Wavelength Selection and Hysteresis in Mathematical Models of Banded Vegetation

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Drylands, Deserts and Desertification, 17-20 November 2014

This talk can be downloaded from my web site

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Outline

Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.

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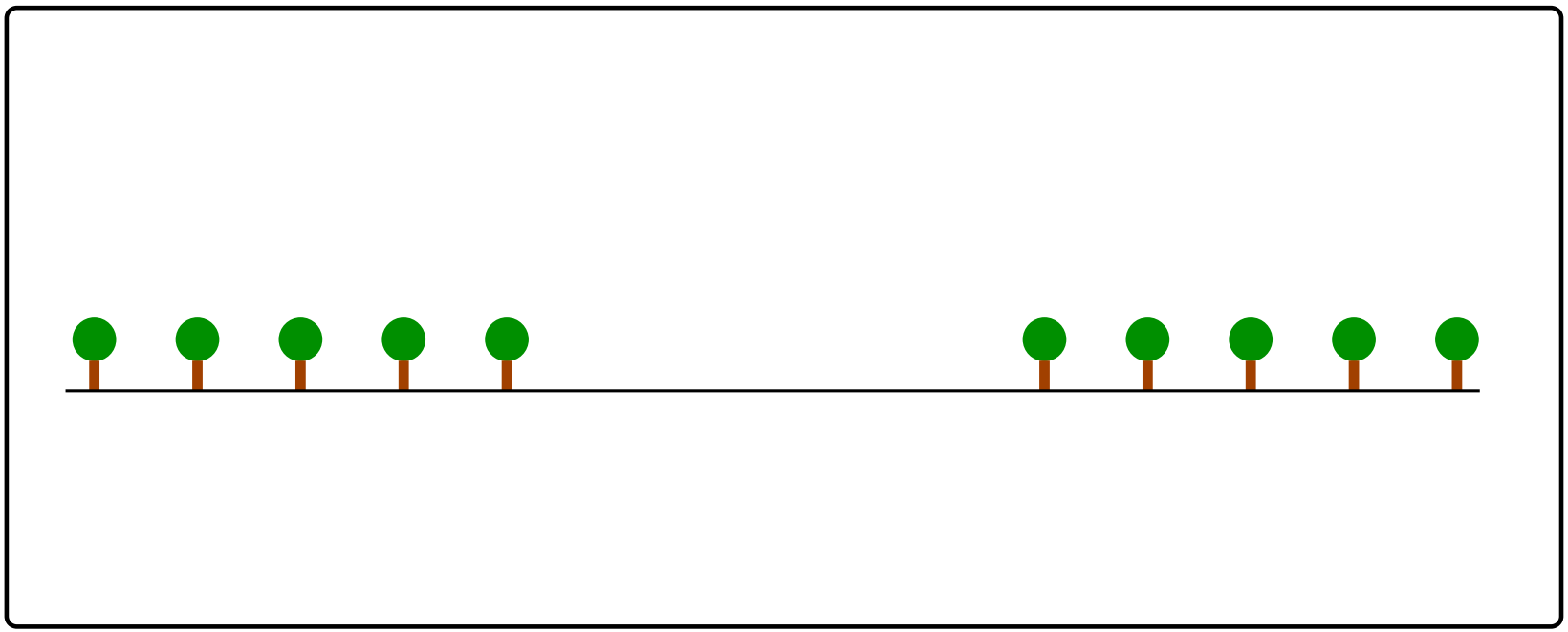

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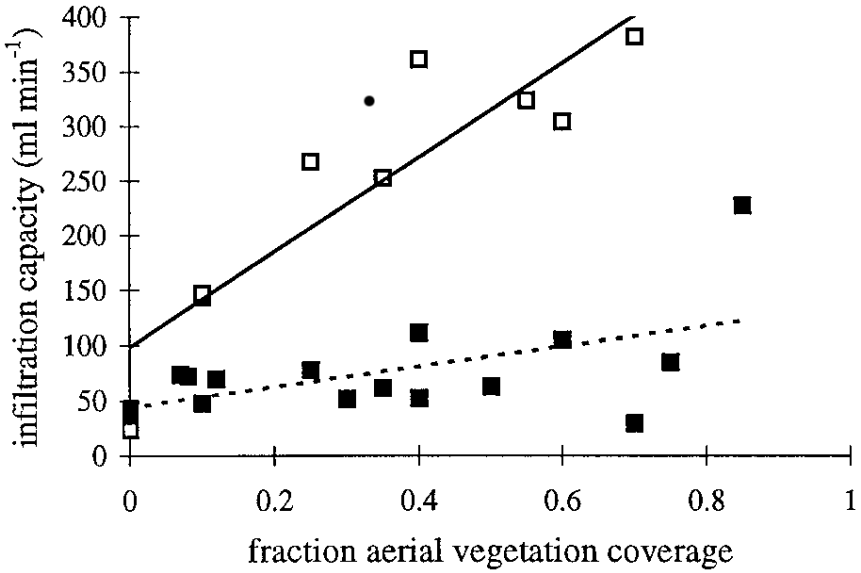

Data from Burkina Faso

Rietkerk et al
Plant Ecology 148: 207-224, 2000

More plants $\Rightarrow$ more roots and organic matter in soil
$\Rightarrow$ more infiltration of rainwater

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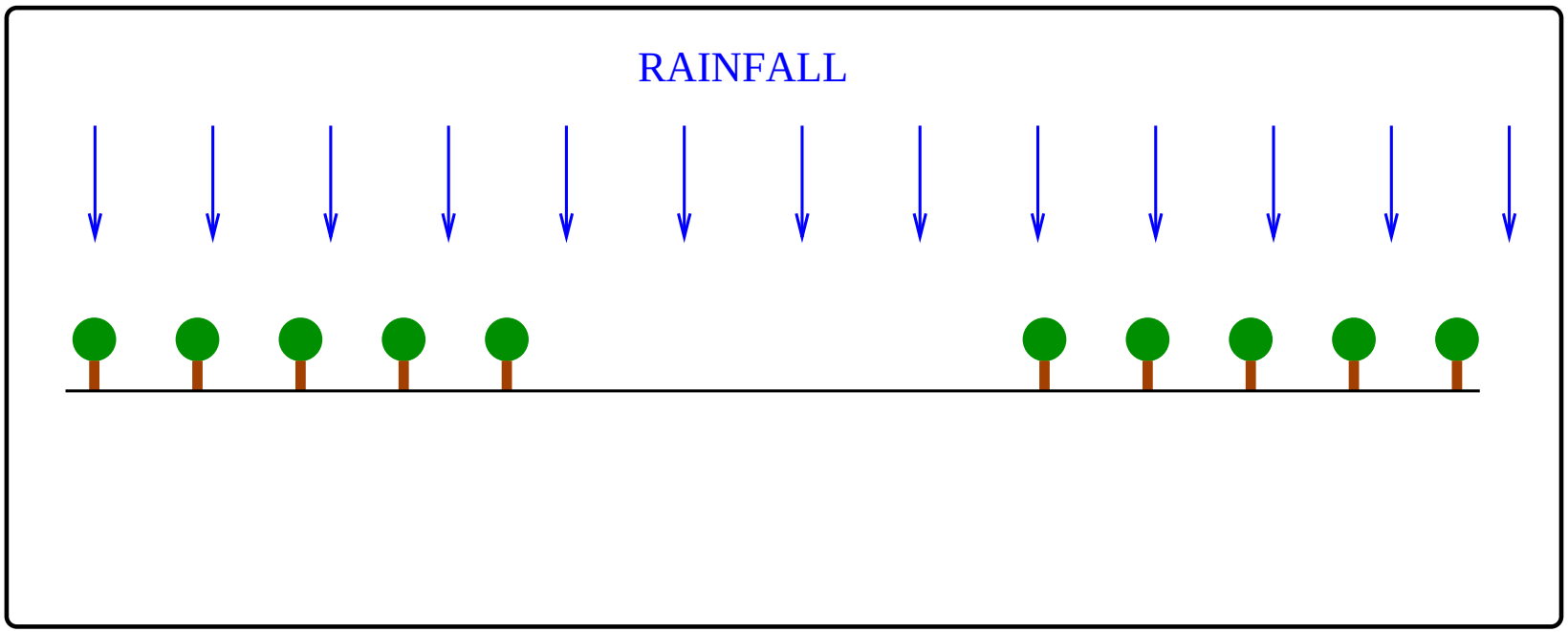

