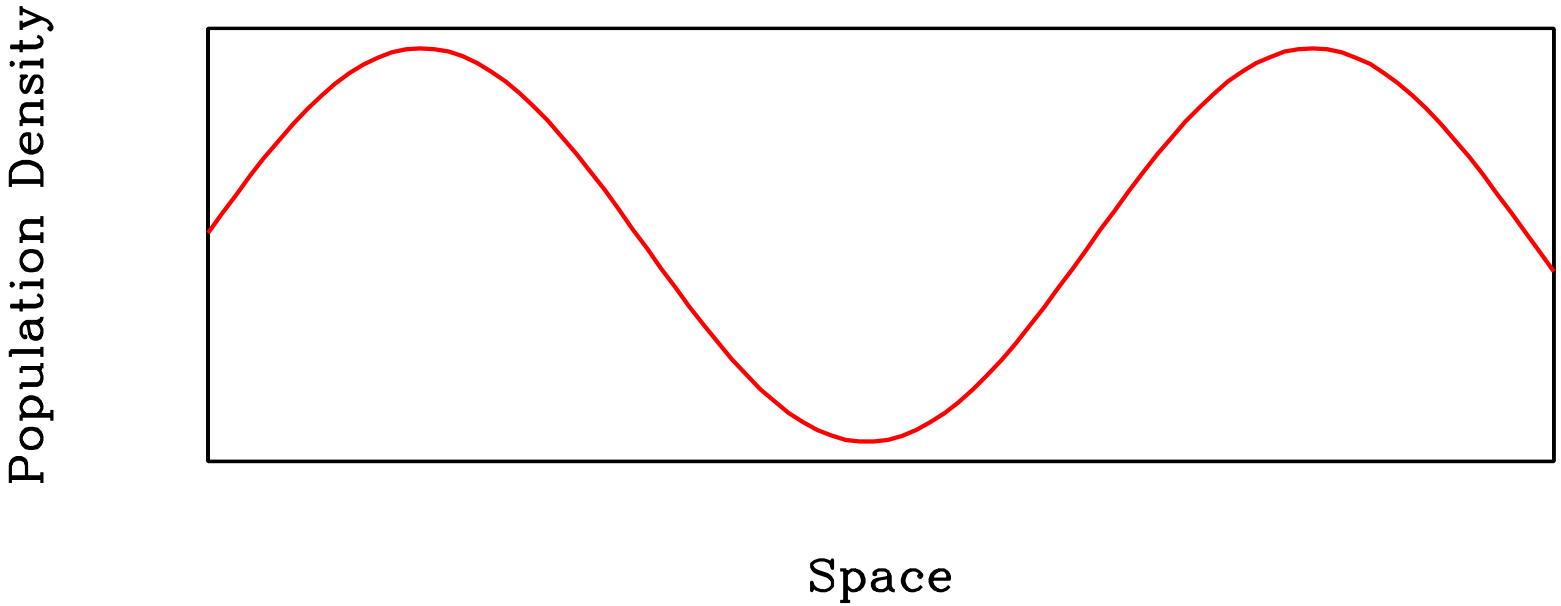

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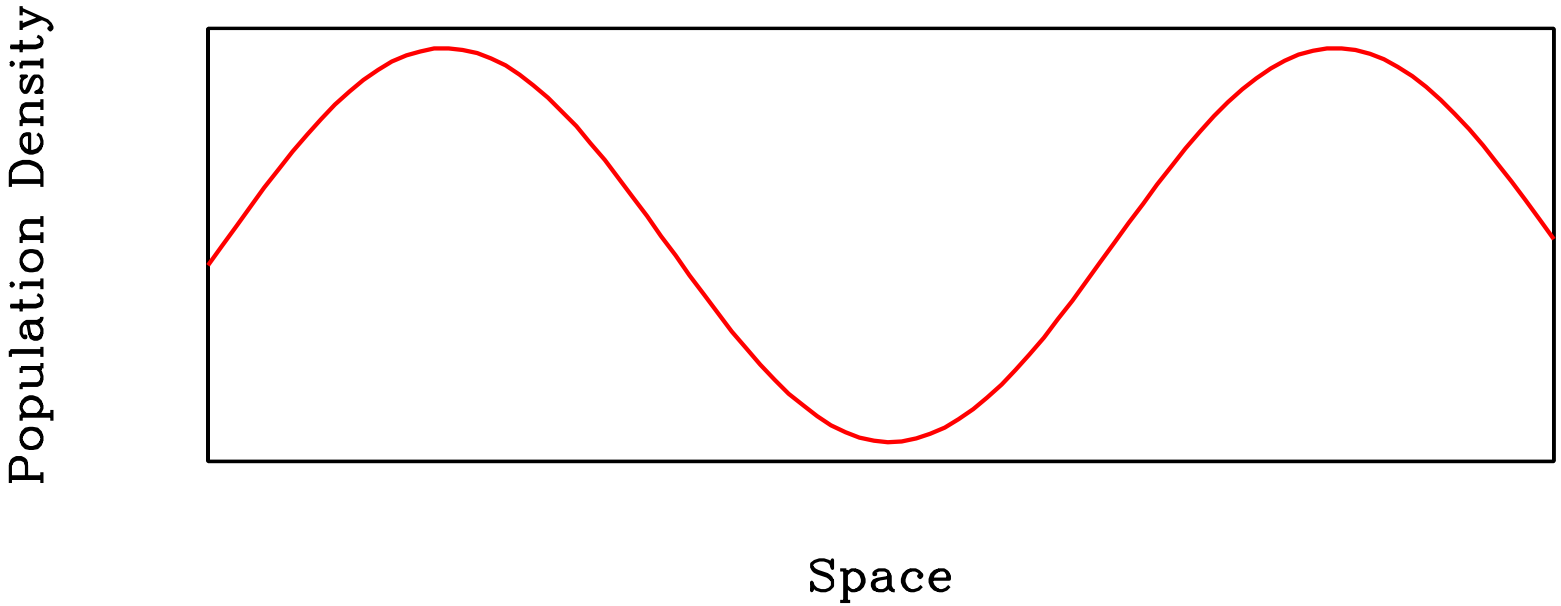

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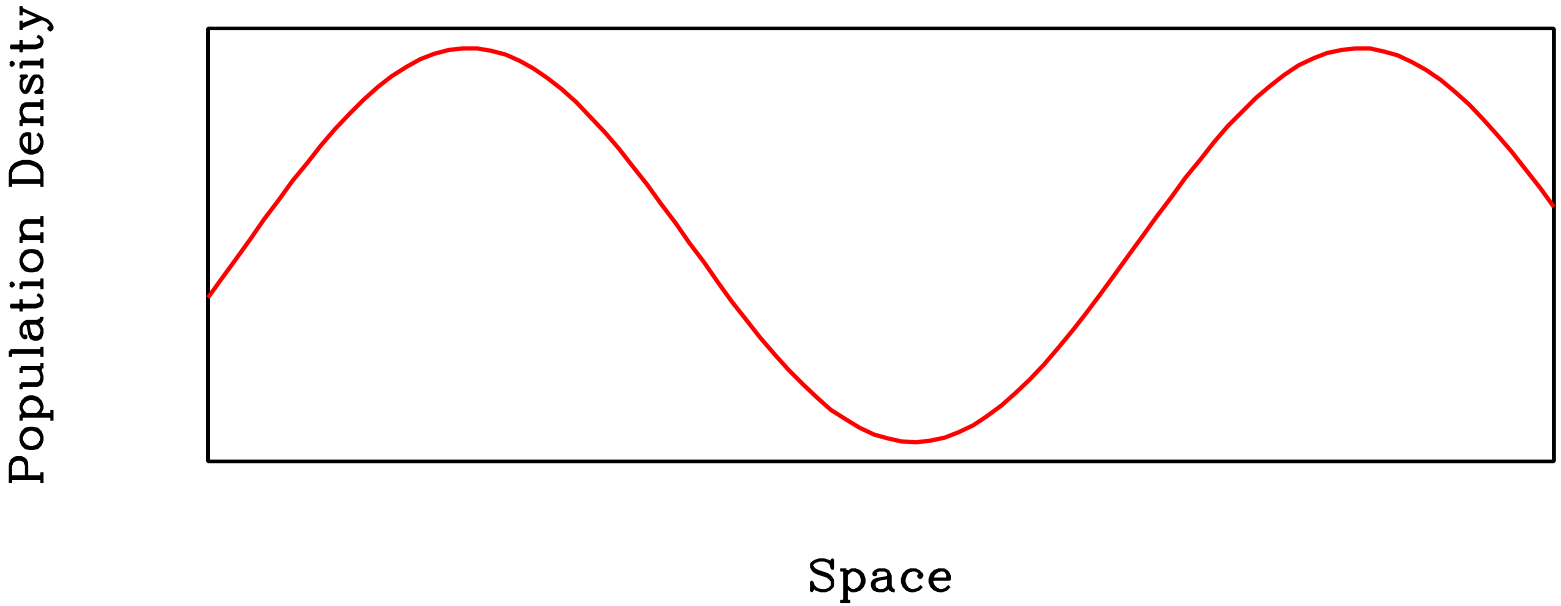

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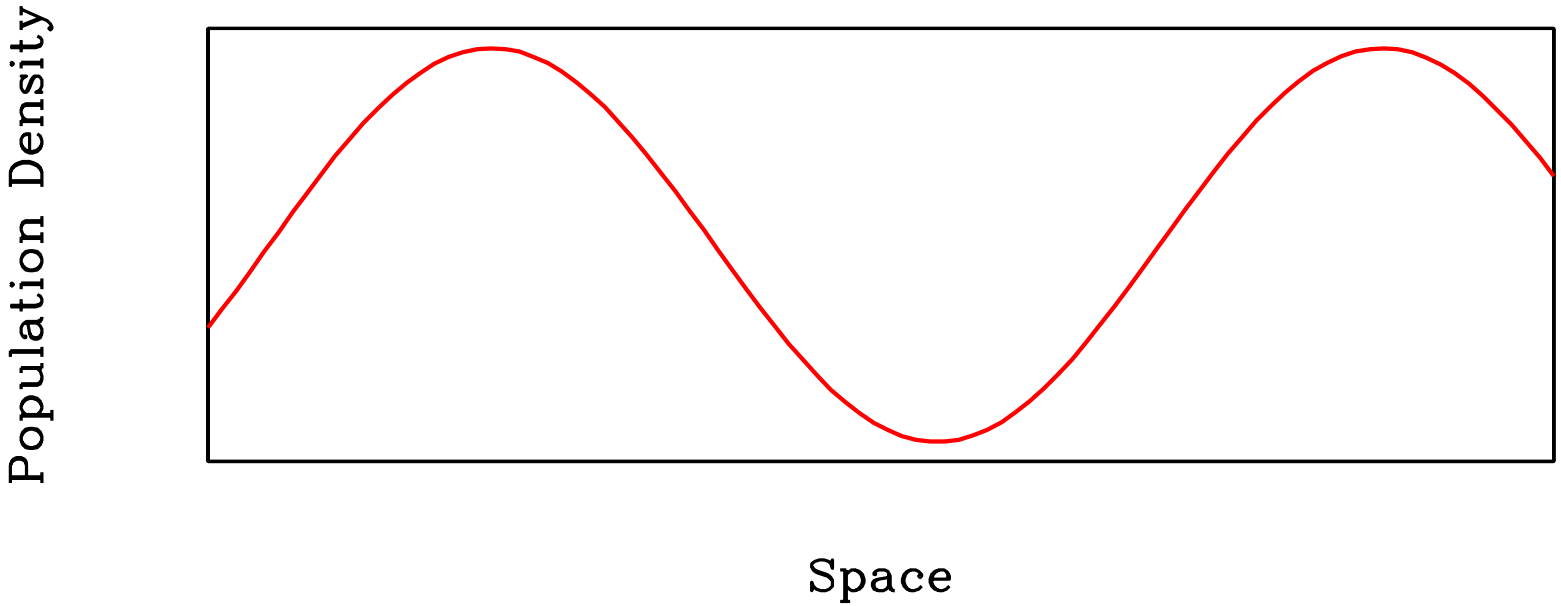

What is a Wavetrain?

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There is an extensive literature on wavetrains
in oscillatory reaction-diffusion equations

$$\begin{aligned} \partial u/\partial t &= D_u\, \partial^2 u/\partial x^2 + f(u,v) \\ \partial v/\partial t &= D_v\, \partial^2 v/\partial x^2 + \underbrace{g(u,v)}_{\substack{\text{kinetics have} \\ \text{a stable} \\ \text{limit cycle}}} \end{aligned}$$

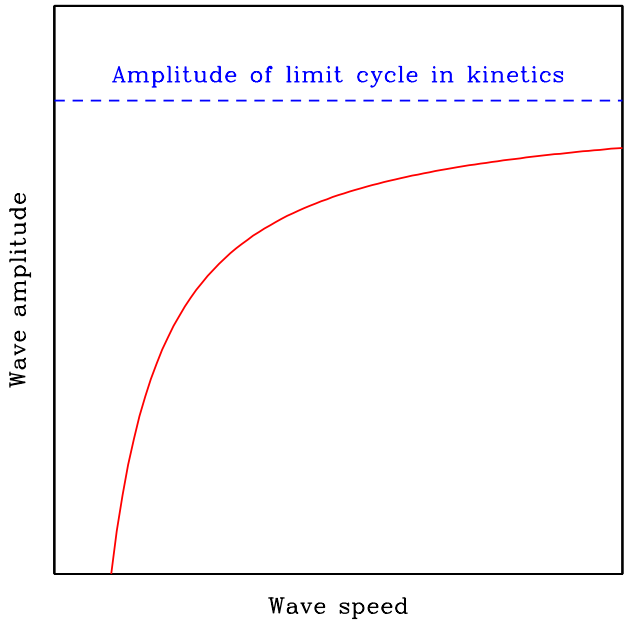
What is a Wavetrain?

A wavetrain is a soln of form $f(x \pm st)$, with $f(.)$ periodic.

An oscillatory reaction-diffusion system has a one-parameter family of wavetrain solutions

(if the diffusion coefficients are sufficiently close to one another)

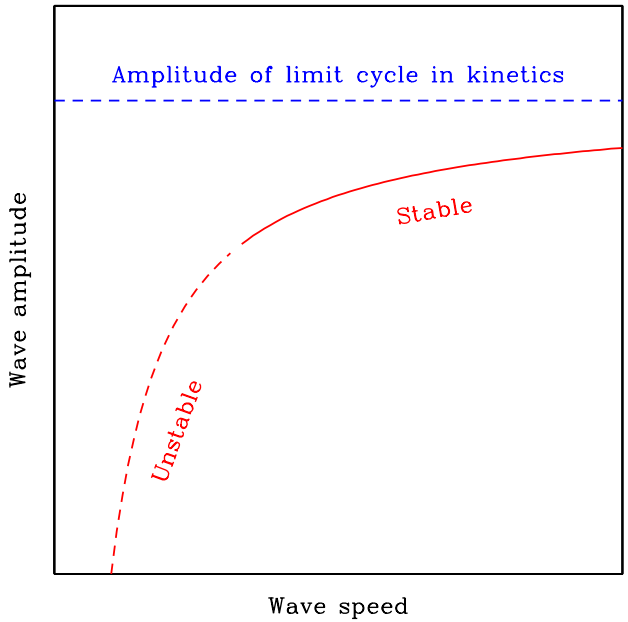
(Kopell, Howard (1973) *Stud Appl Math* 52:291)



What is a Wavetrain?

A wavetrain is a soln of form $f(x \pm st)$, with $f(.)$ periodic.

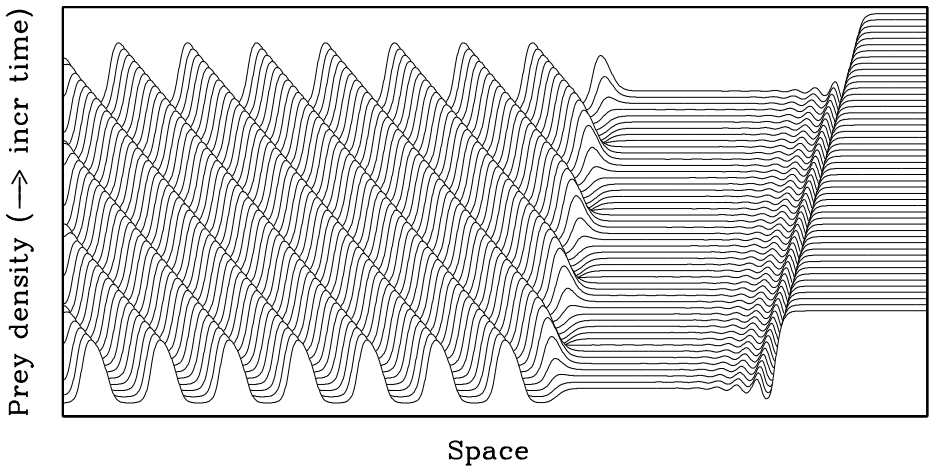
Some members of the wavetrain family are stable as solutions of the partial differential equations, while others are unstable.



Outline

The Wavetrain Band

The invasion process selects a particular member of the wavetrain family (Sherratt (1998) *Physica D* 117:145).



The Wavetrain Band

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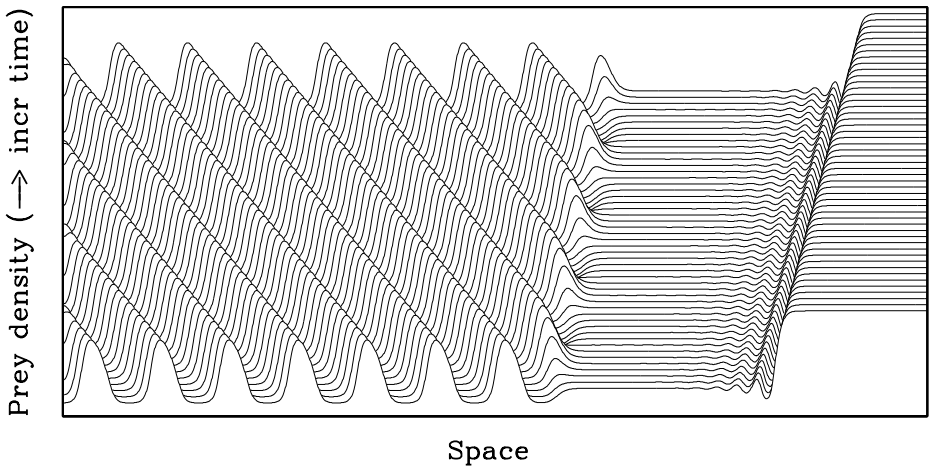
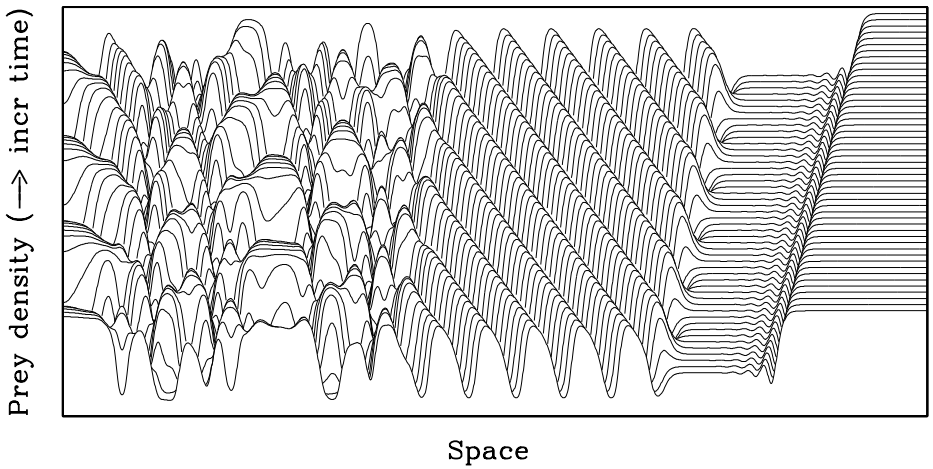
For these parameters, the selected wavetrain is stable.