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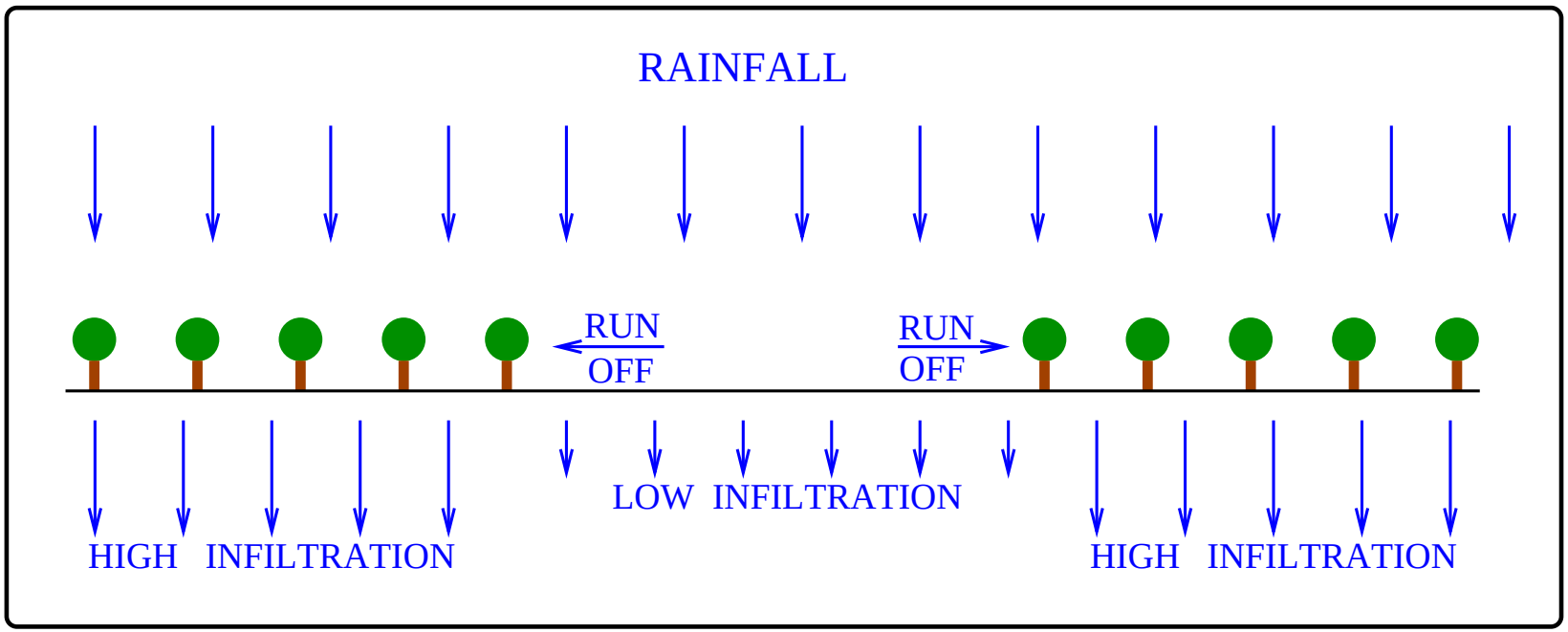

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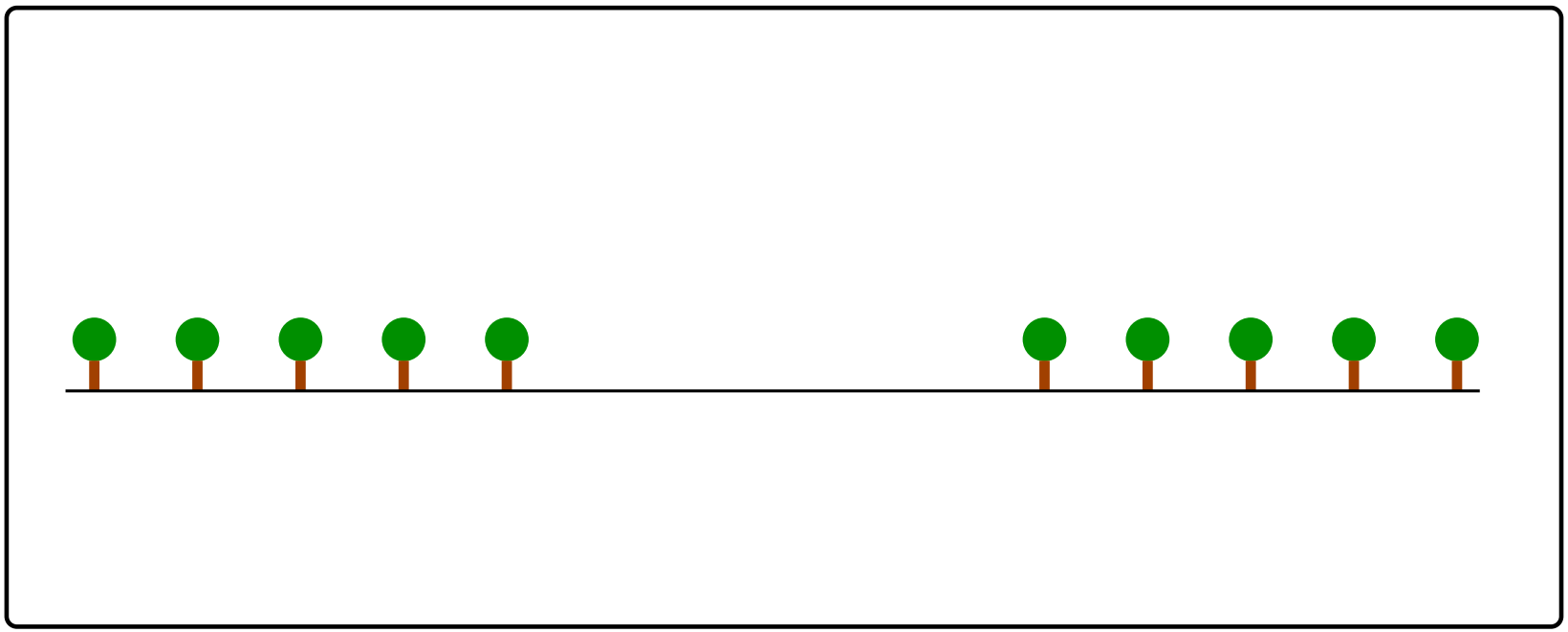

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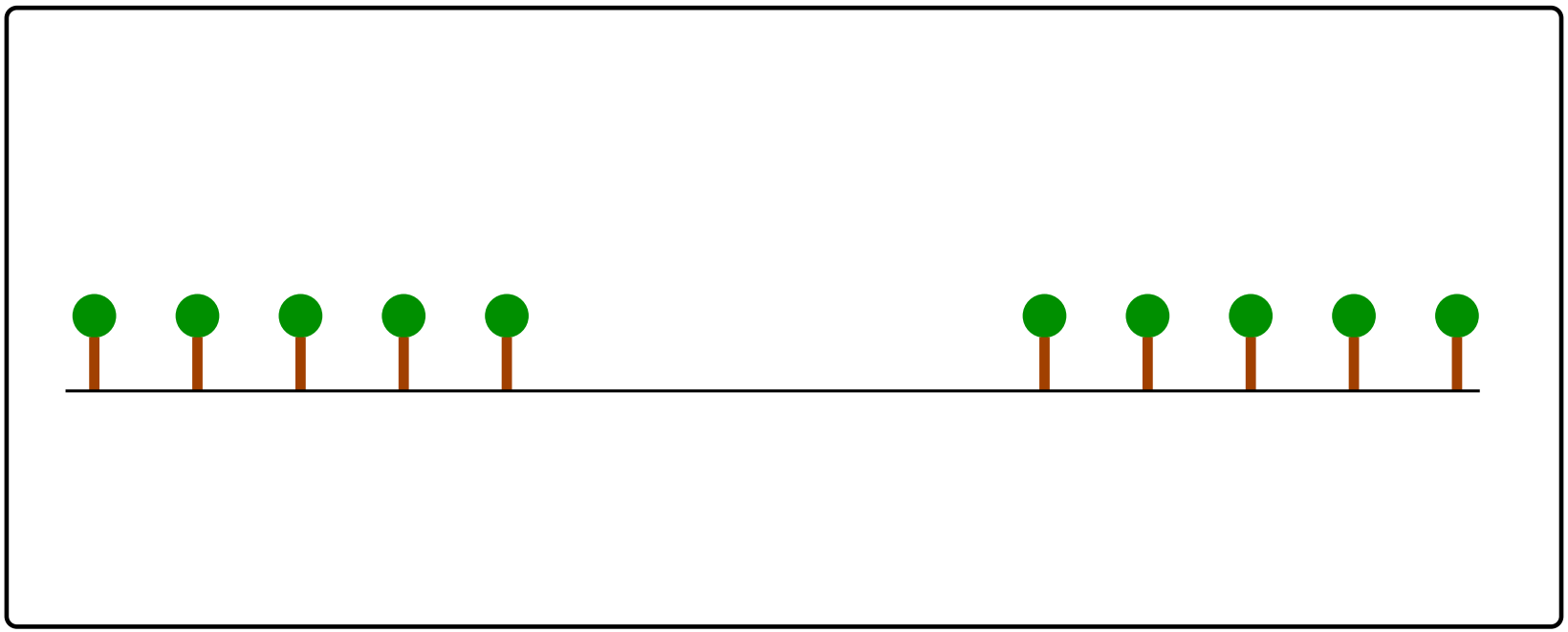

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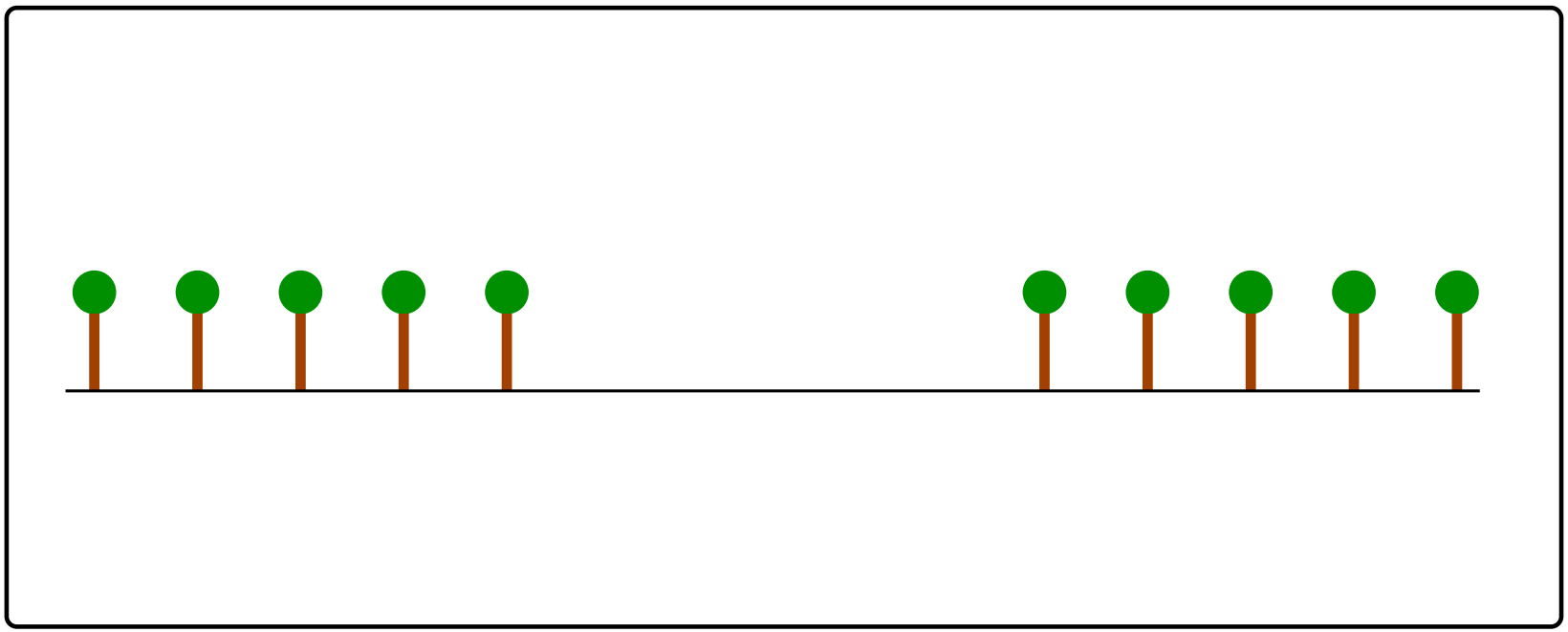