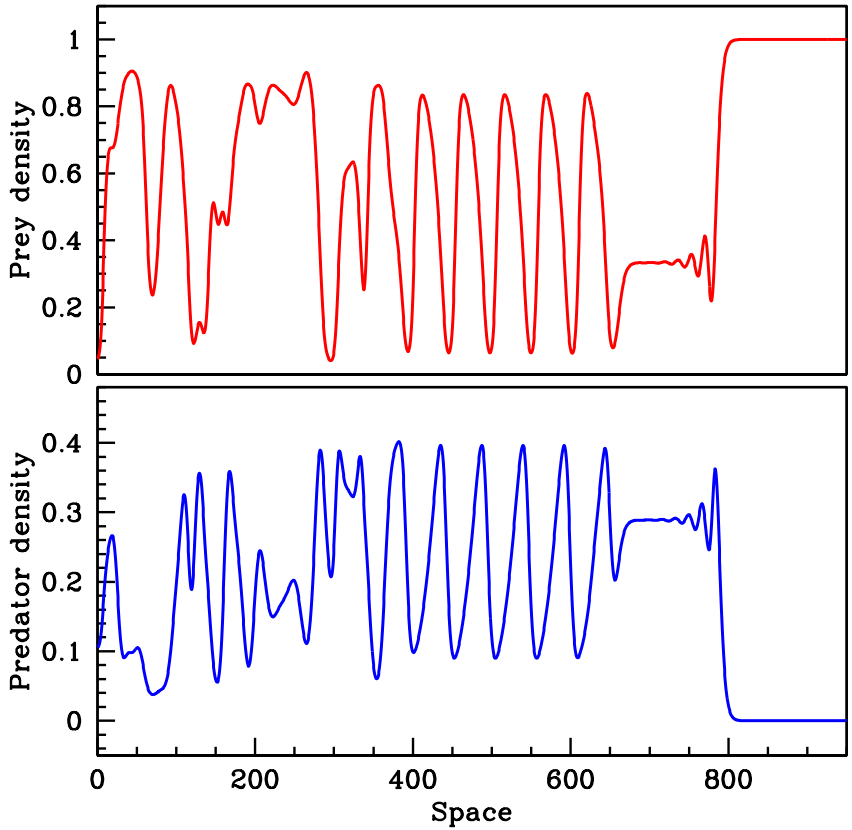
The Wavetrain Band

The invasion process selects a particular member of the wavetrain family (Sherratt (1998) *Physica D* 117:145).

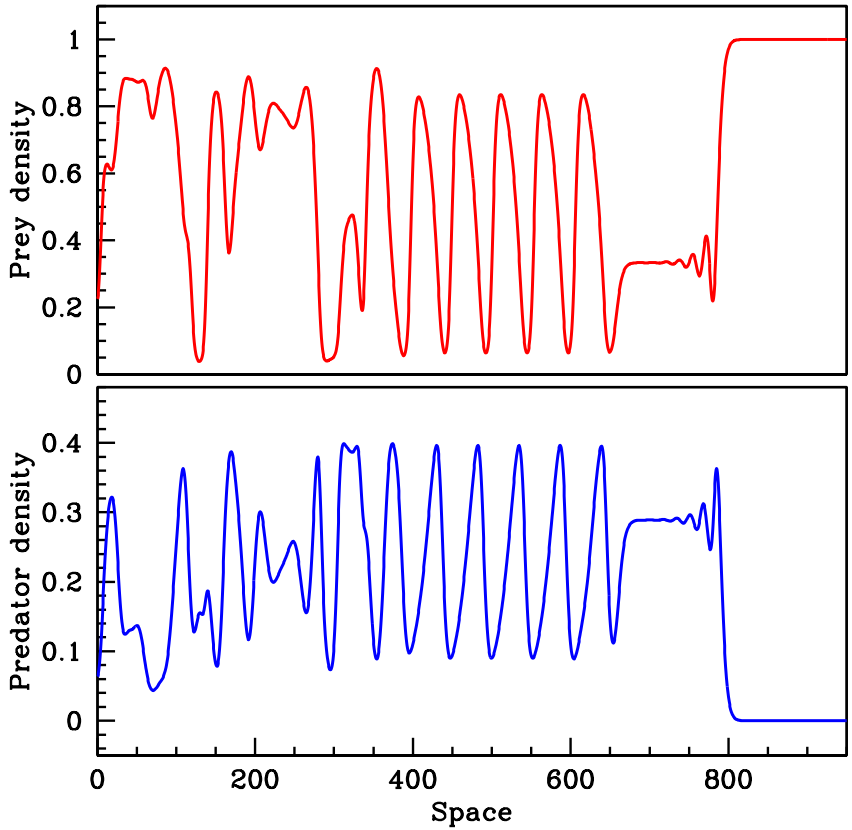
A "wavetrain band" occurs when the selected wavetrain is unstable.



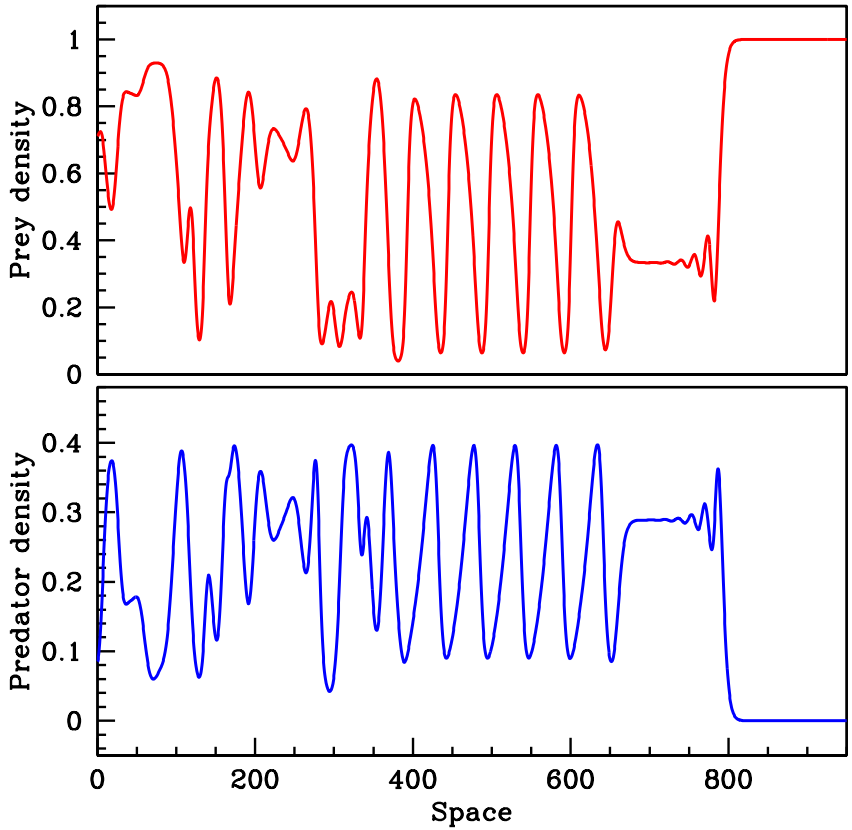
The Wavetrain Band: Animation



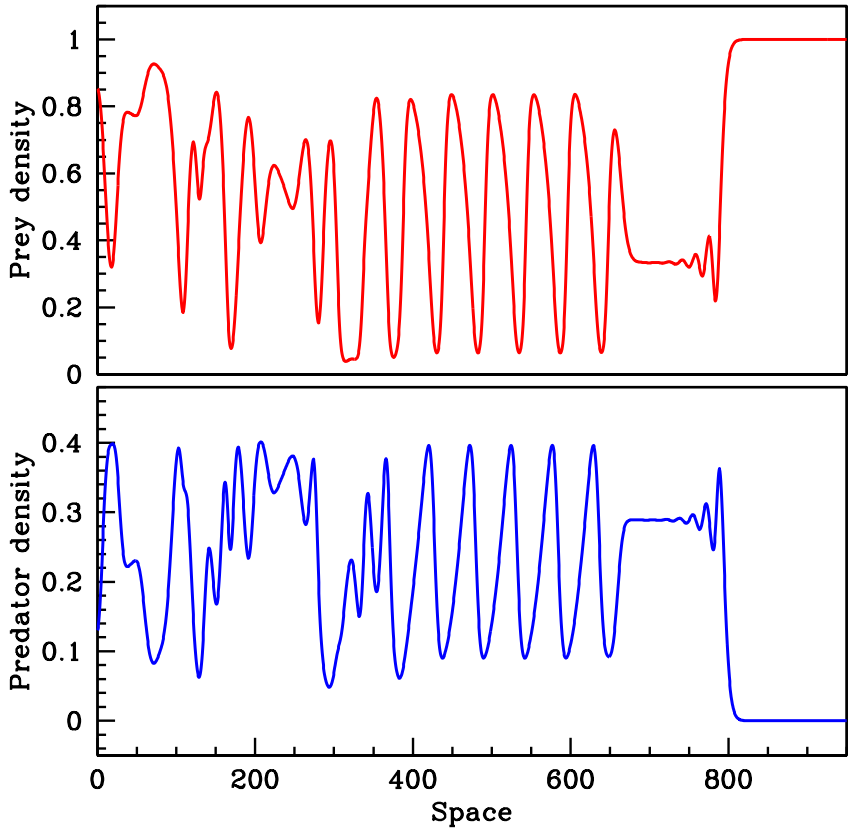
The Wavetrain Band: Animation



The Wavetrain Band: Animation



The Wavetrain Band: Animation



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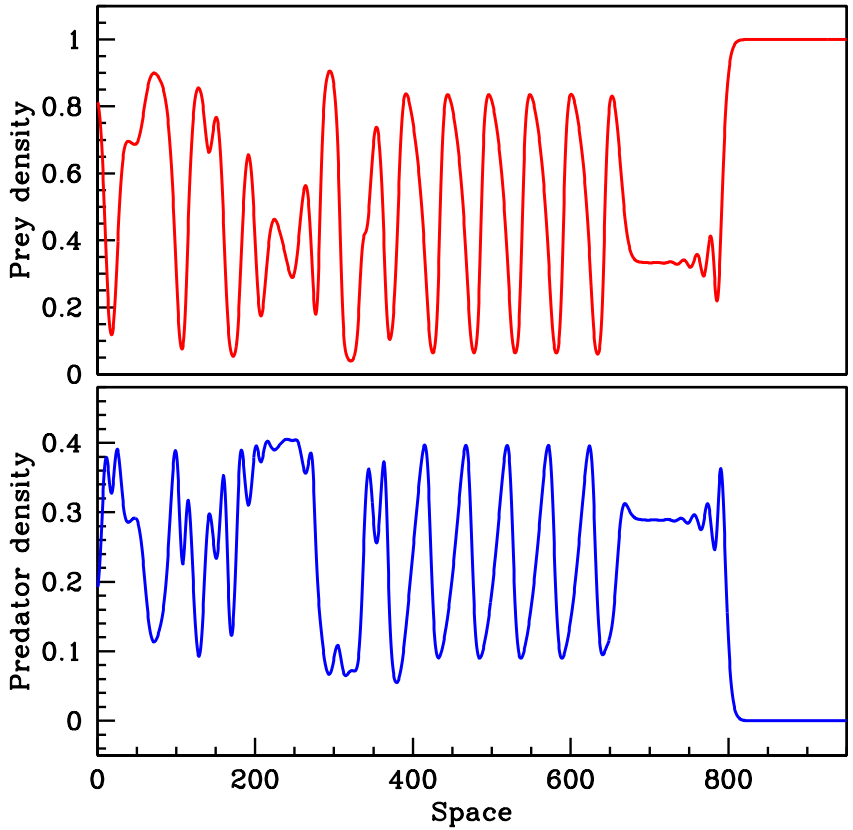

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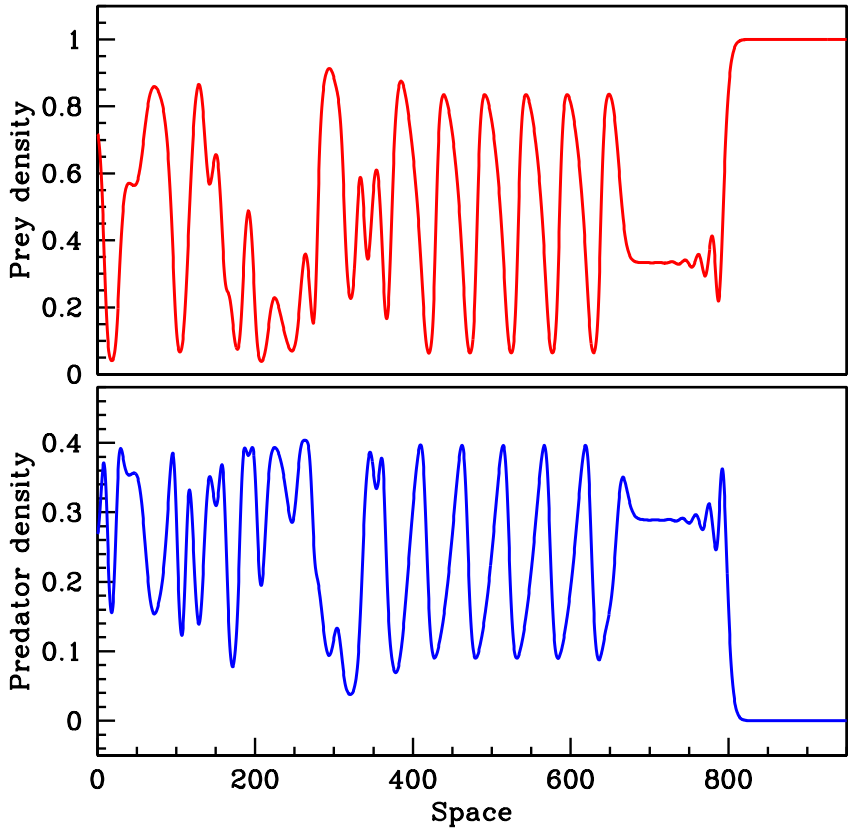

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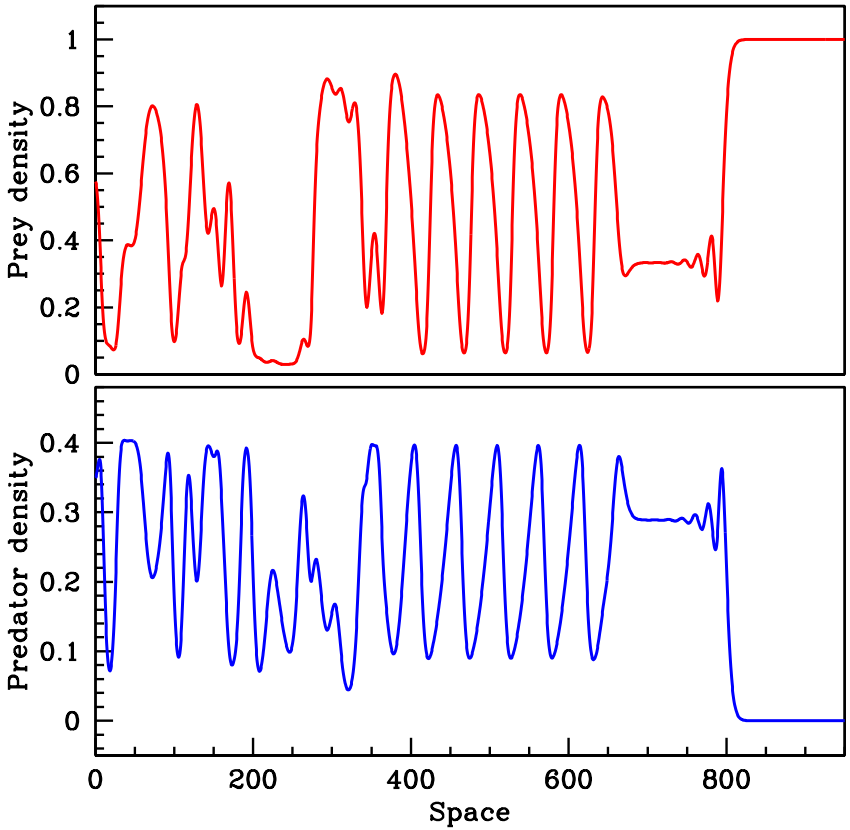

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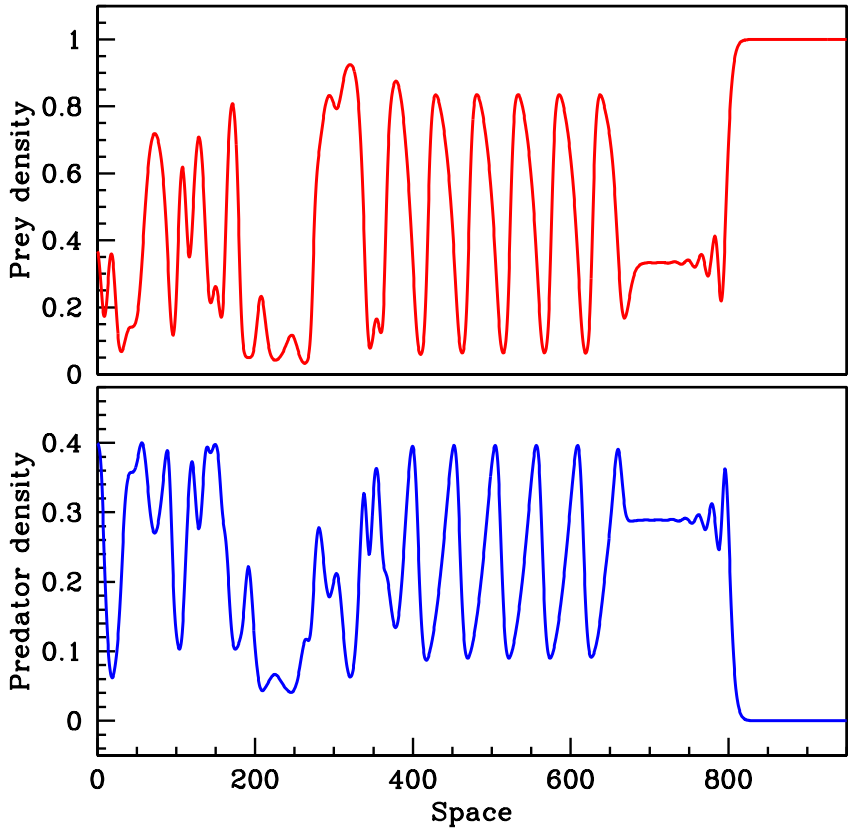

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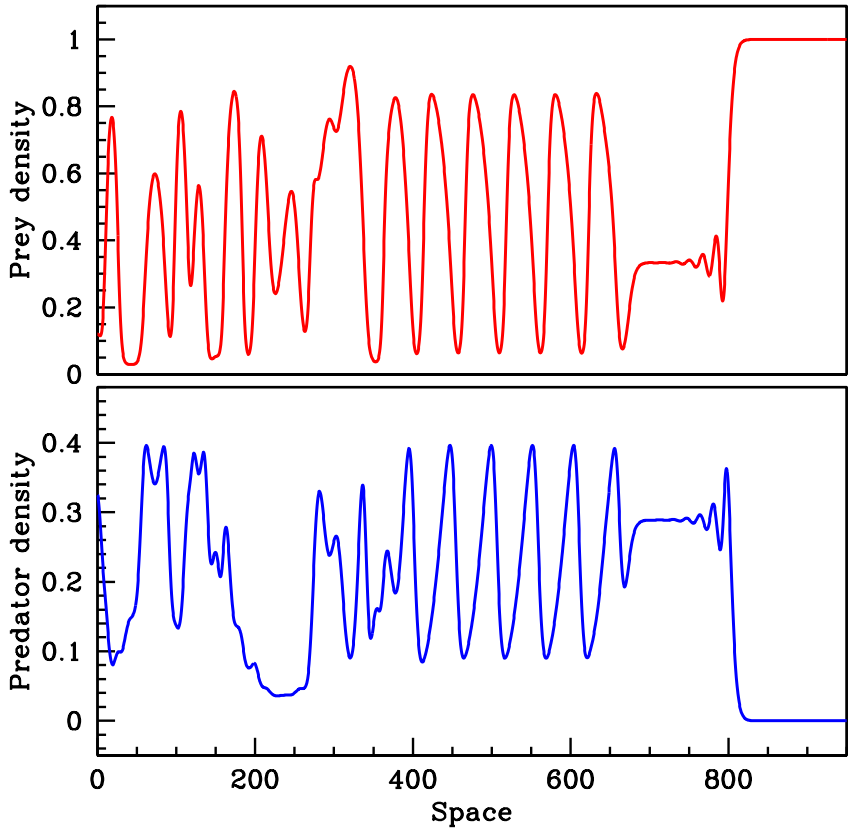

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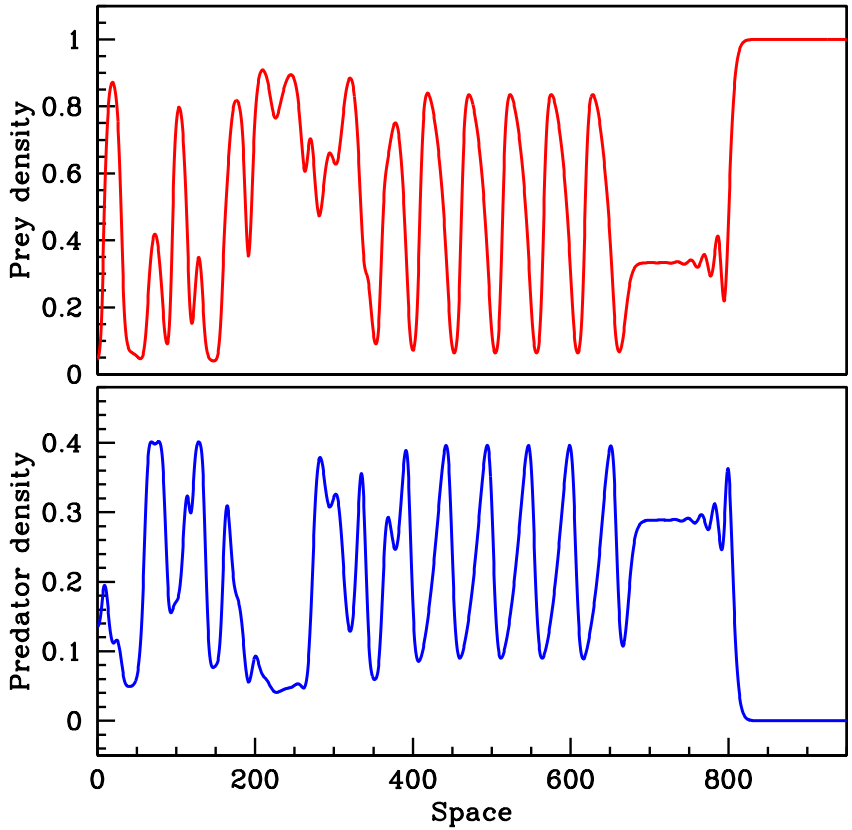

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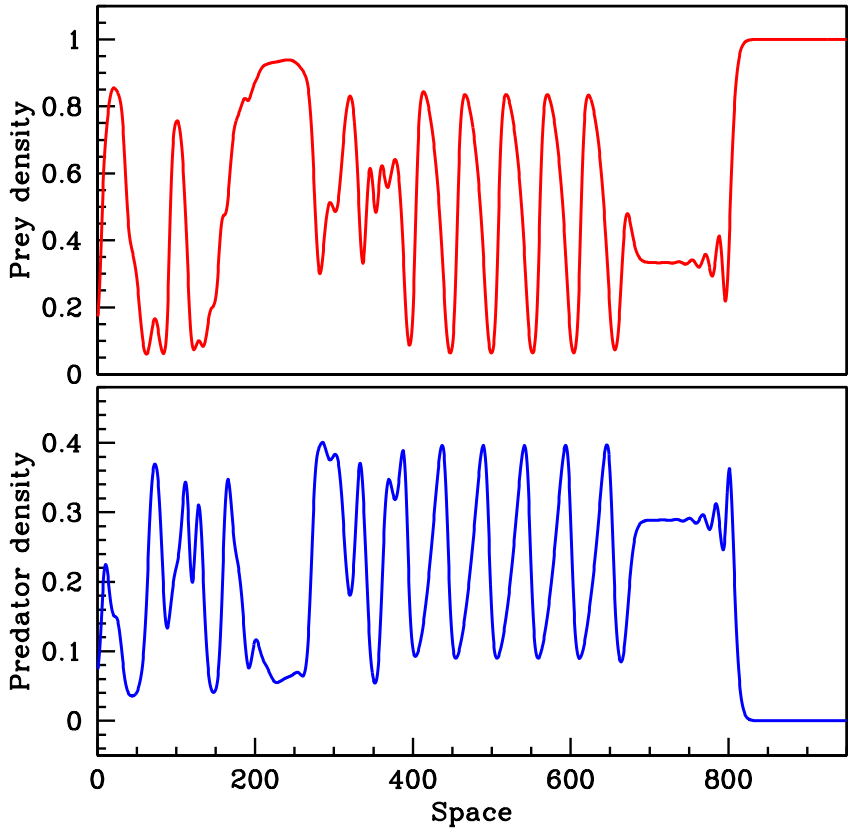

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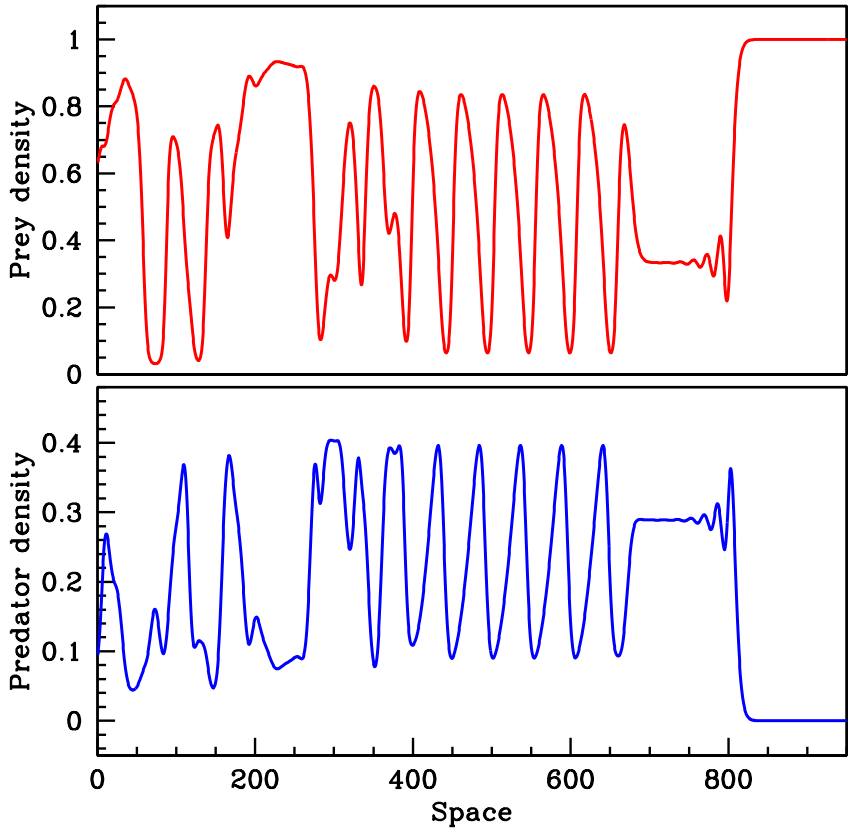

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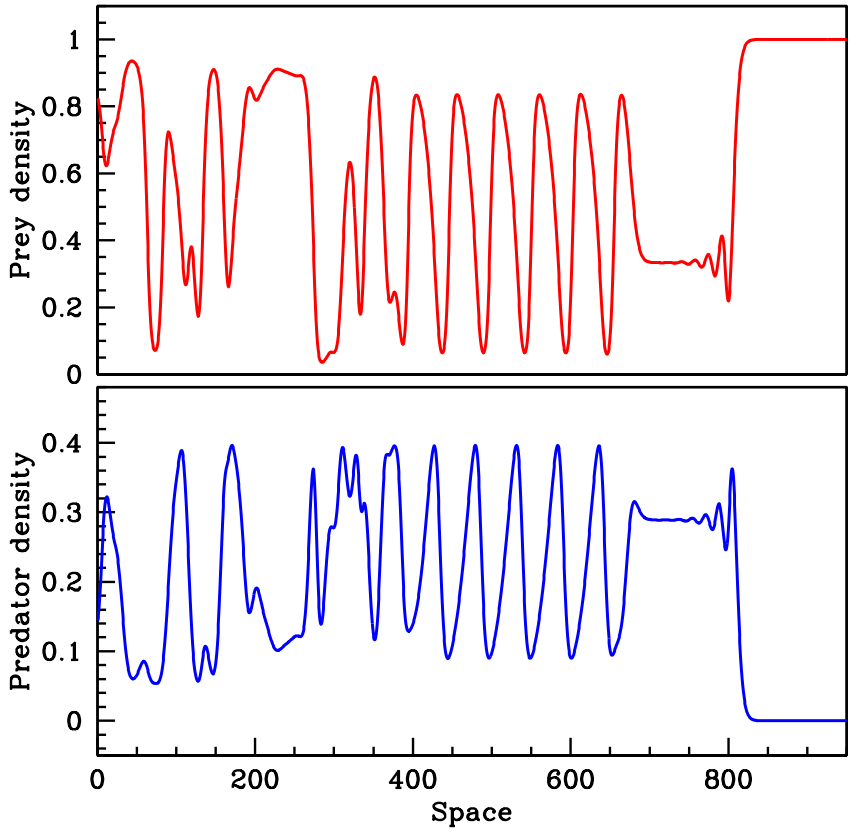

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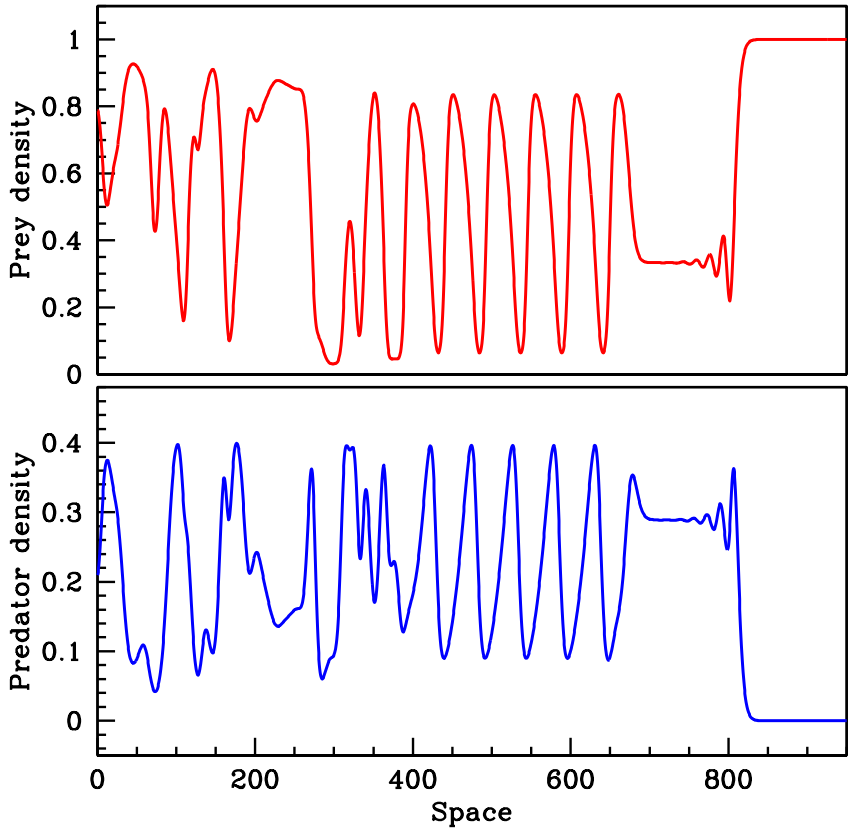

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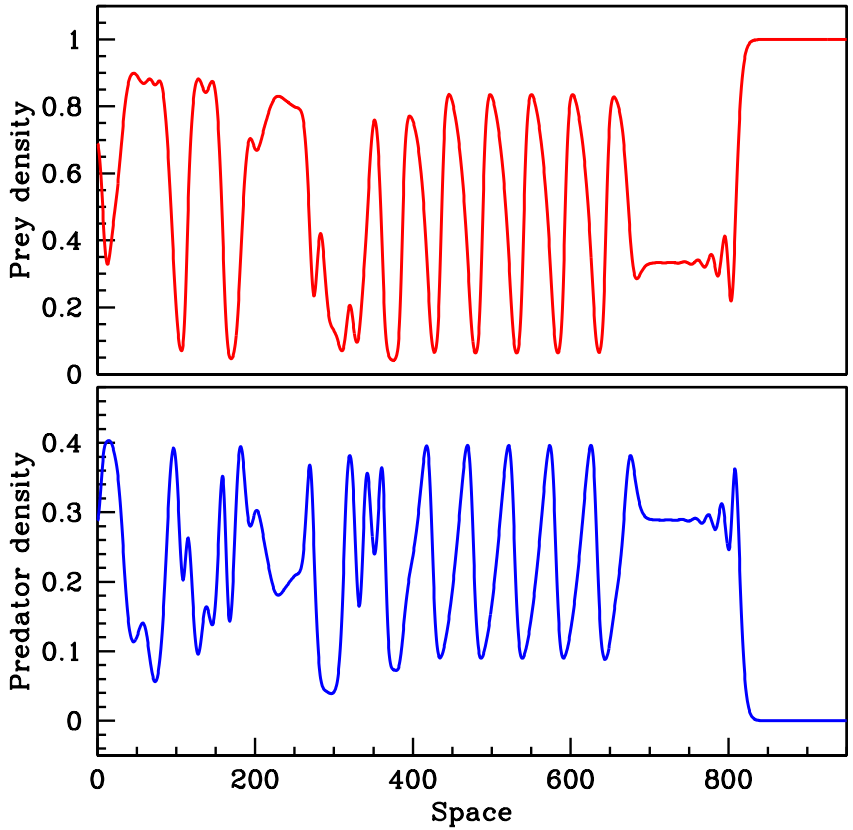

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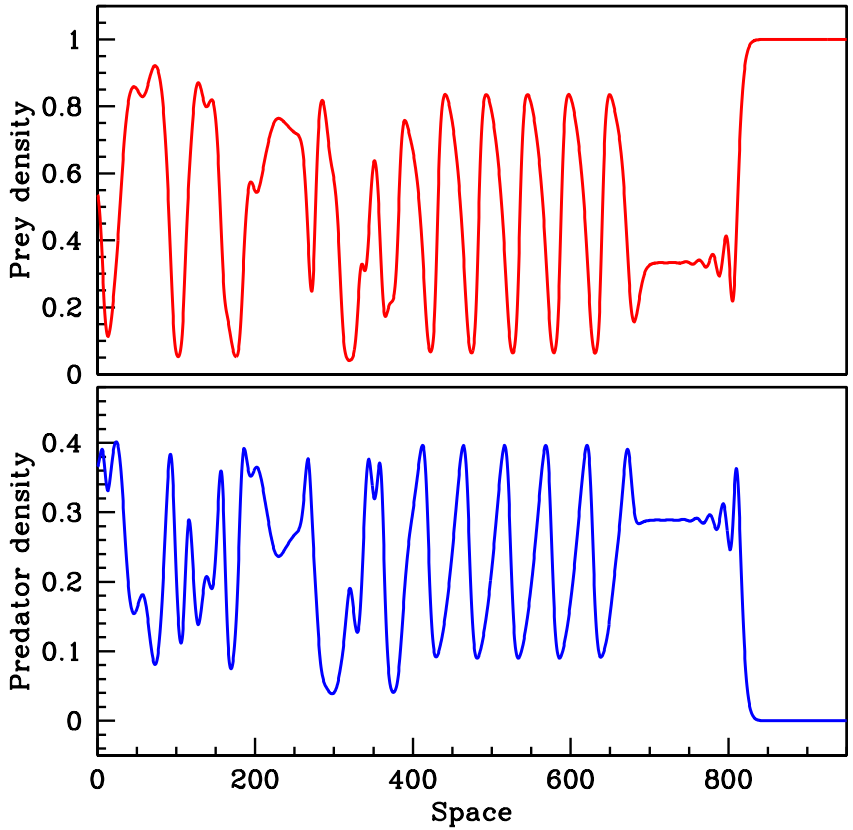

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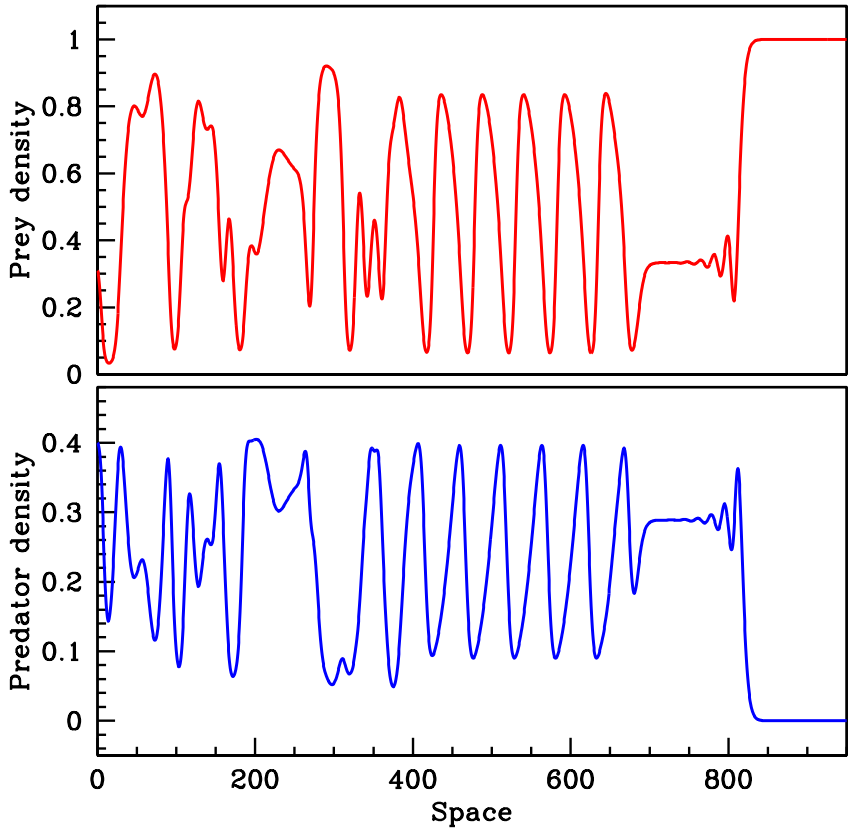

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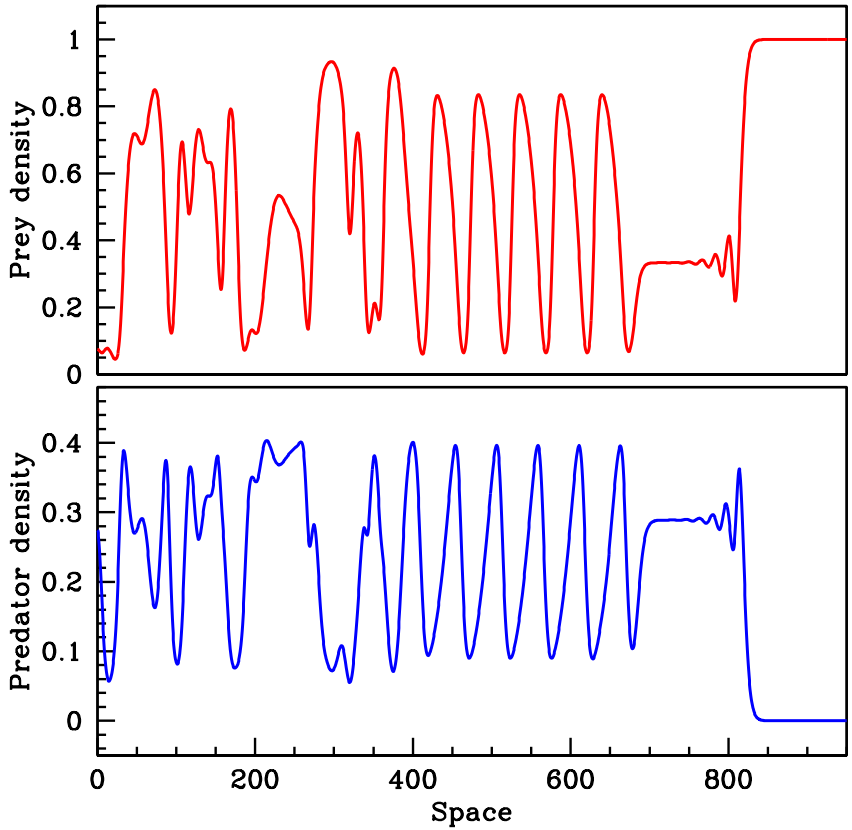