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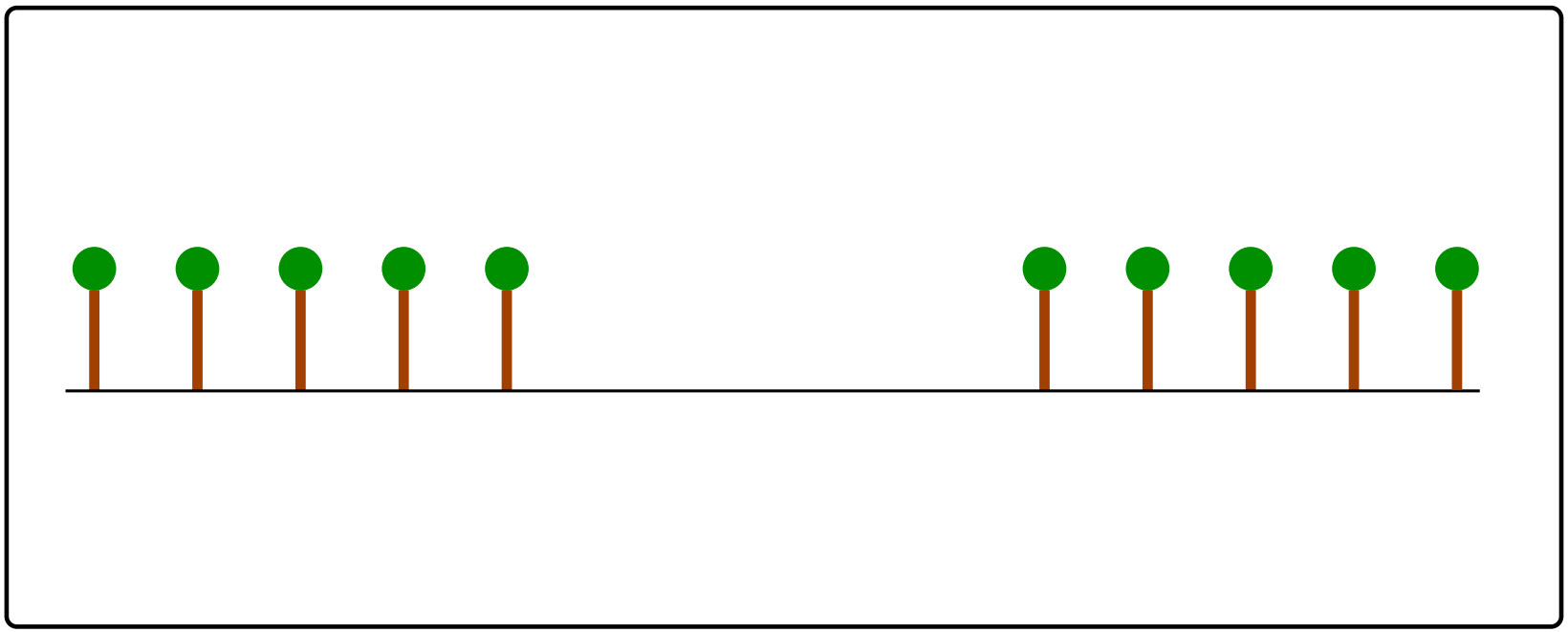

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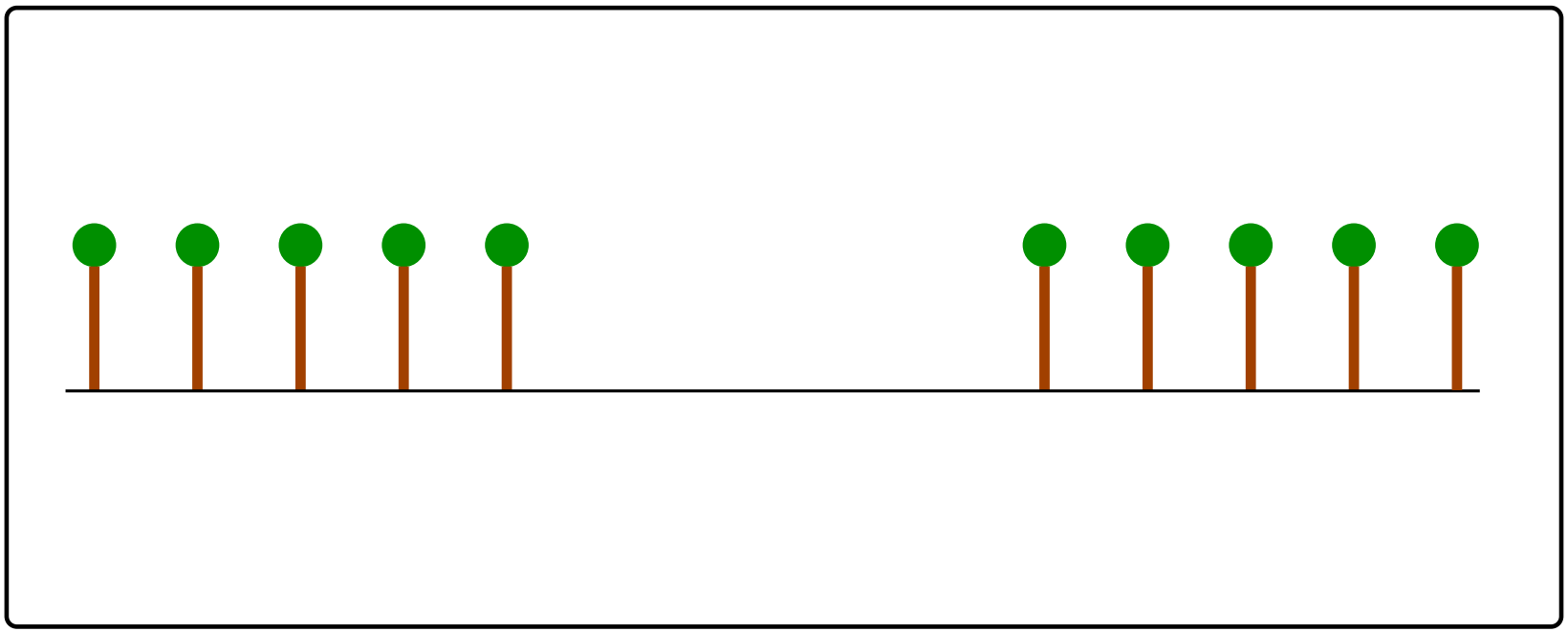

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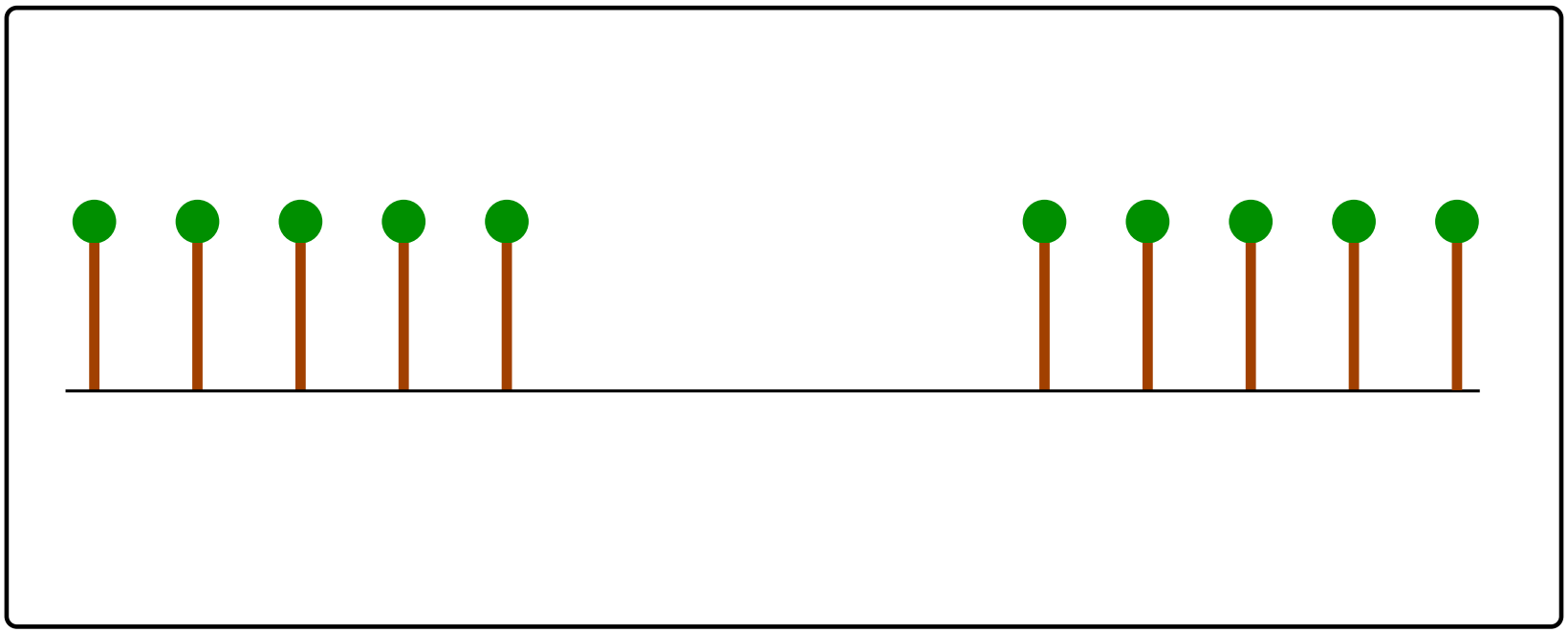

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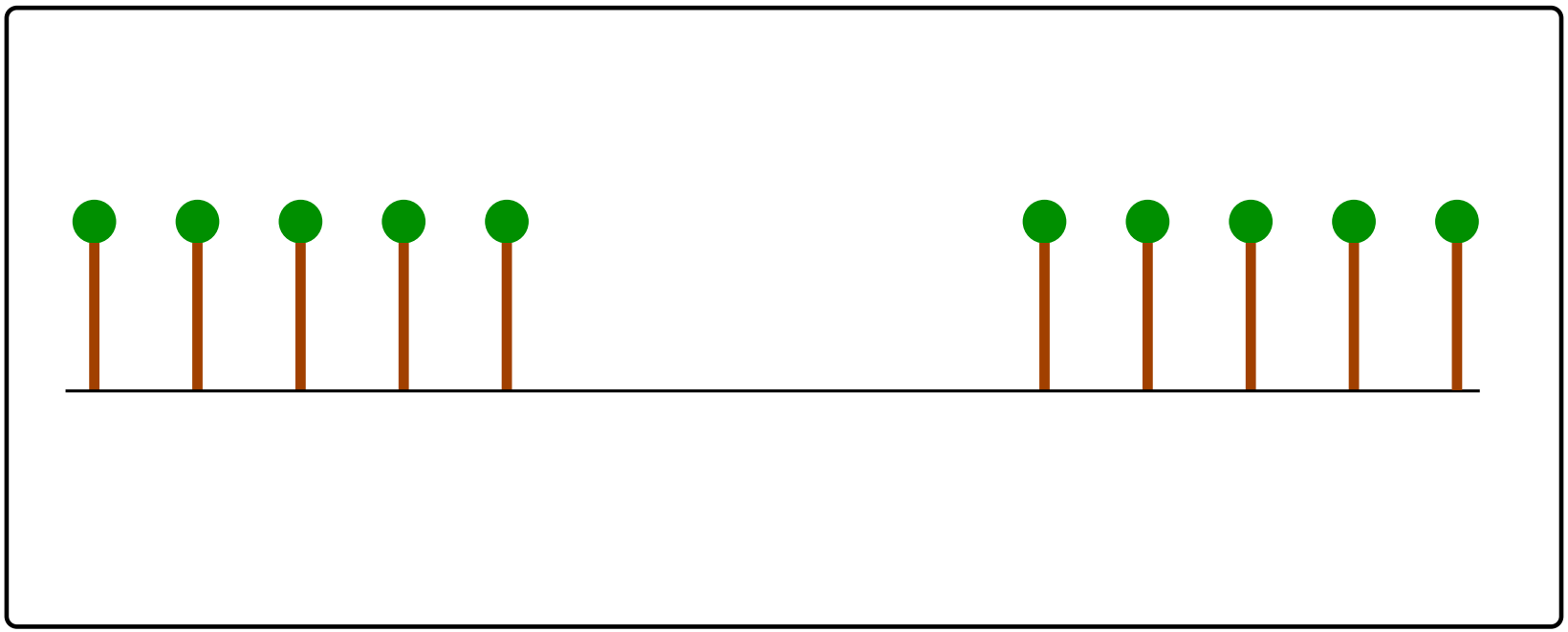

Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia

(Western New South Wales)

Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia, Mexico and S-W USA.

Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia

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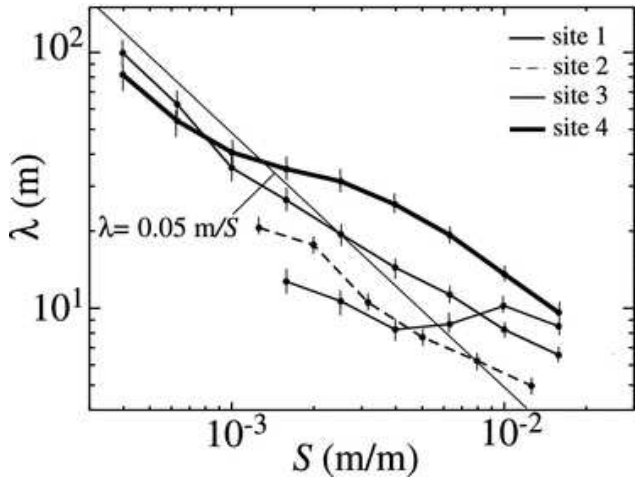
Wavelength can be measured via remote sensing.