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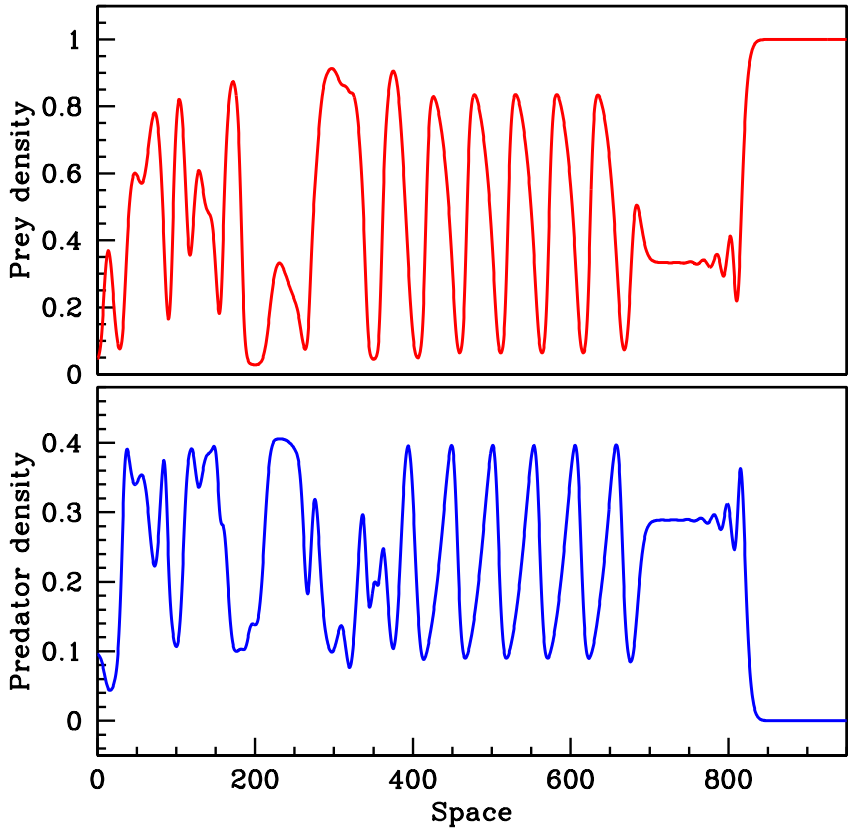

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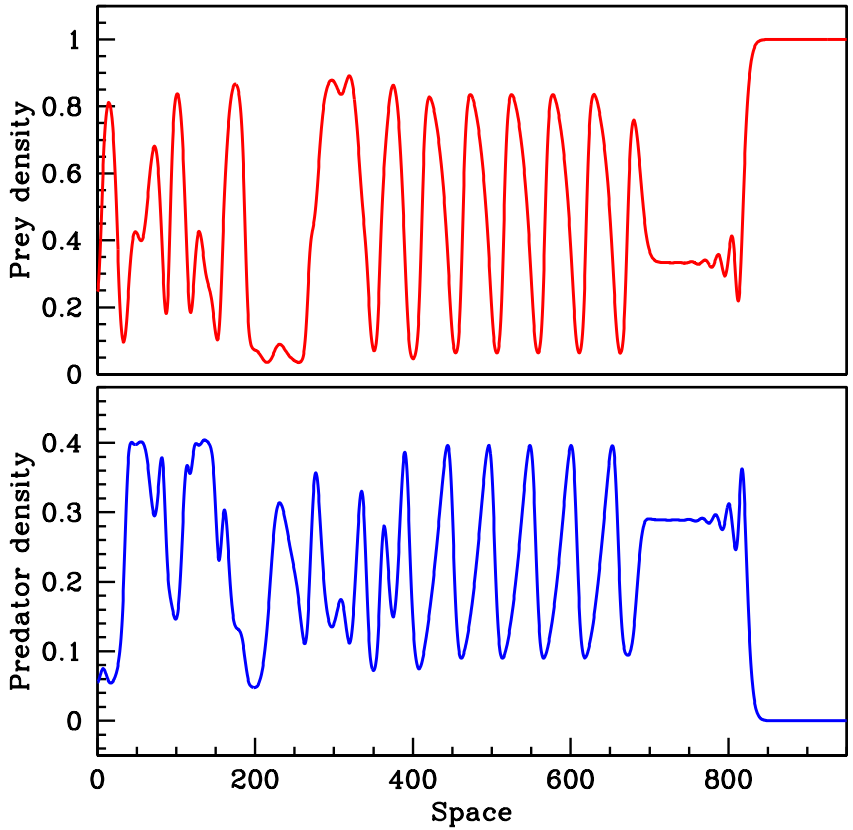

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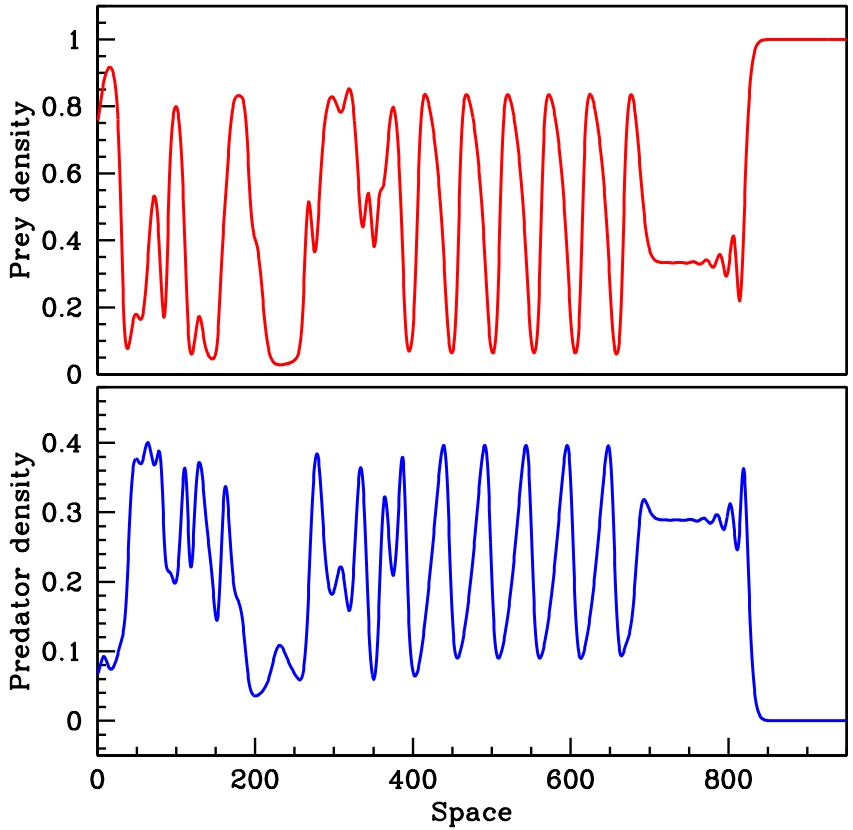

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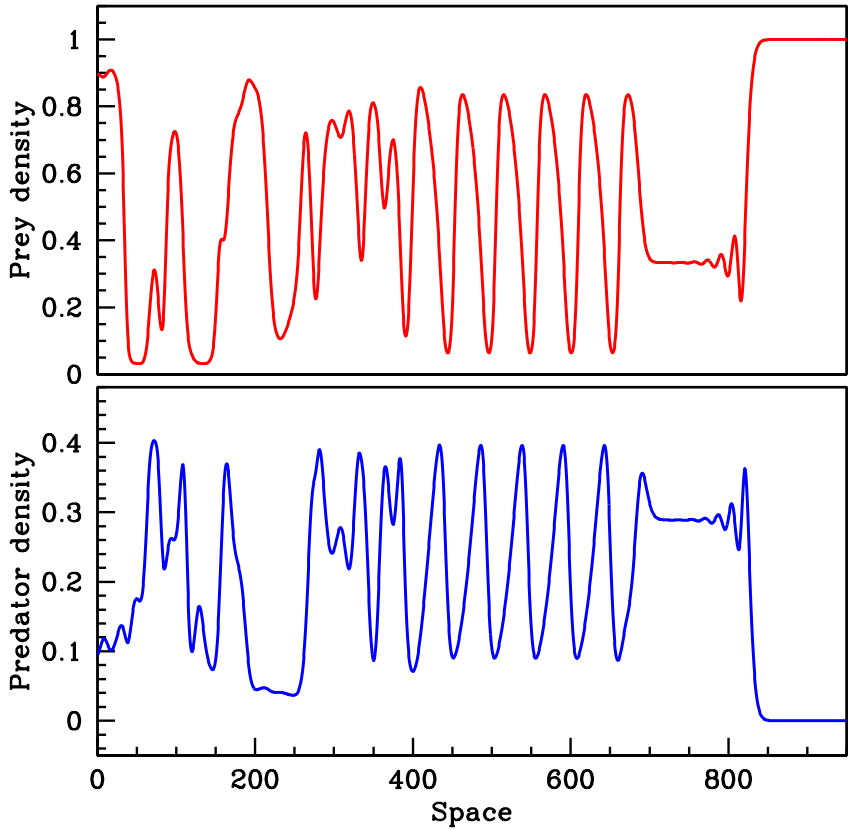

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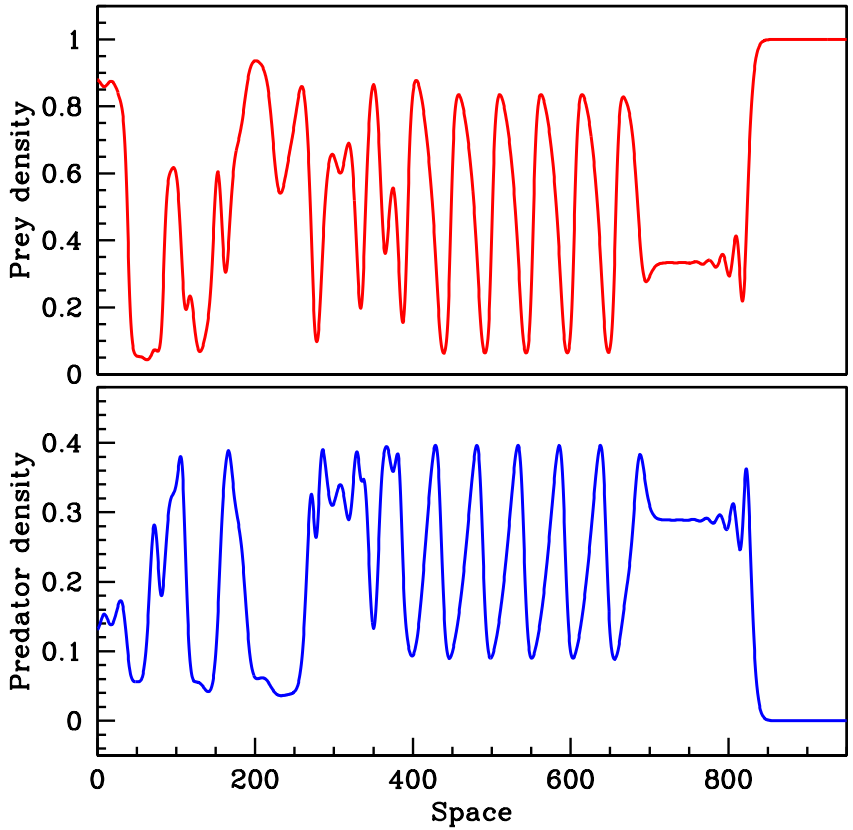

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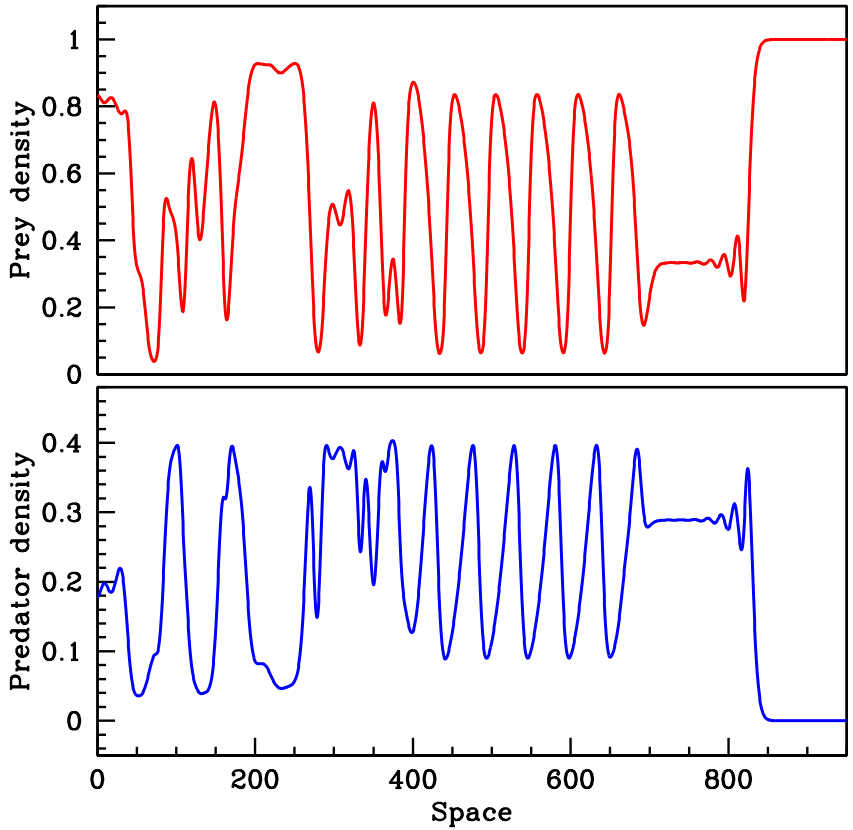

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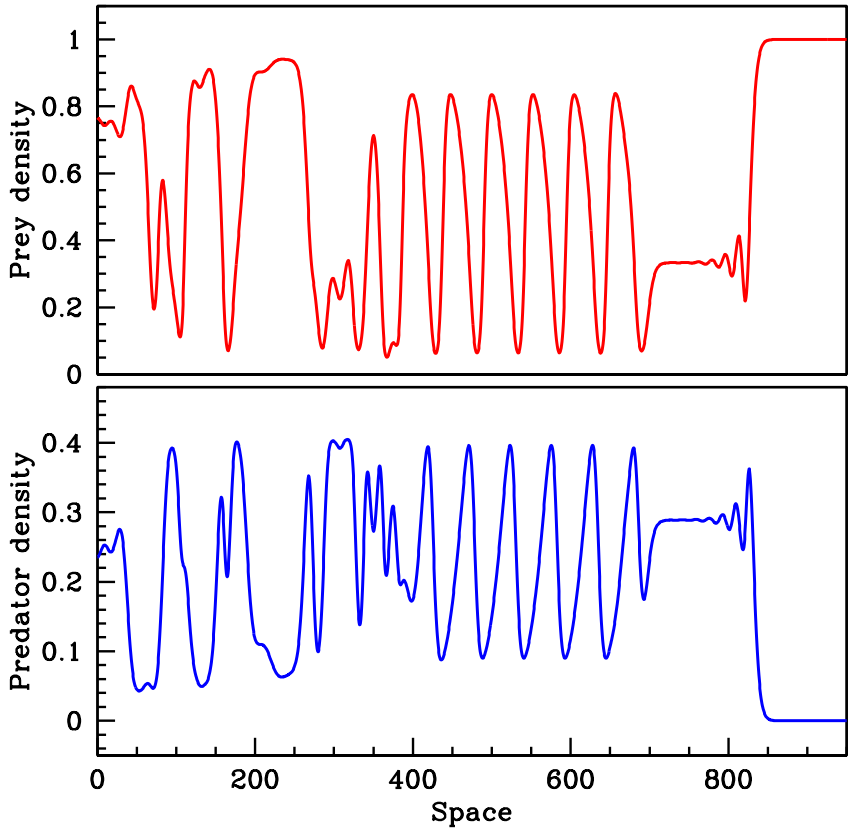

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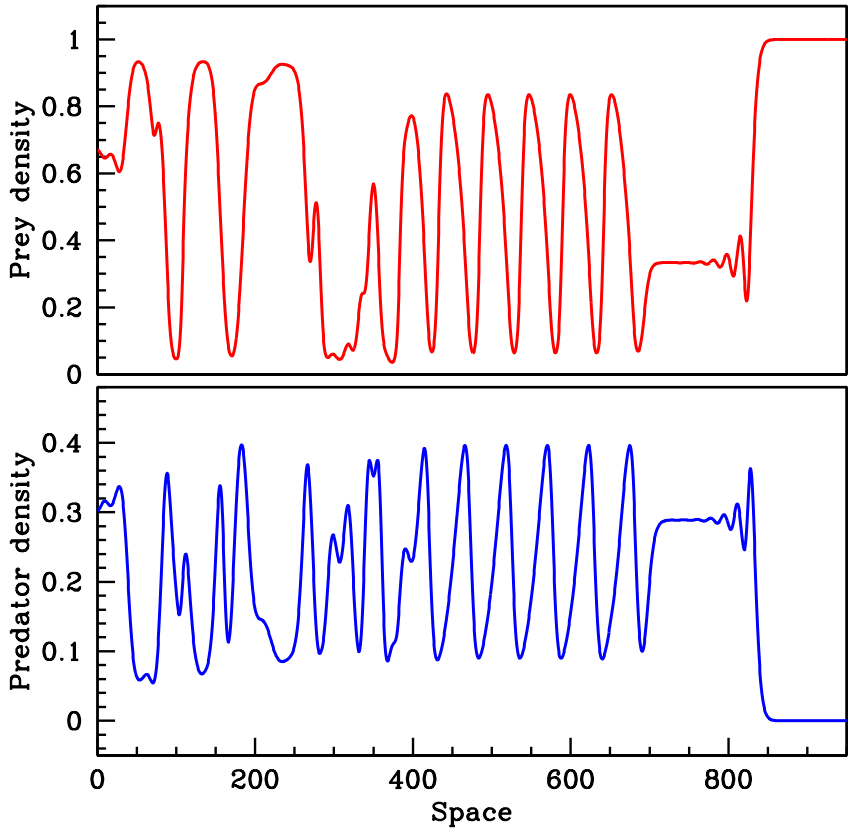

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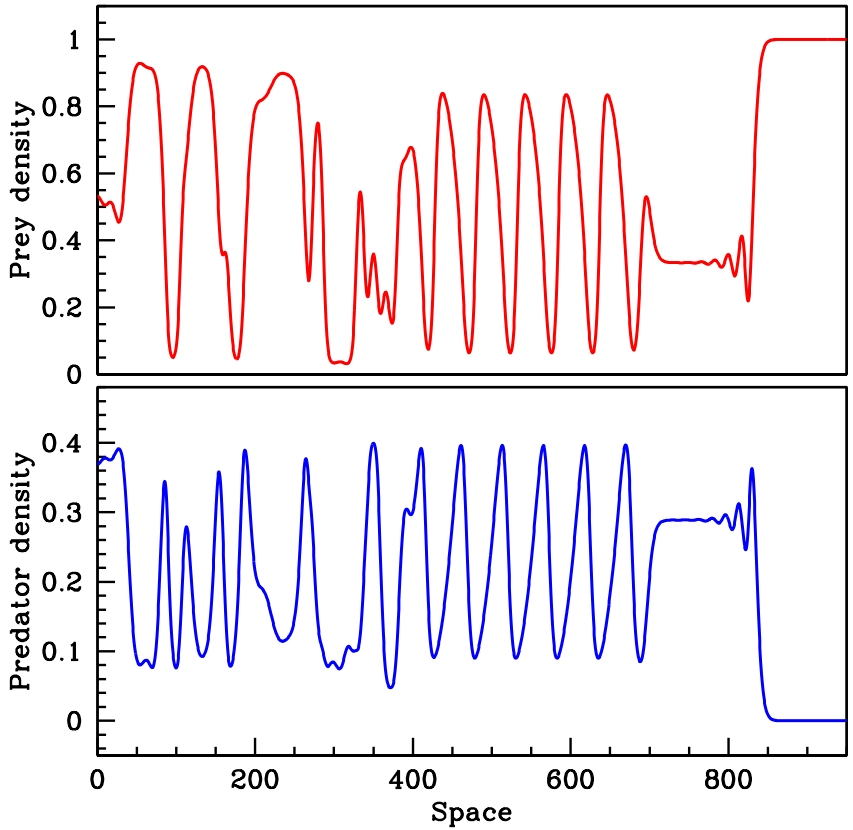

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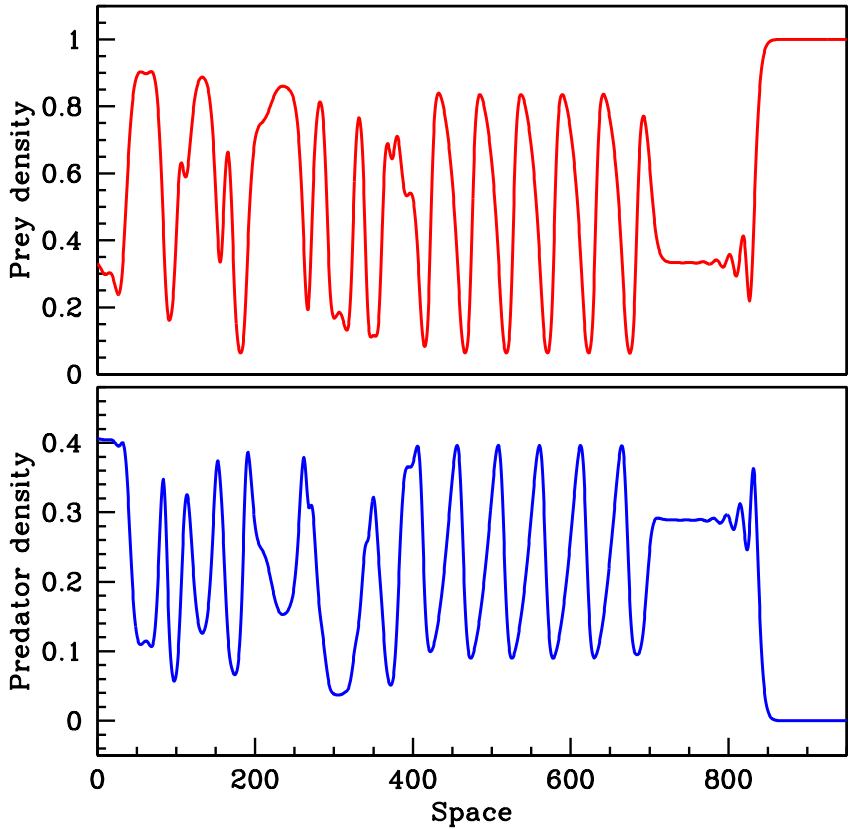