Data on Wavelength vs Slope

Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.



Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

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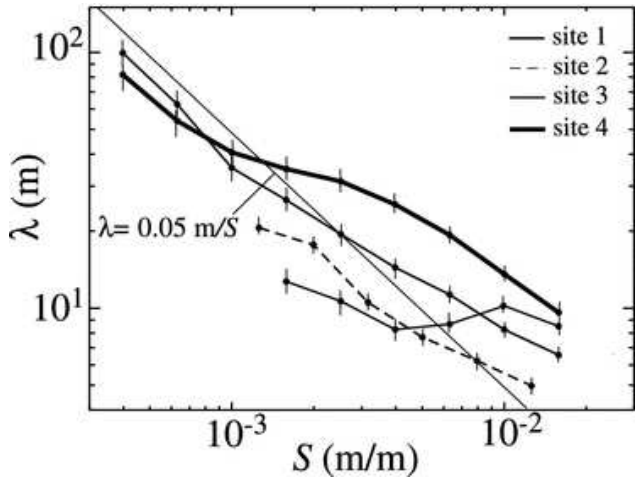

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How does this compare with predictions of mathematical models?

Outline

Mathematical Model of Klausmeier

$$\partial u/\partial t = \overbrace{wu^2}^{\text{plant growth}} - \overbrace{Bu}^{\text{plant loss}} + \overbrace{\partial^2 u/\partial x^2}^{\text{plant dispersal}}$$

$$\partial w/\partial t = \underbrace{A}_{\substack{\text{average}\\ \text{rainfall}}} - \underbrace{w}_{\substack{\text{evaporation}\\ \text{\& drainage}}} - \underbrace{wu^2}_{\substack{\text{uptake}\\ \text{by plants}}} + \underbrace{\nu\,\partial w/\partial x}_{\substack{\text{flow}\\ \text{downhill}}} + \underbrace{D\,\partial^2 w/\partial x^2}_{\substack{\text{diffusion}\\ \text{of water}}}$$

(Klausmeier, Science 284: 1826-8, 1999)

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The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.

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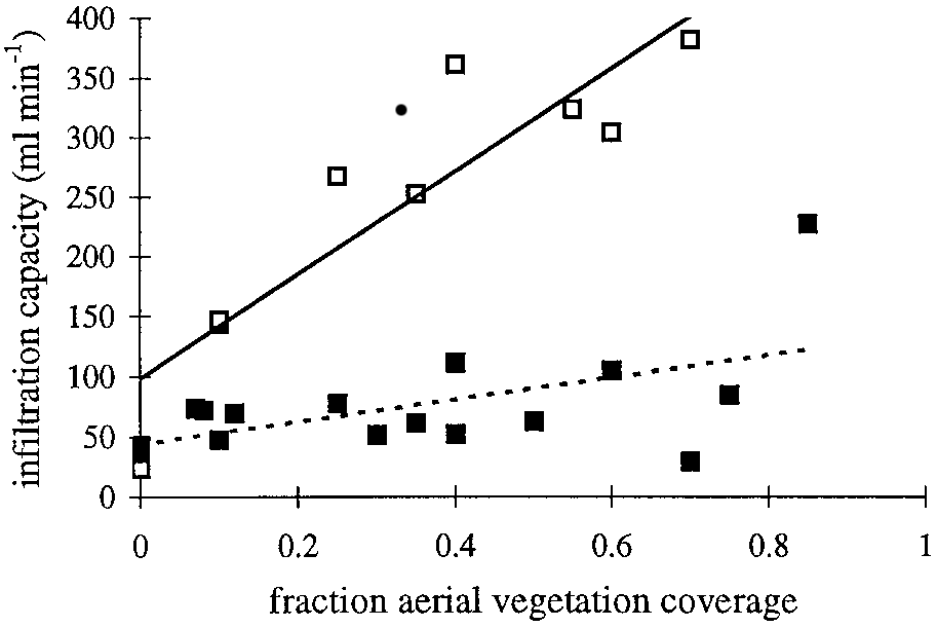

Water uptake=
Water density
$\times$ Plant density
$\times$ $\begin{pmatrix}\text{infiltration}\\ \text{rate}\end{pmatrix}$

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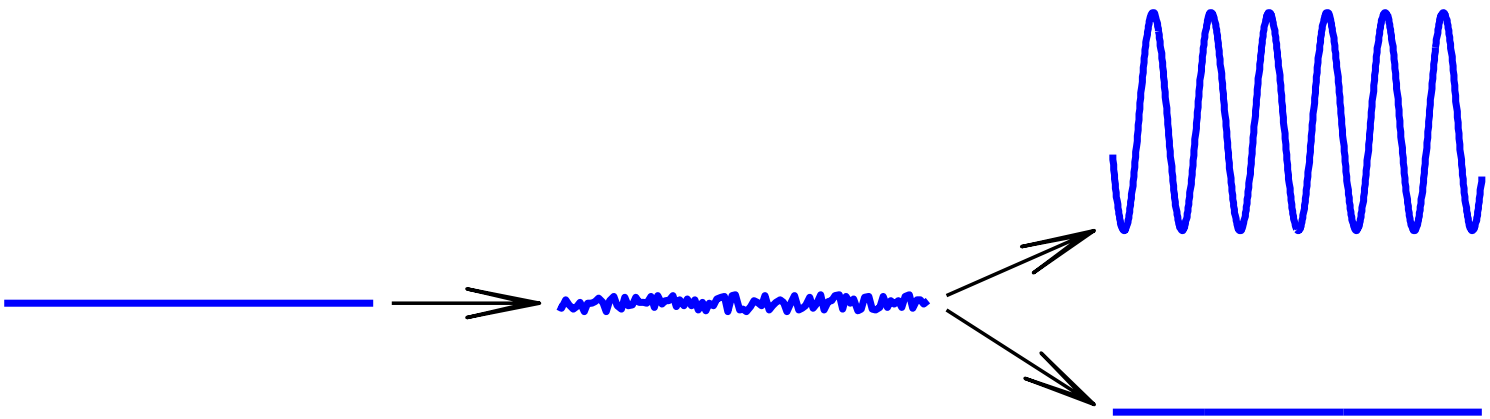
Predicting Pattern Wavelength: Textbook Approach

The standard approach to predicting pattern wavelength is to apply a small perturbation to a uniform equilibrium state.



The expected wavelength $\leftrightarrow$ the frequency of noise giving the fastest growth rate.