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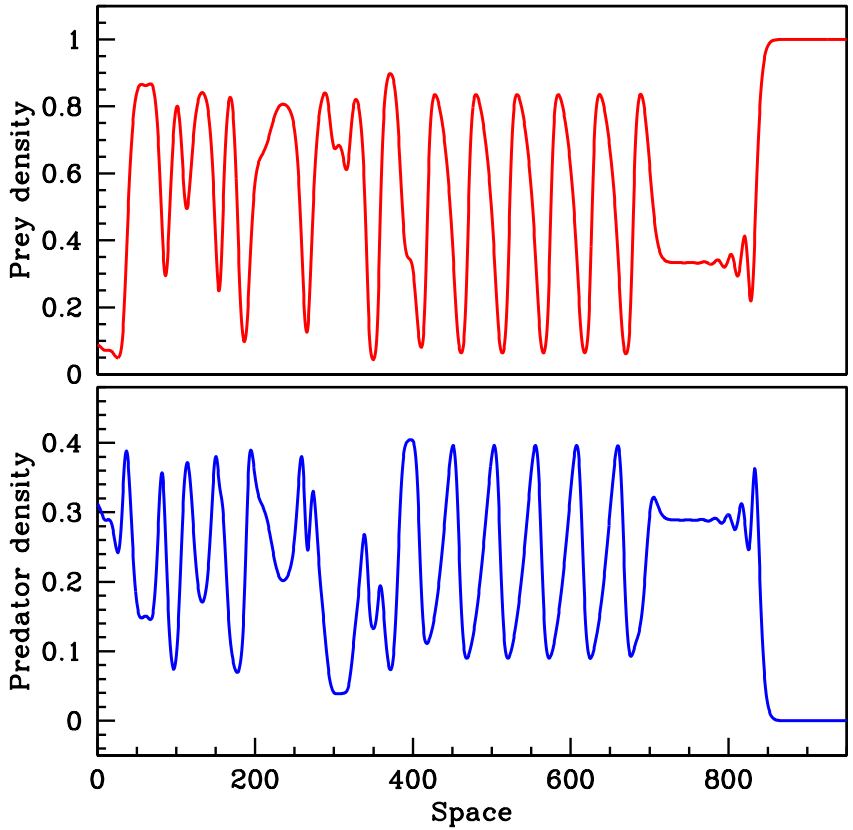

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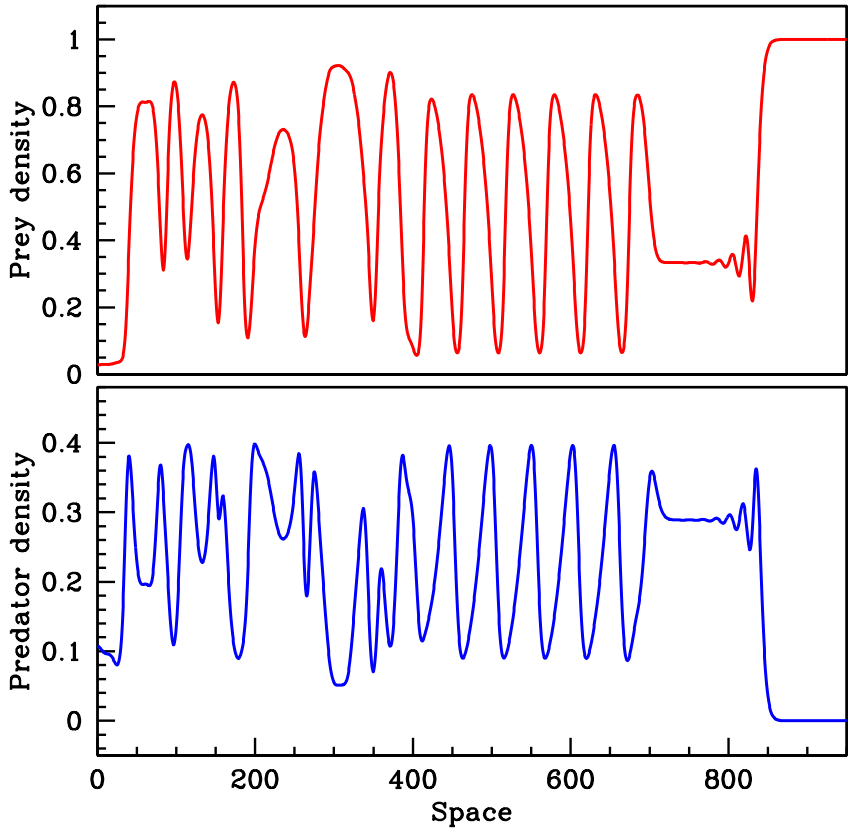

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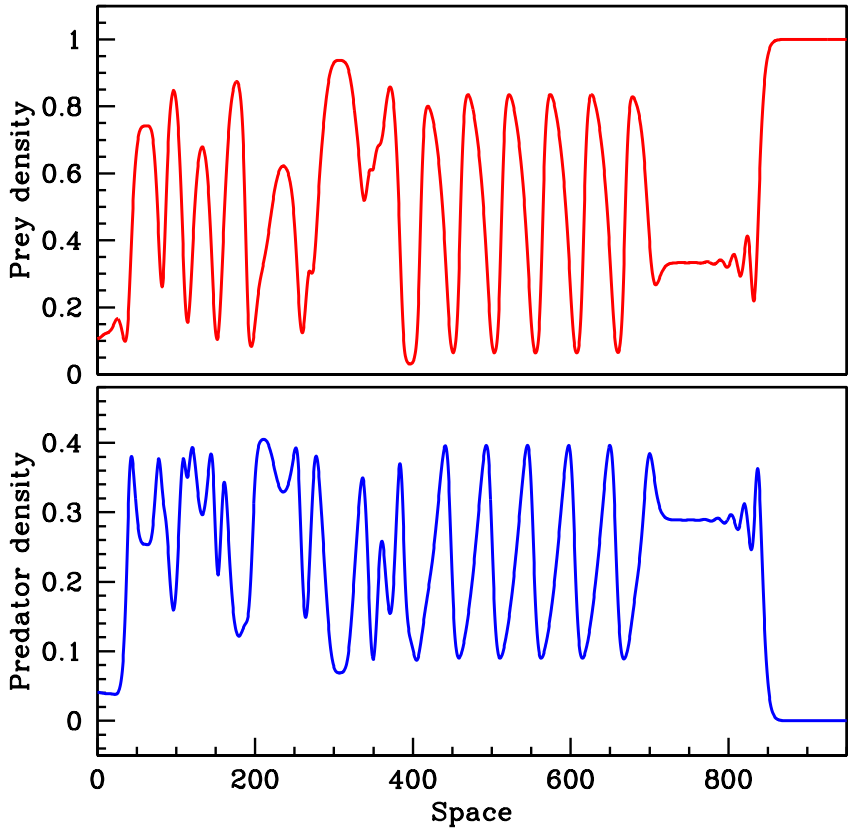

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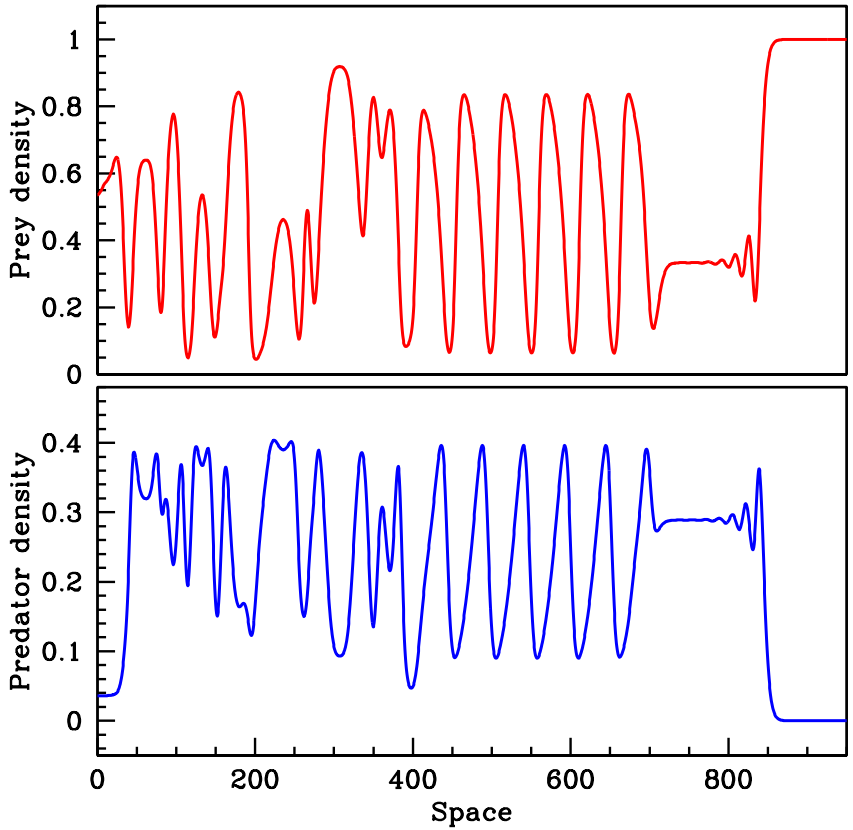

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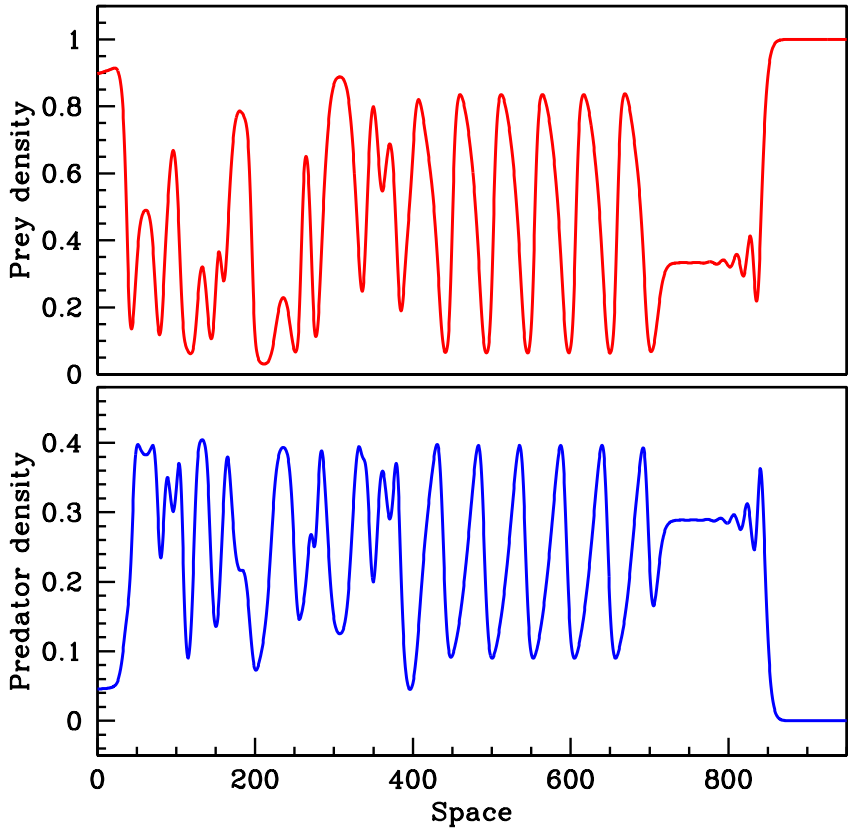

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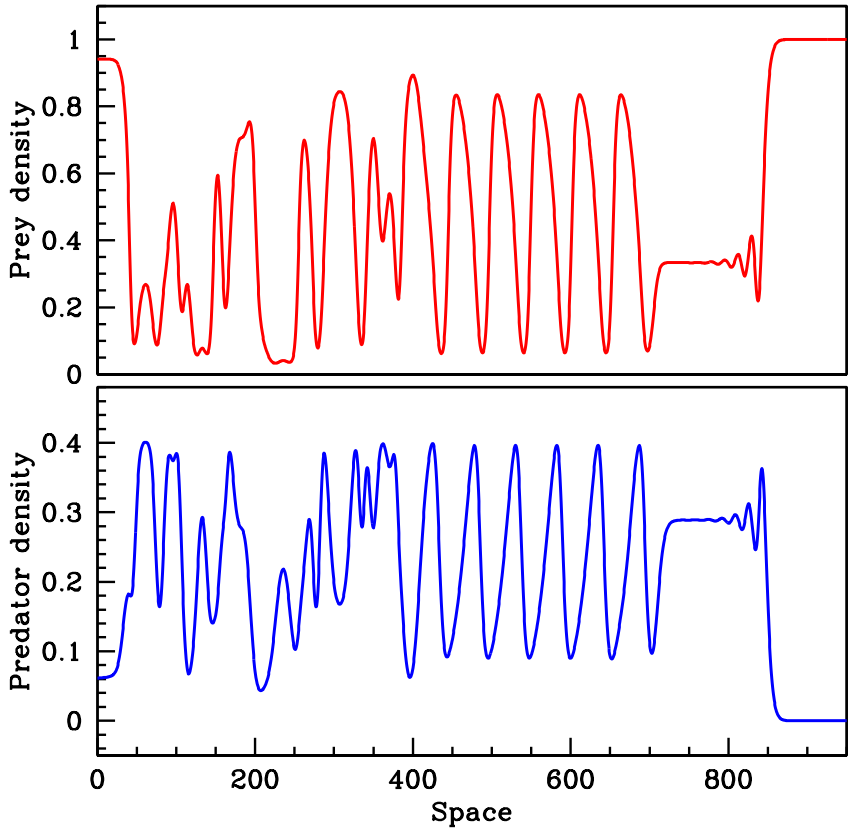

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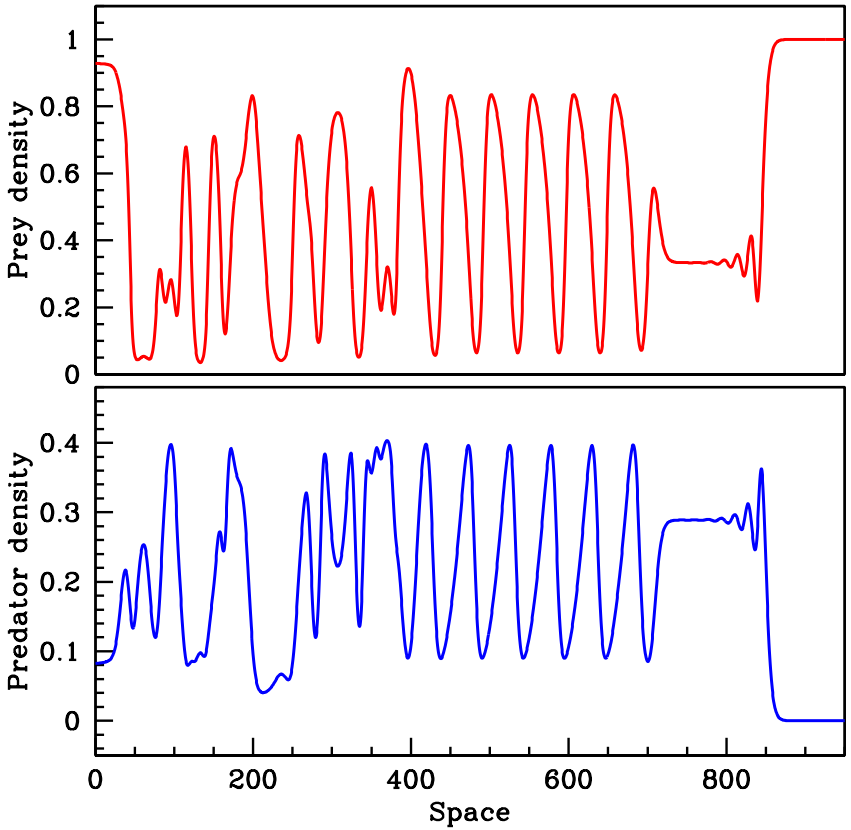

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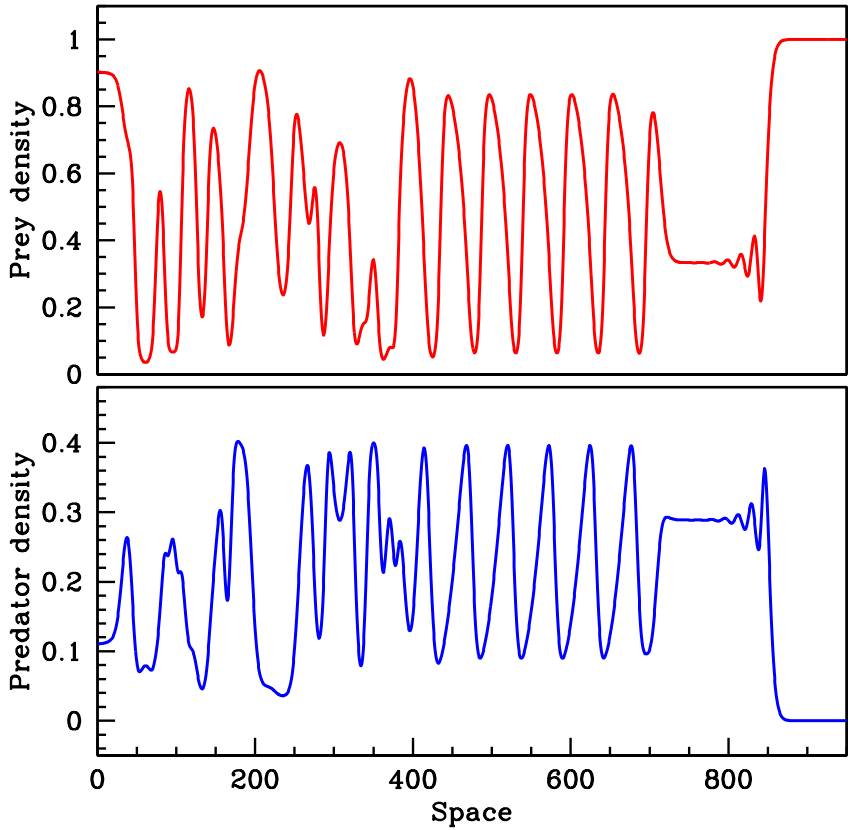

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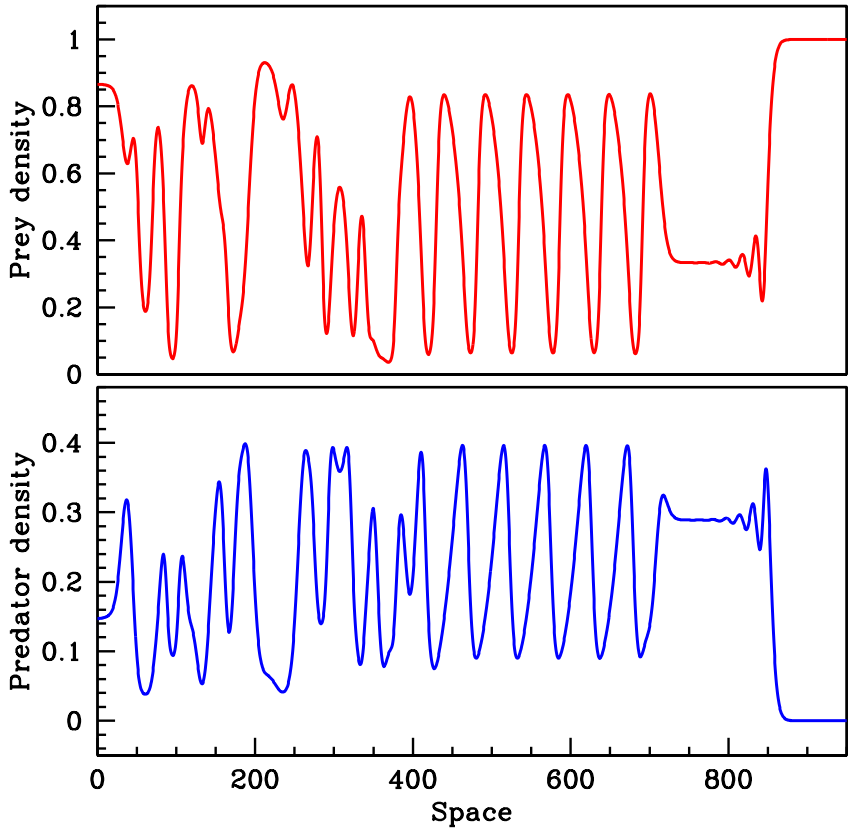

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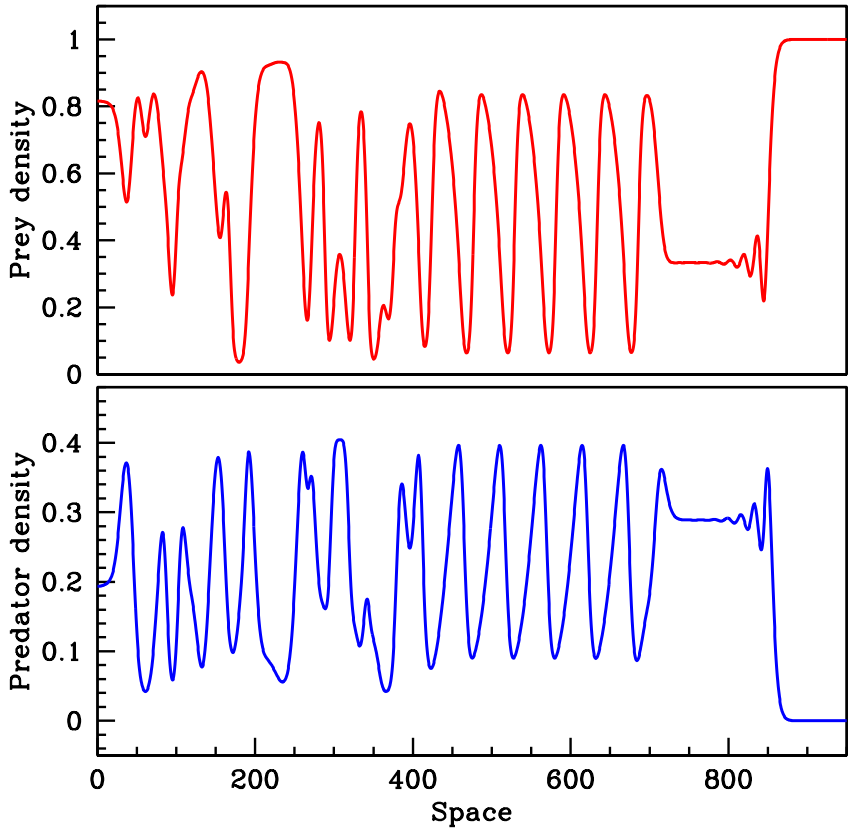

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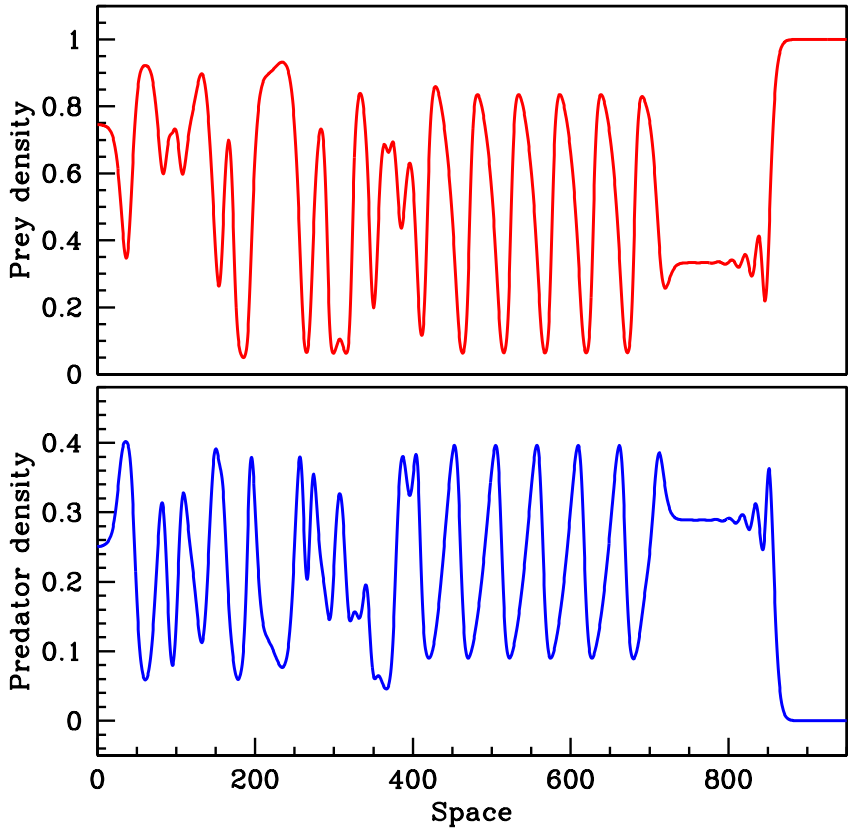

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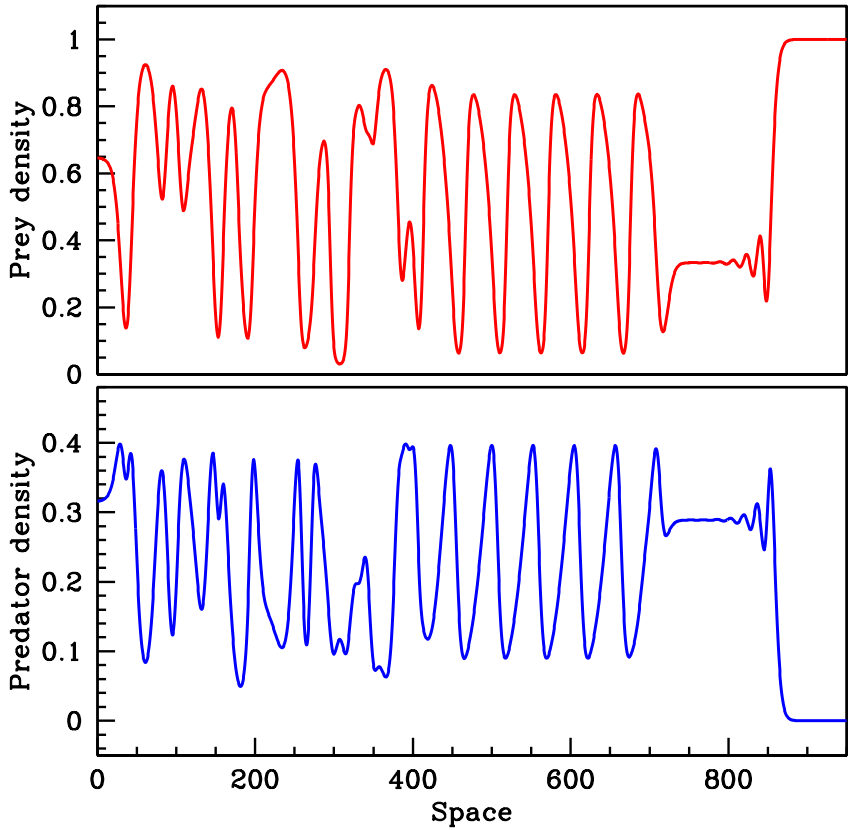

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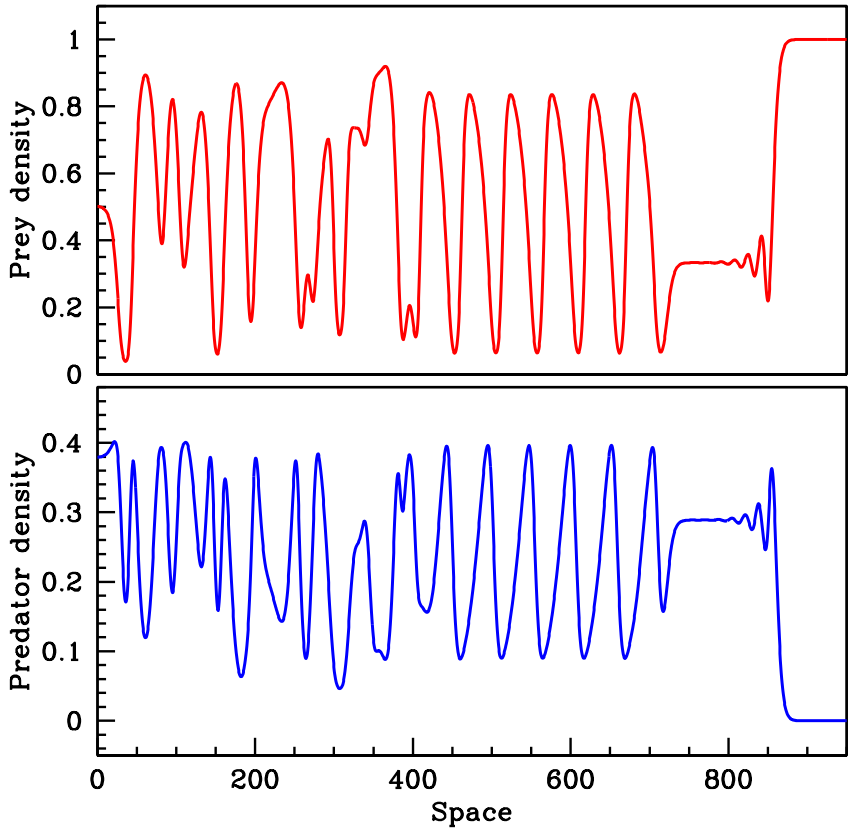

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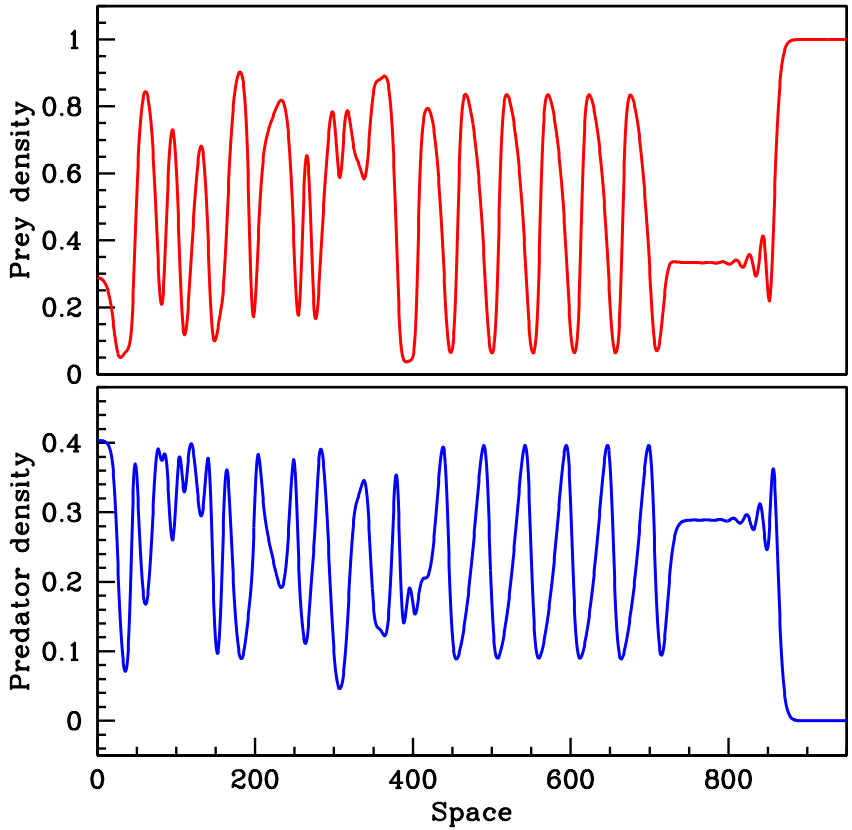

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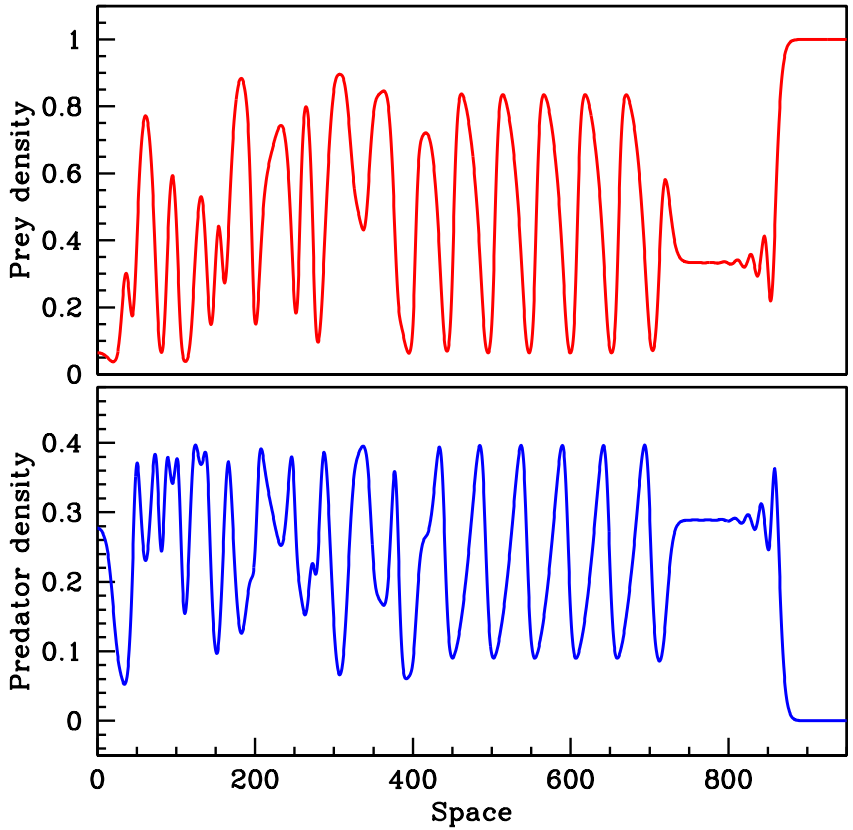

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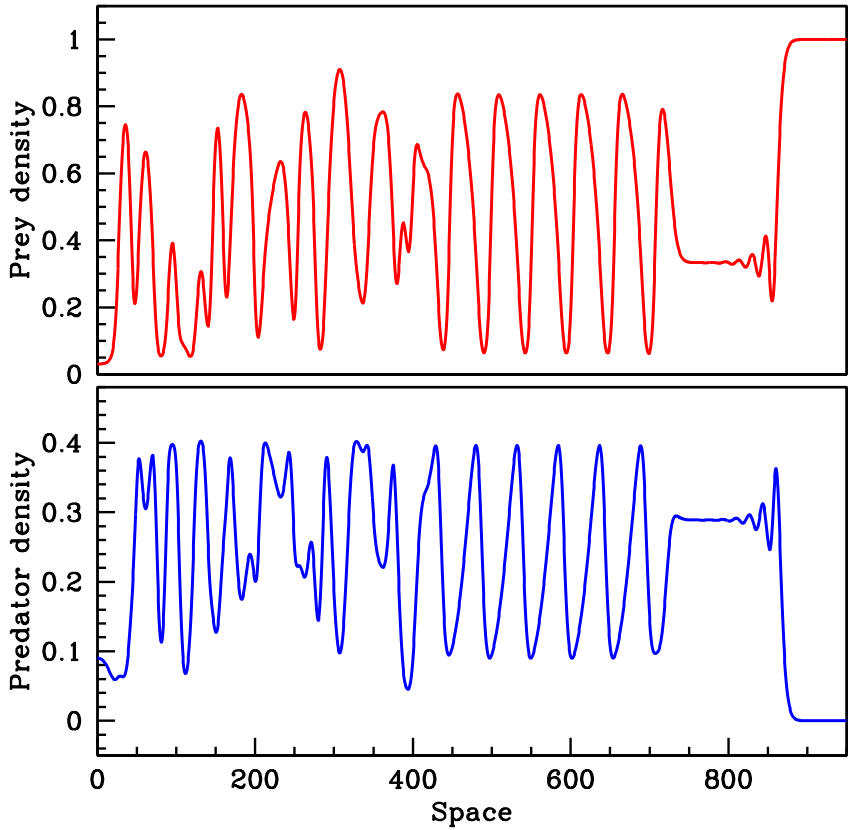

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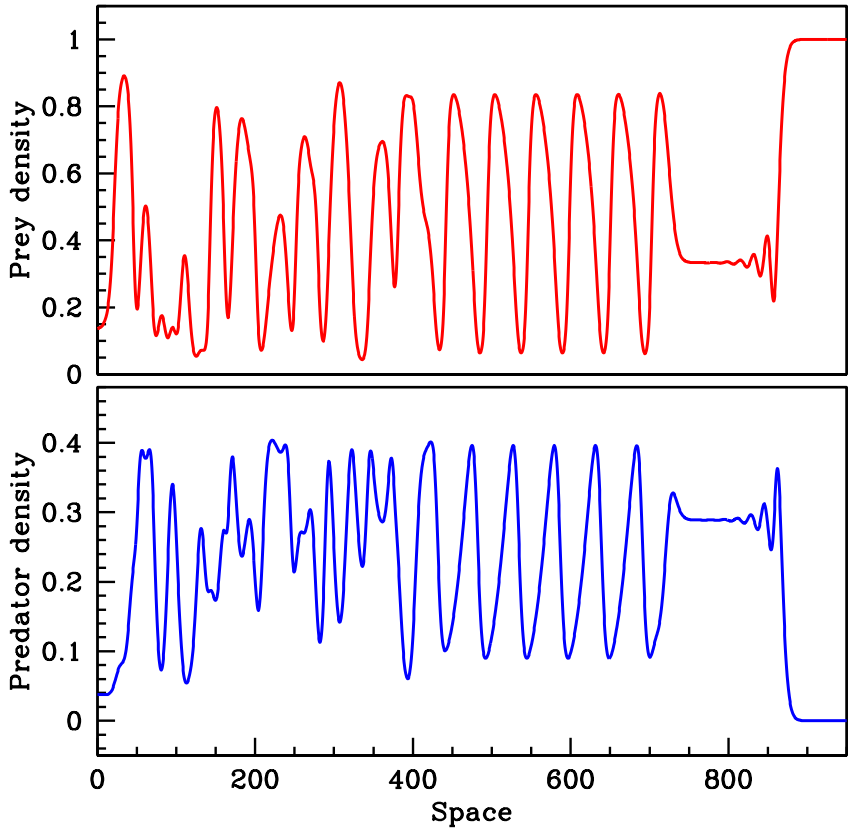

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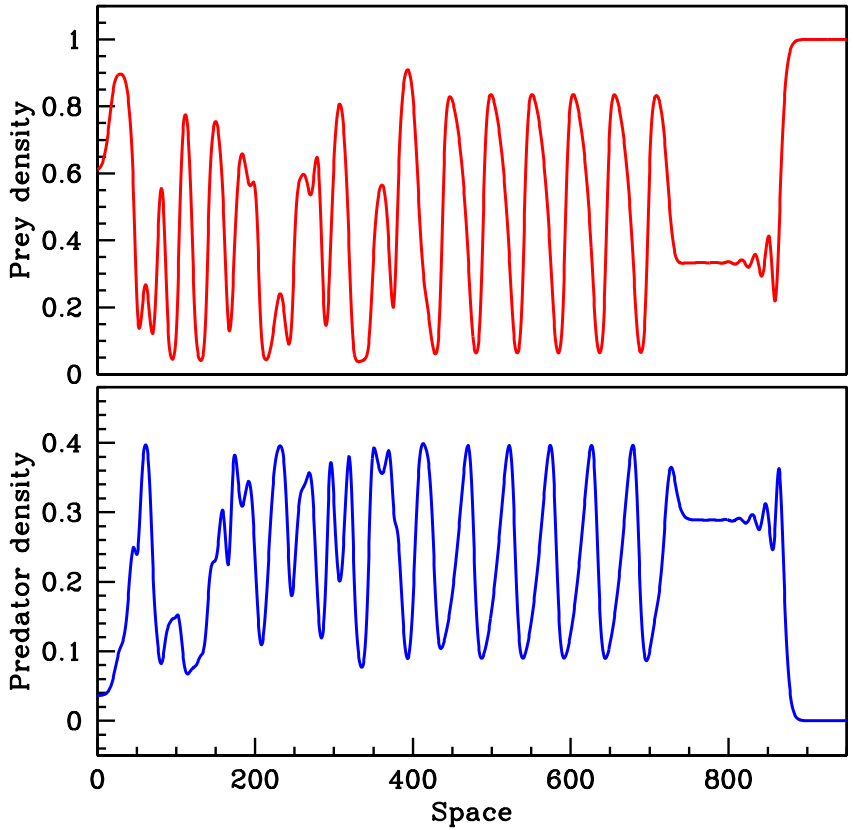