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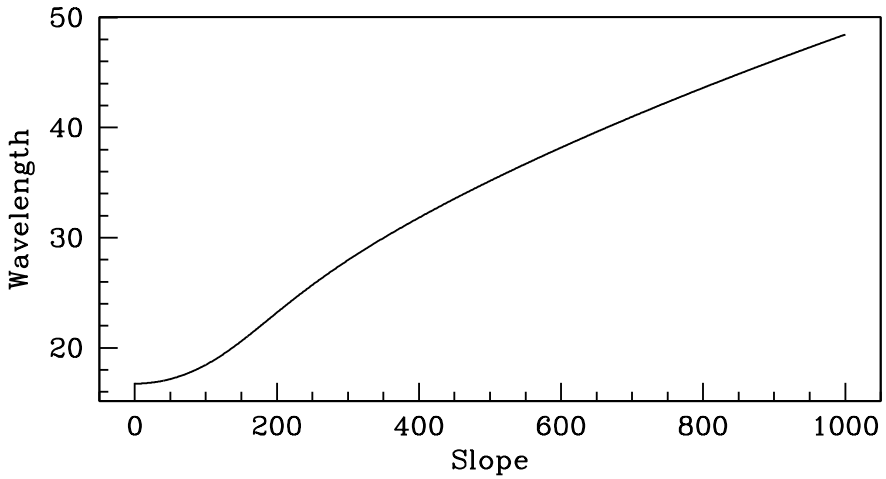


This implies a positive correlation between wavelength and slope, contrary to data.

The Origin of Vegetation Patterns

"Most unstable frequency" assumes that patterns develop from a pre-existing unstable uniform state.

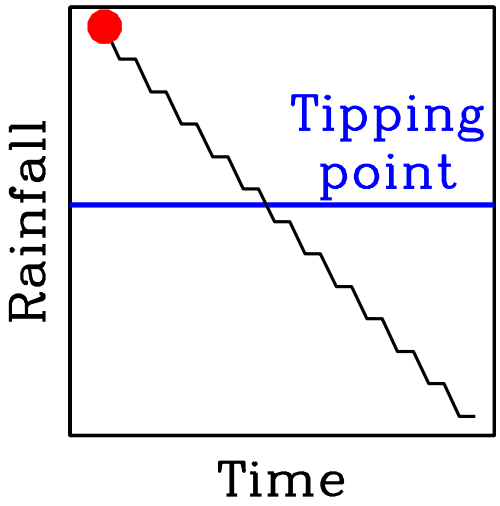
Vegetation patterns develop via

either degradation of uniform vegetation

or colonisation of bare ground

History-Dependent Patterns

At high rainfall levels, vegetation is uniform.
The transition to patterns is a “tipping point”



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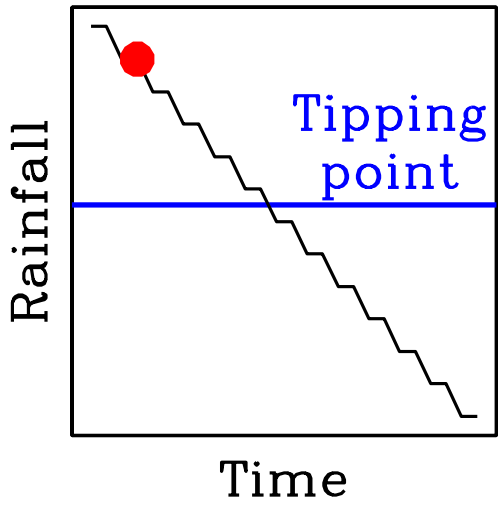

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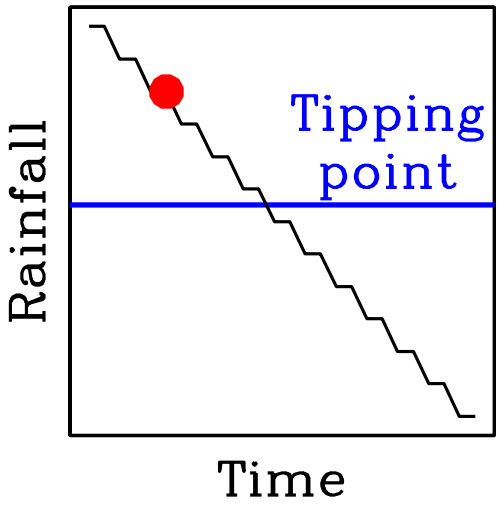

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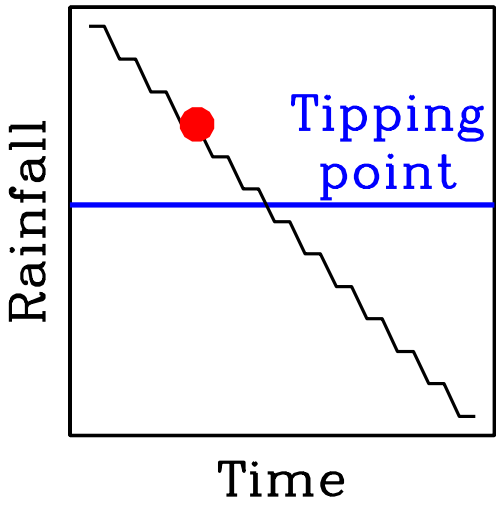

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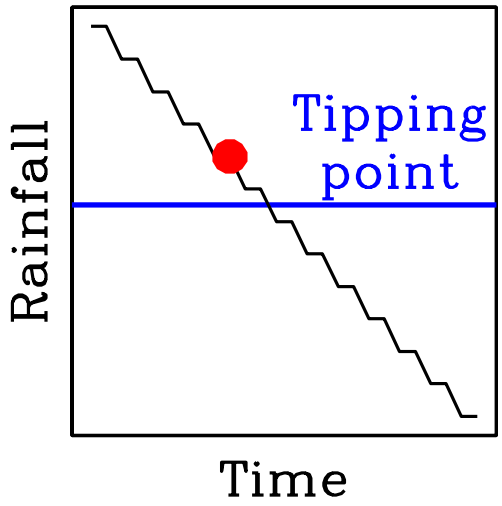

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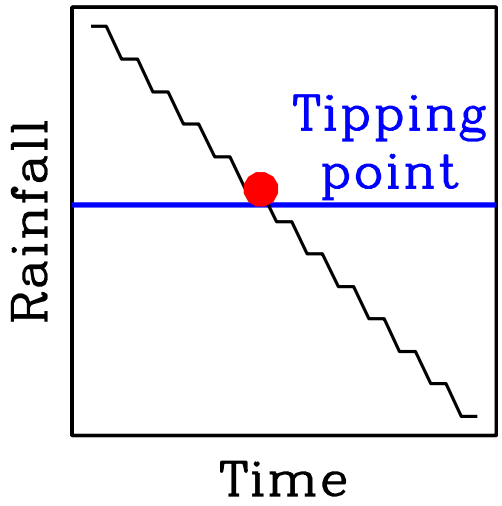

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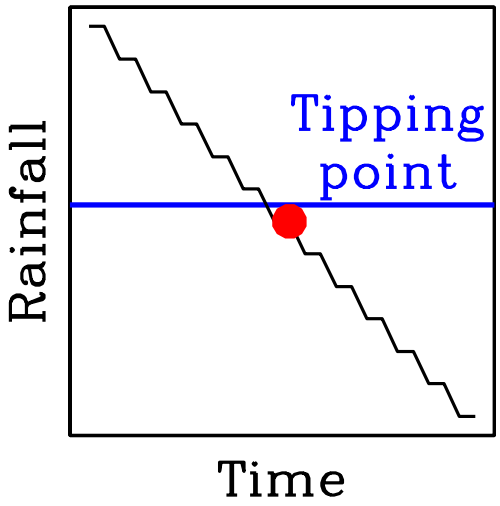