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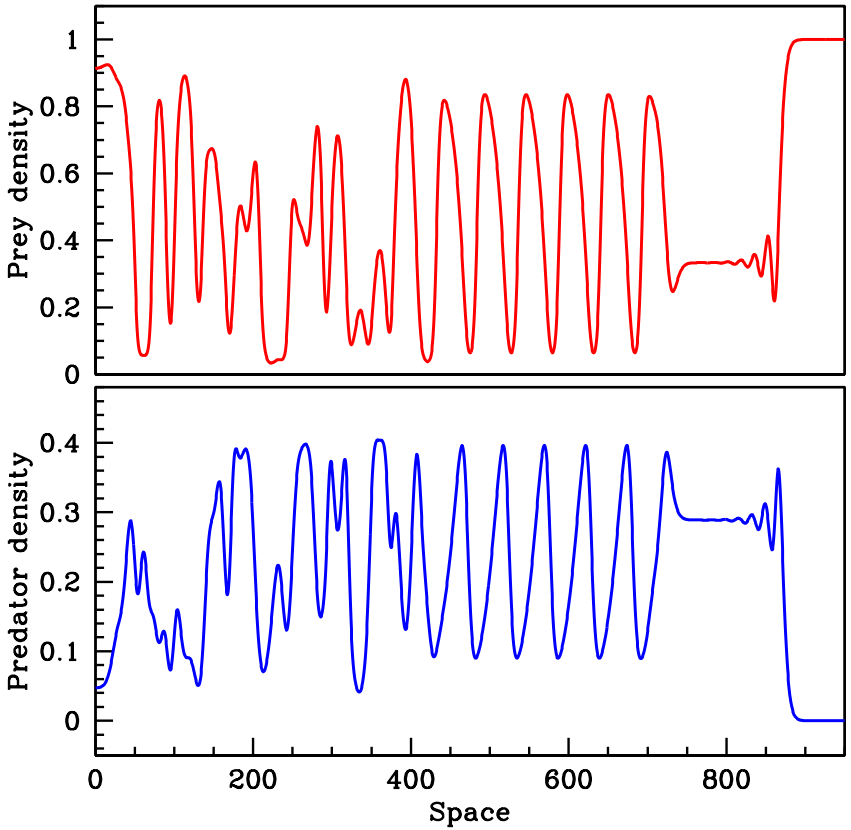

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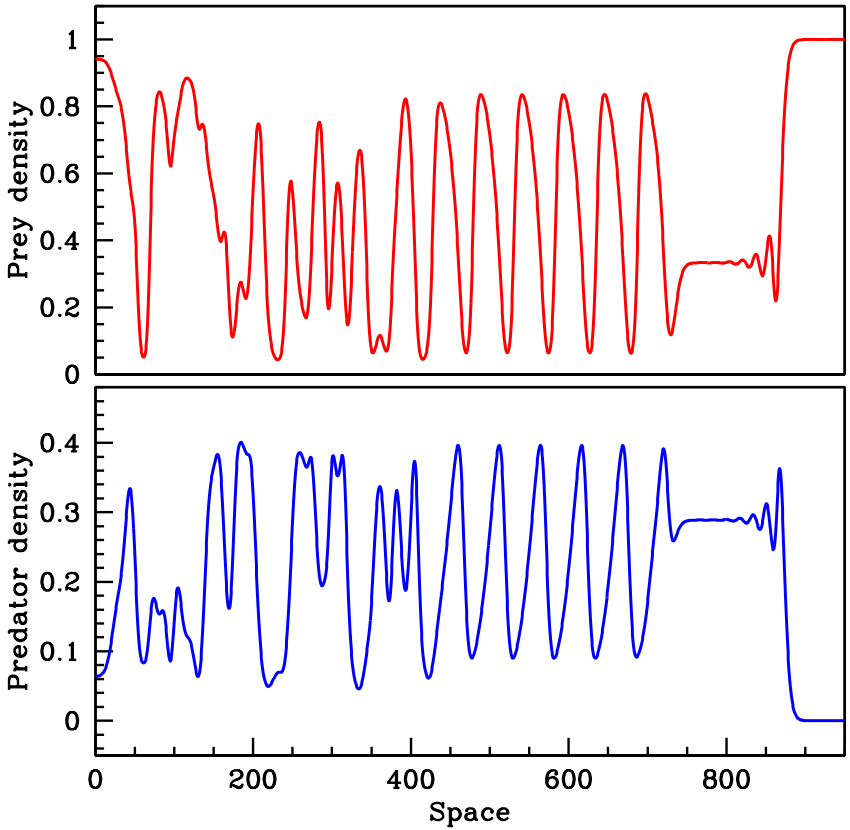

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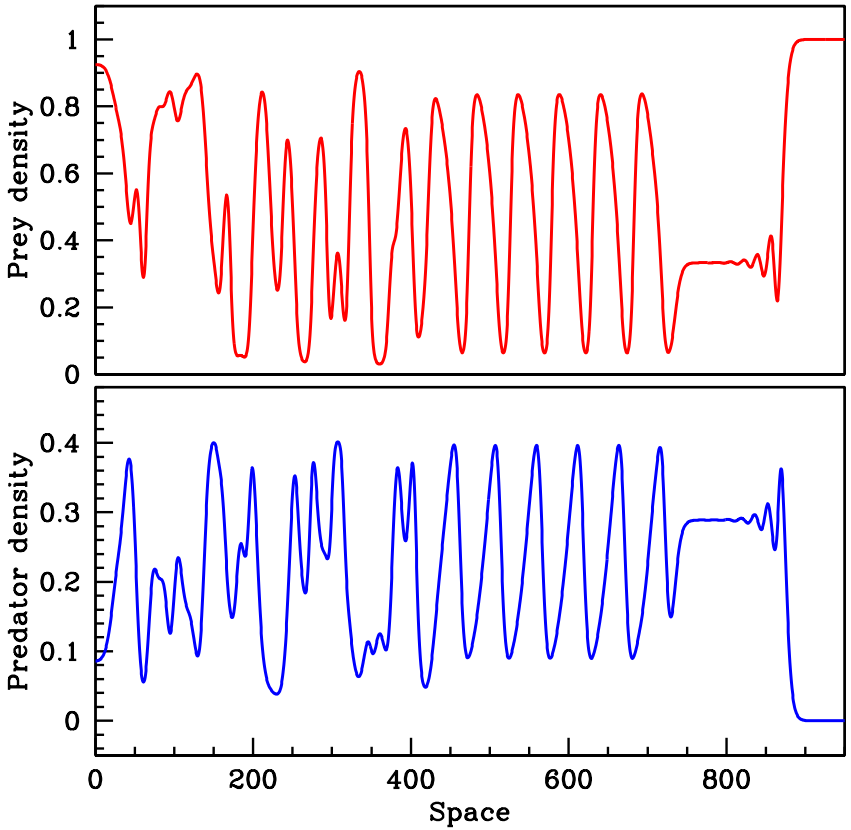
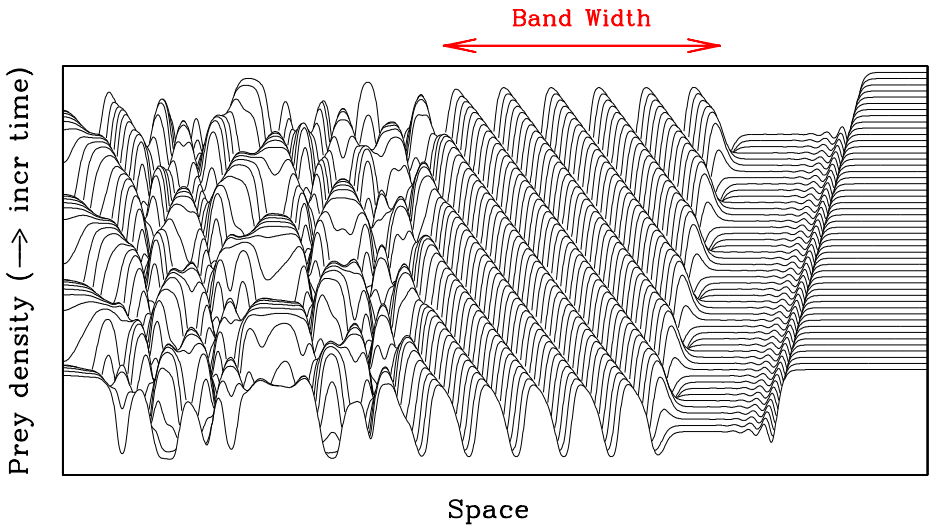


What is the Wavetrain Bandwidth?

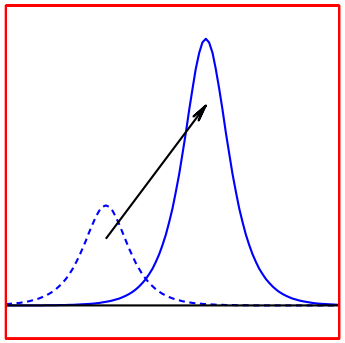
Question: what is the wavetrain band width?



Outline

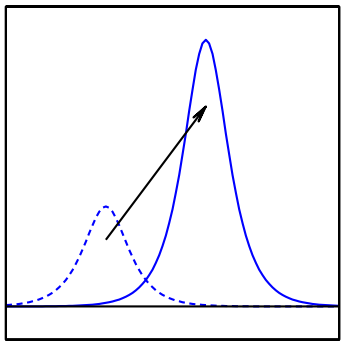
Convective and Absolute Stability

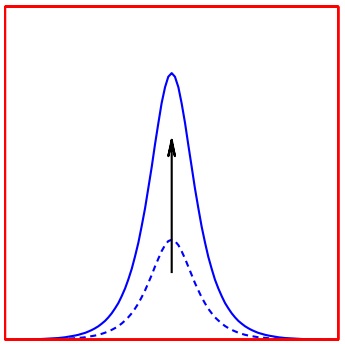
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving.
  This is "convective instability".



Convective and Absolute Stability

- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving.
  This is “convective instability”.
- Alternatively, a solution can be unstable with perturbations growing without moving. This is “absolute instability”.





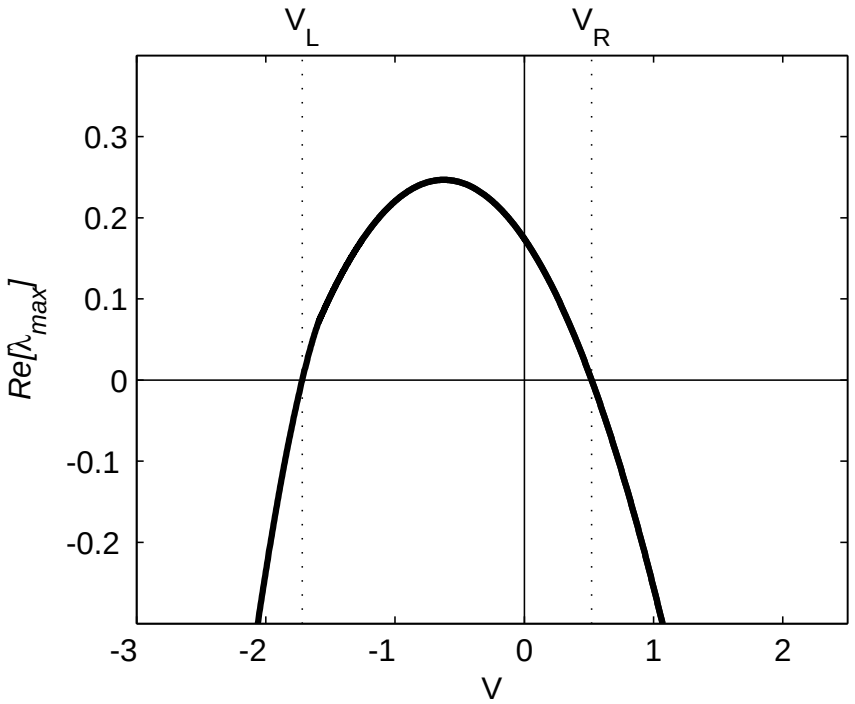
Absolute Stability in a Moving Frame of Reference

Absolute stability refers to the growth/decay of stationary perturbations.

We must consider the growth/decay of perturbations moving with a specified velocity V, i.e. absolute stability in a frame of reference moving with velocity V.

$$\begin{aligned} \text{Define } \lambda_{max}(V) &= \text{temporal eigenvalue of the most unstable linear mode} \\ \nu_{max}(V) &= \text{the corresponding spatial eigenvalue} \end{aligned}$$

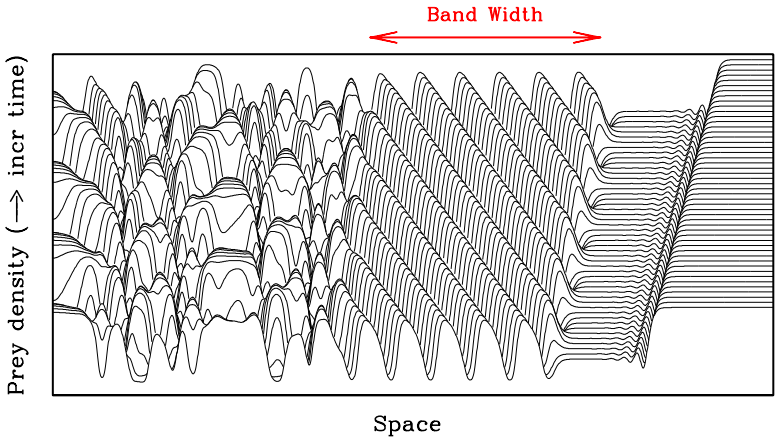
Absolute Stability in a Moving Frame of Reference



A tutorial guide to calculating absolute stability is freely available at

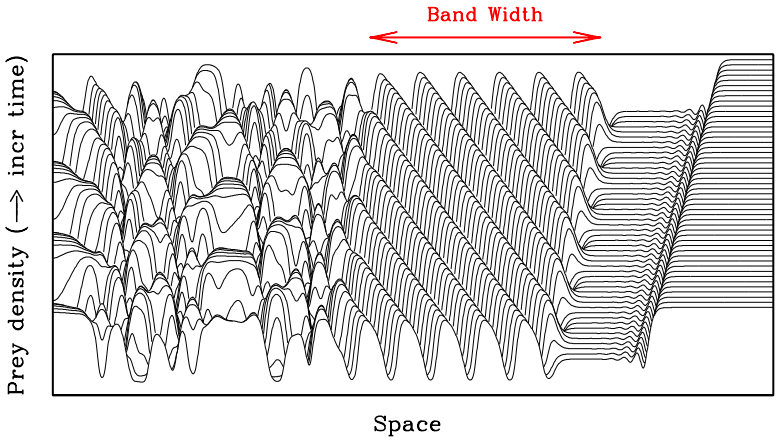
`http://research.microsoft.com/en-us/projects/loptw/tutorial.aspx`

Defining the Band Width



- We define the left-hand edge of the wavetrain band as where unstable linear modes first become amplified by a factor $\mathcal{F}$.

Defining the Band Width



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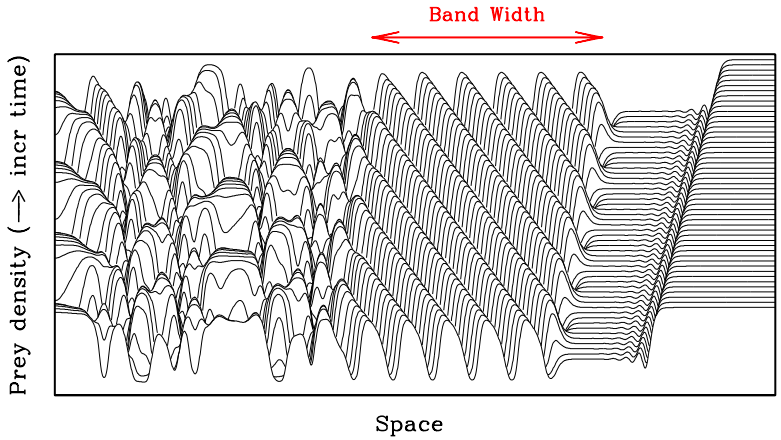
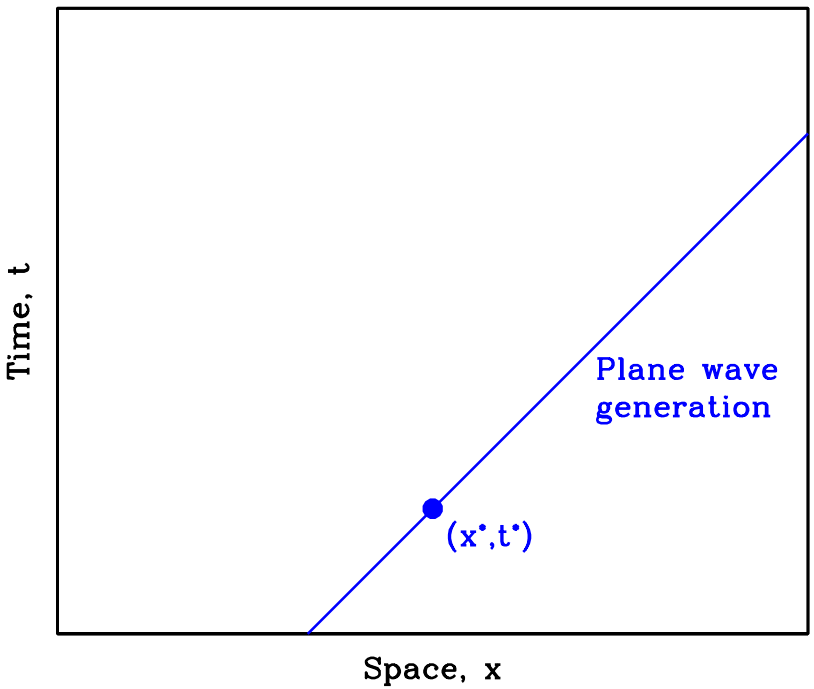


- We define the left-hand edge of the wavetrain band as where unstable linear modes first become amplified by a factor $\mathcal{F}$.
- Our calculations $\Rightarrow$ band width $= \log(\mathcal{F}) \cdot \underbrace{\mathcal{W}}_{\text{“band width coefficient”}}$
- The dependence on ecological parameters is via $\mathcal{W}$.

The Band Width Formula



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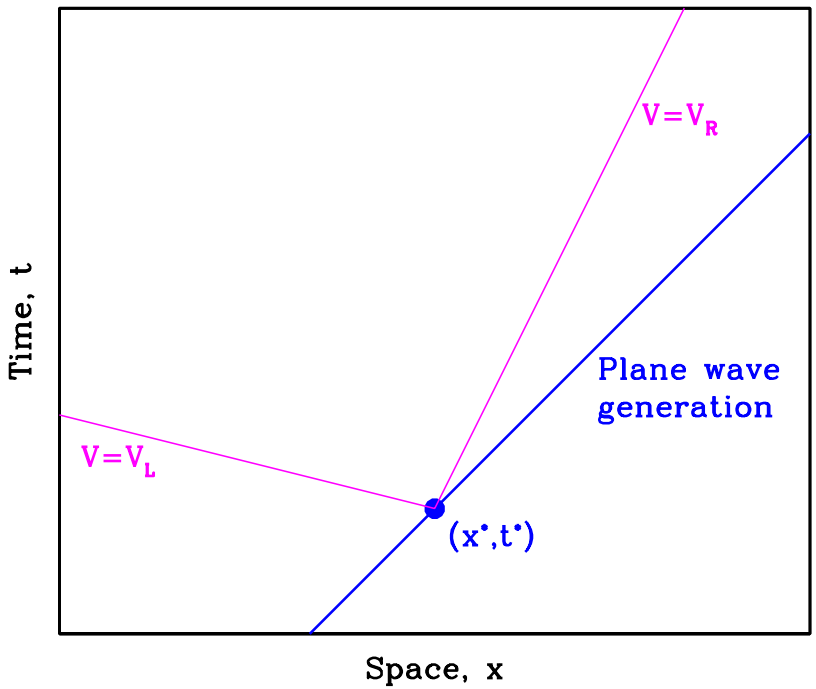

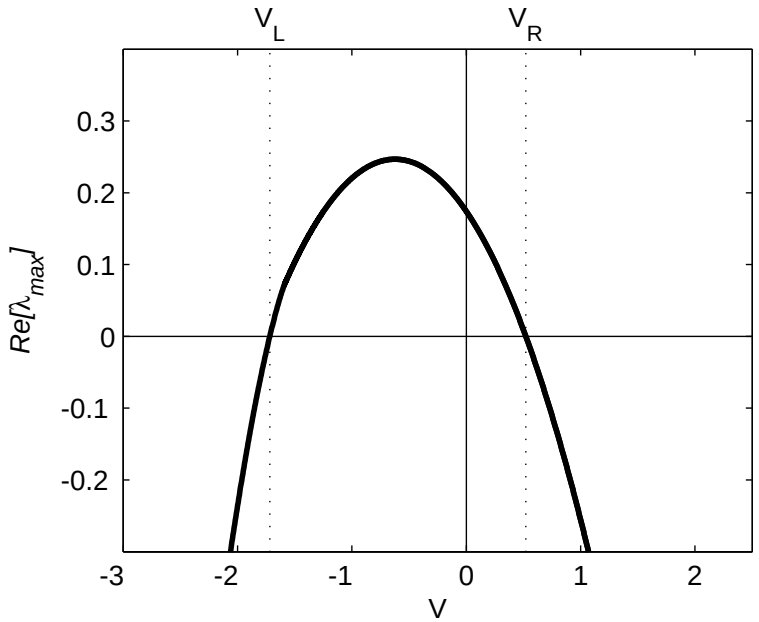

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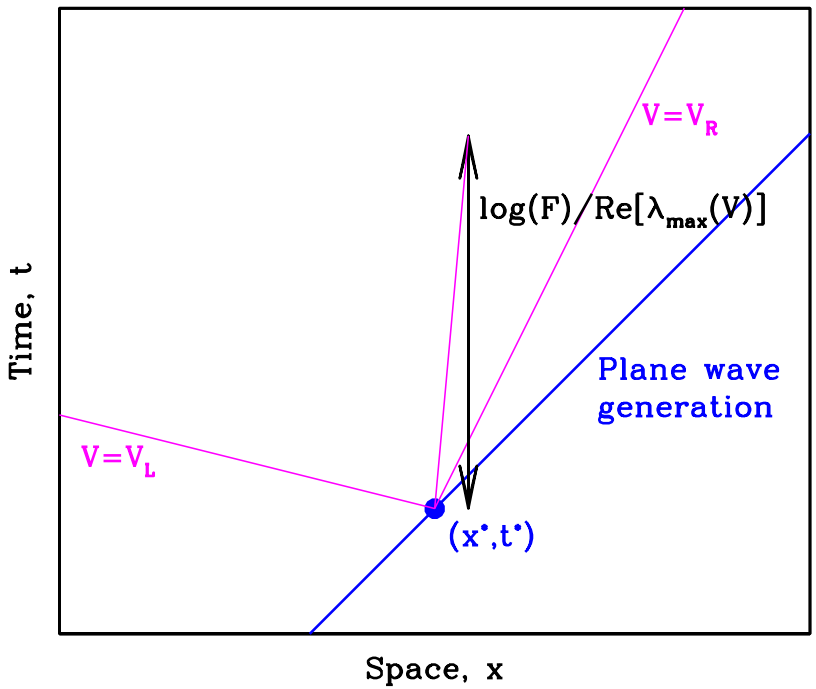
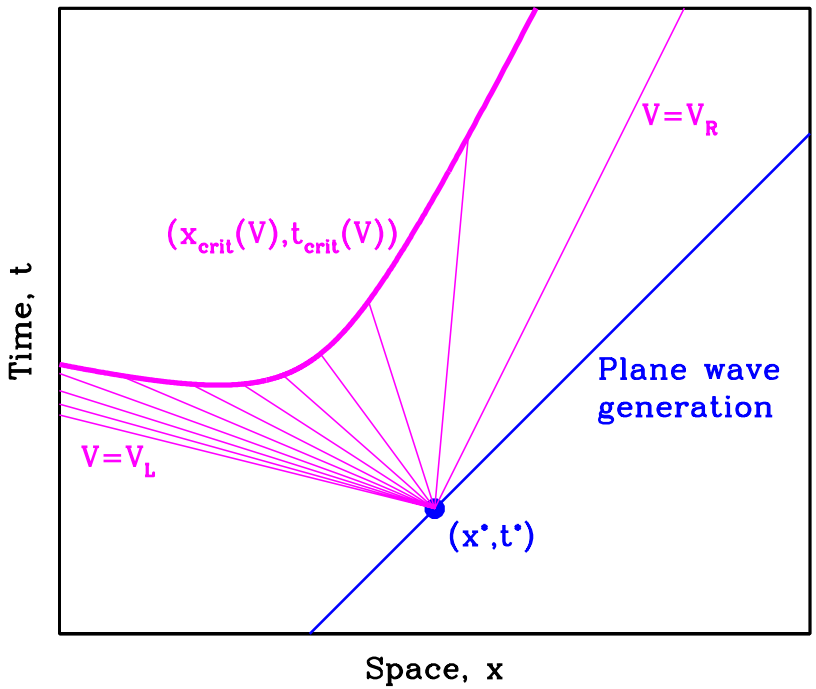


Perturbations moving with velocity V grow as $\exp[\mathrm{Re}\,(\lambda_{max}(V)) \cdot t]$

$\Rightarrow$ amplified by the factor $\mathcal{F}$ after time $\log(\mathcal{F})/\mathrm{Re}\,(\lambda_{max}(V))$

The Band Width Formula



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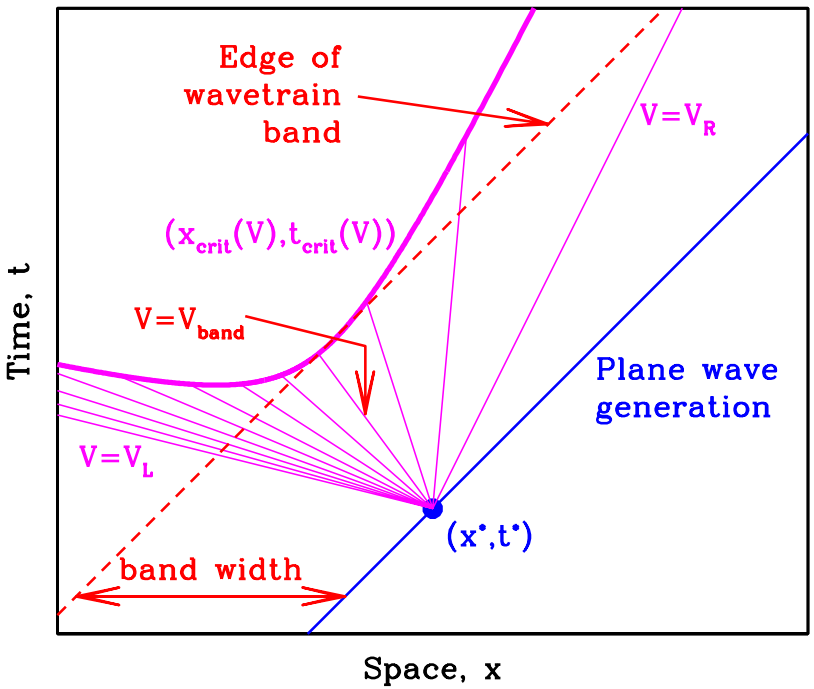

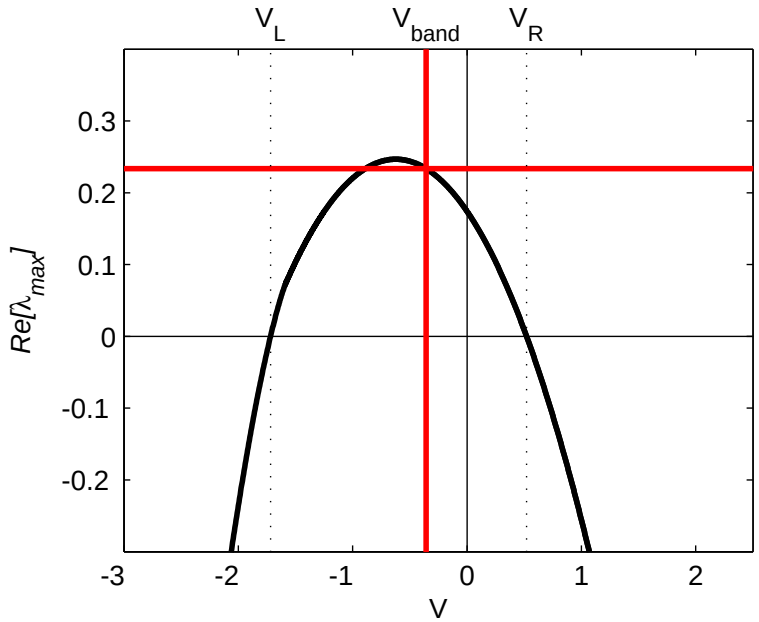

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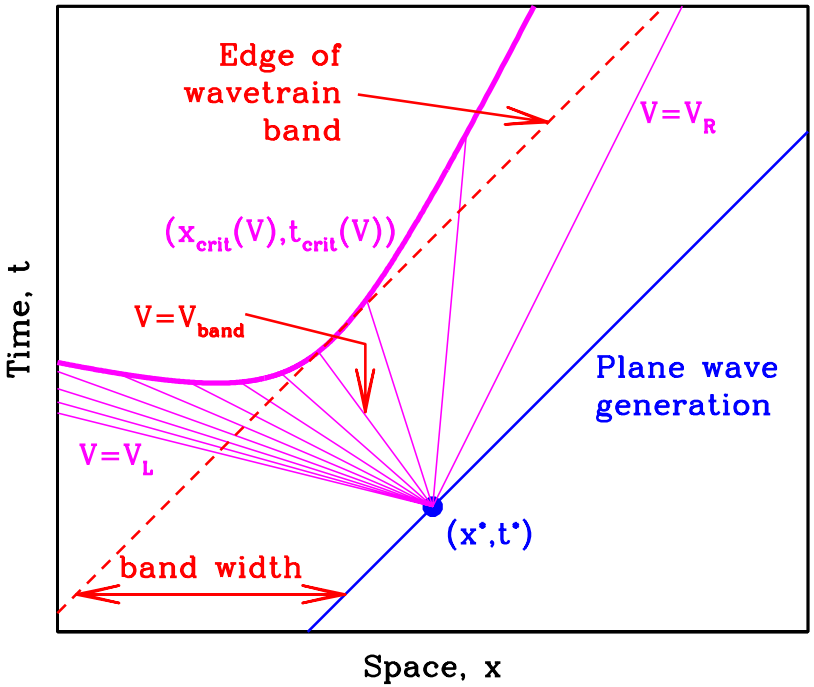
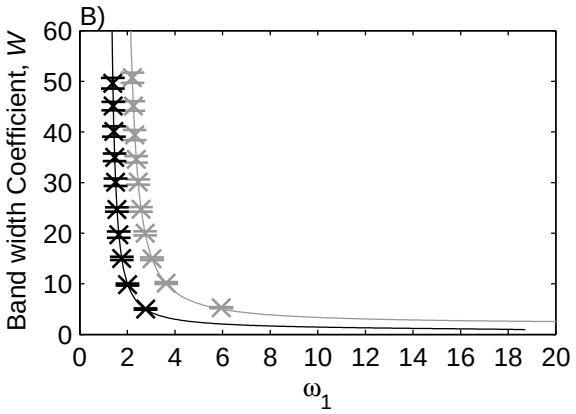
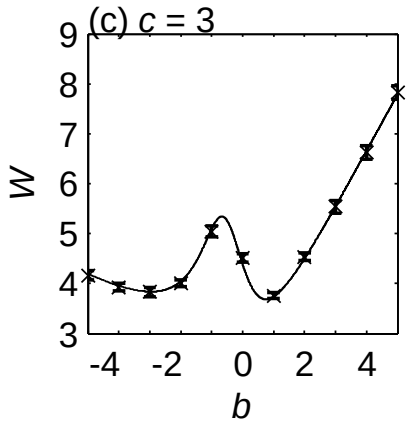


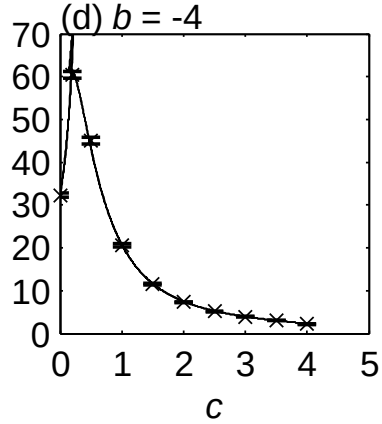
$$\mathcal{W} = 1/\mathrm{Re}\,[\nu_{max}(V_{band})]$$

where $(V_{band} - c_{inv})\mathrm{Re}\,[\nu_{max}(V_{band})] = \mathrm{Re}\,[\lambda_{max}(V_{band})]$

The Form of W



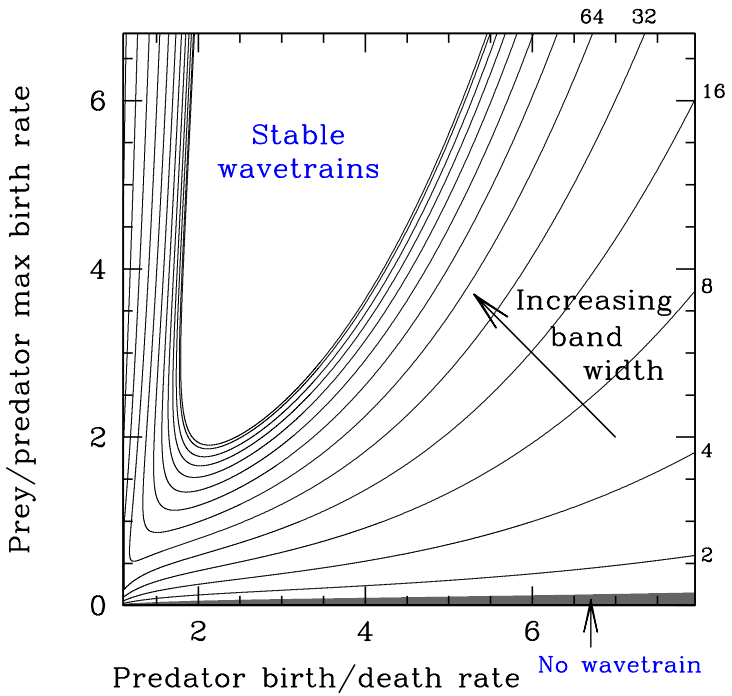




Outline

Band Width Sensitivity

Our formula gives band width vs ecological parameters.



Band Width Sensitivity

Our formula gives band width vs ecological parameters.

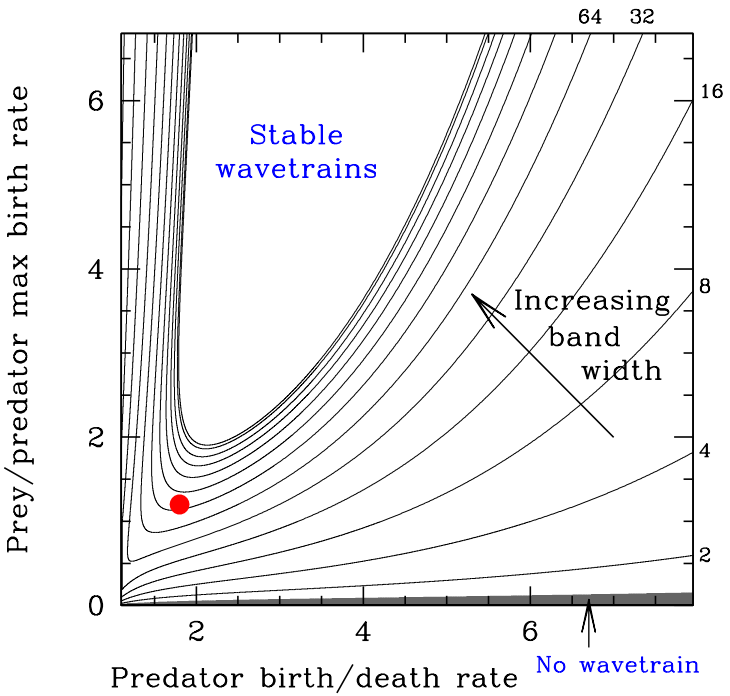
Example: vole – weasel interaction in Fennoscandia

vole

weasel

Band Width Sensitivity

Our formula gives band width vs ecological parameters.



● = weasel–vole parameters.

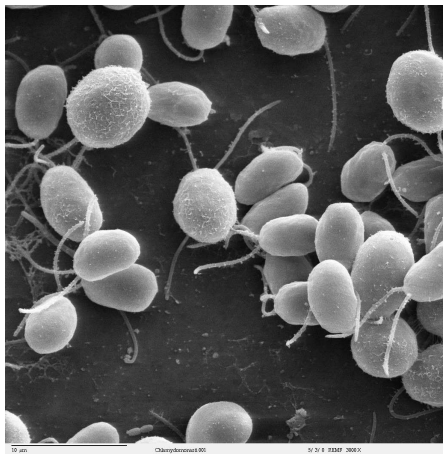
5%↑ in vole birth rate
⇒ 22%↑ in band width.

Band Width Sensitivity

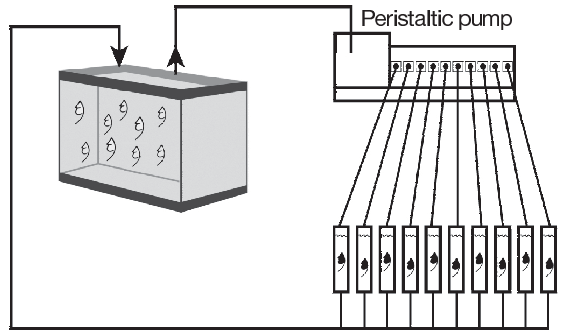
Our formula gives band width vs ecological parameters.

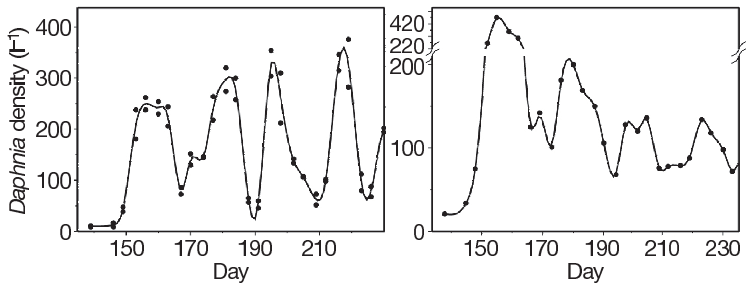
Example: *Daphnia pulex–Chlamydomonas reinhardii* interaction

Daphnia pulex



Chlamydomonas reinhardii

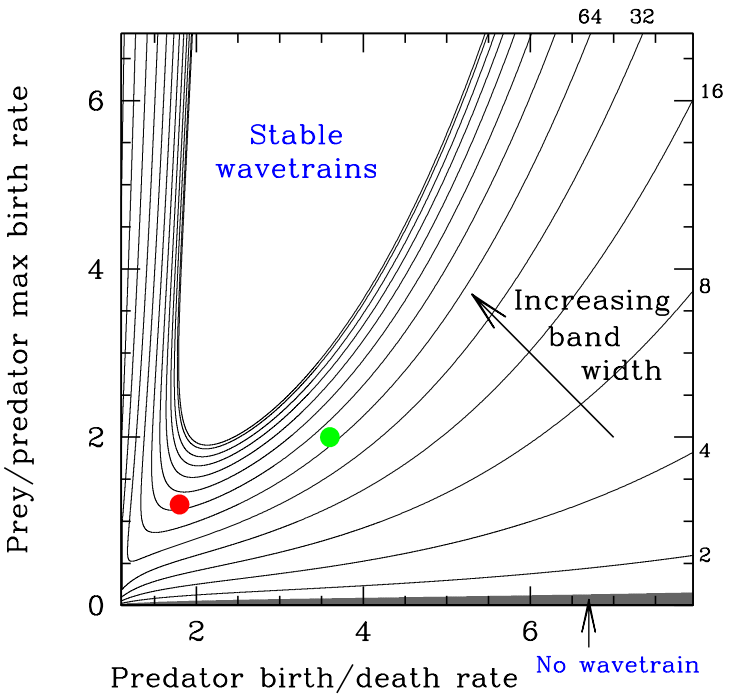




(from McCauley *et al* (2008), *Nature* 455:1240, 2008)

Band Width Sensitivity

Our formula gives band width vs ecological parameters.



● = plankton parameters
(*Daphnia pulex–Chlamydomonas reinhardii*).

5.2%↓ in Daphnia
birth rate
⇒ doubling of band width.

Ecological Implications

- Climate change $\Rightarrow$ more frequent invasions.
- It is known that climate change is significantly affecting the parameters of oscillatory ecological systems
  (e.g. Ims *et al* (2008) *TREE* 23:79).
- We have shown that band width depends sensitively on ecological parameters.

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- We have shown that band width depends sensitively on ecological parameters.
- This suggests that the implications of climate change for *spatio* temporal dynamics may be even more dramatic than for purely temporal behaviour.

References

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List of Frames

The Form of V_{band}