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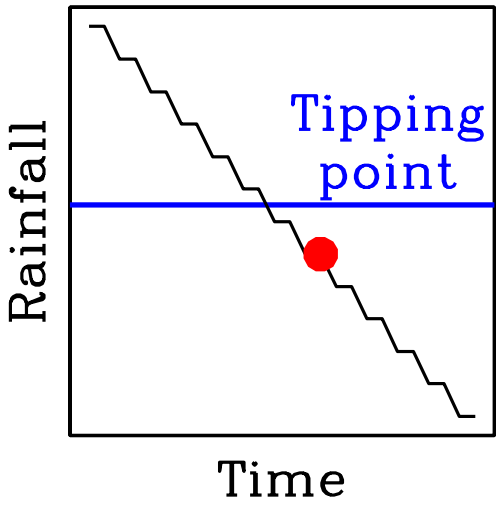

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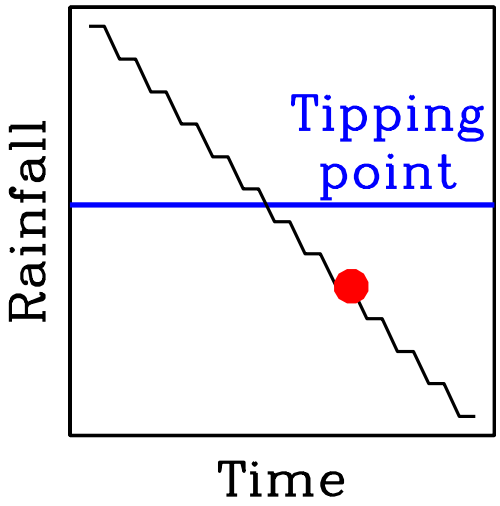

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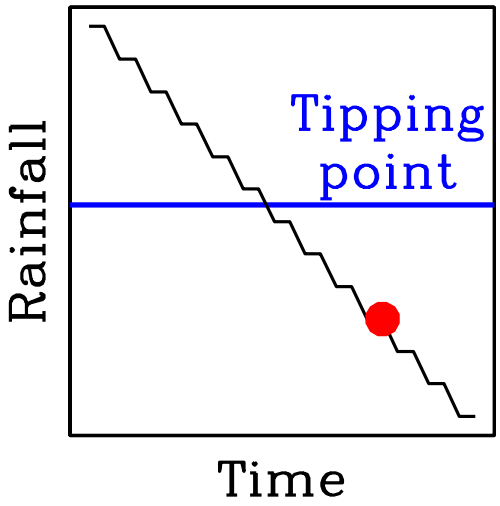

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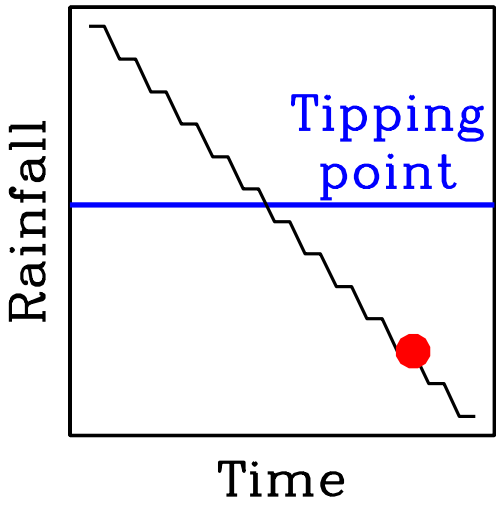

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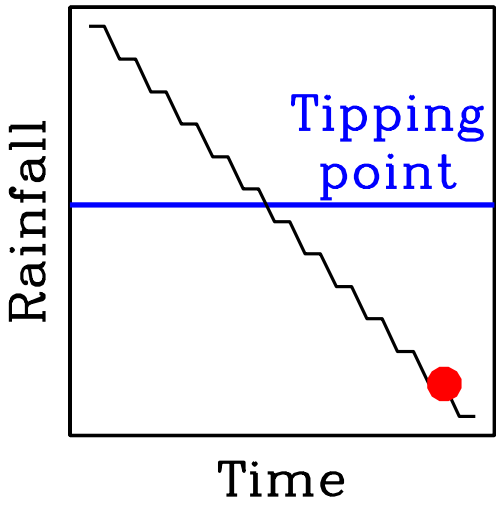

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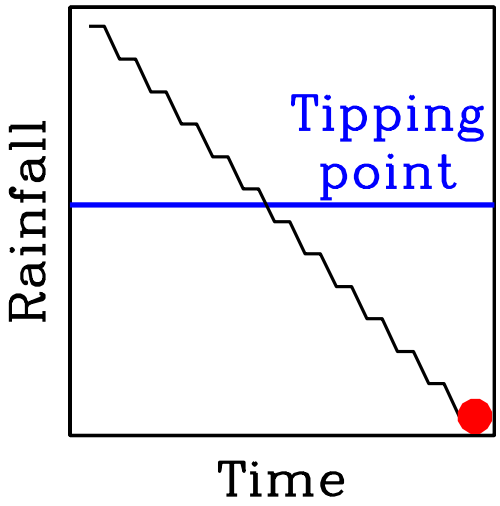

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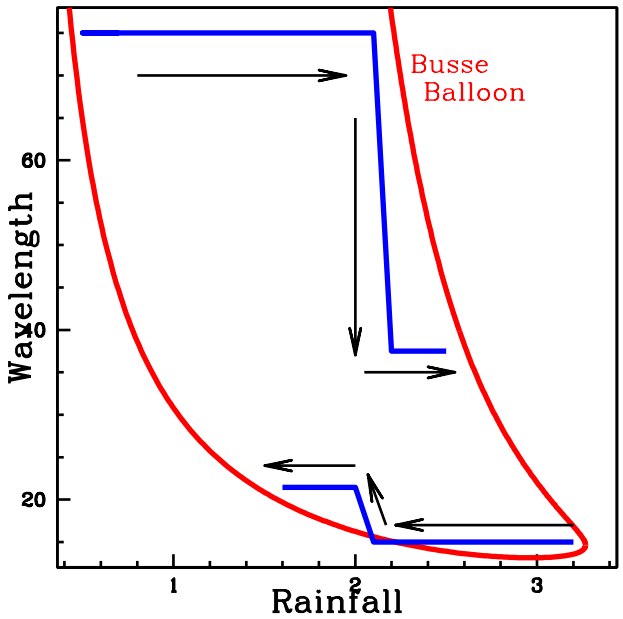
The Extent of History-Dependent Wavelength

Wavelength remains constant until an abrupt change at the edge of the “Busse Balloon”.



Outline

How to Predict Pattern Wavelength

Pattern wavelength

is history-dependent

$\Downarrow$

We must focus on the

onset of patterning

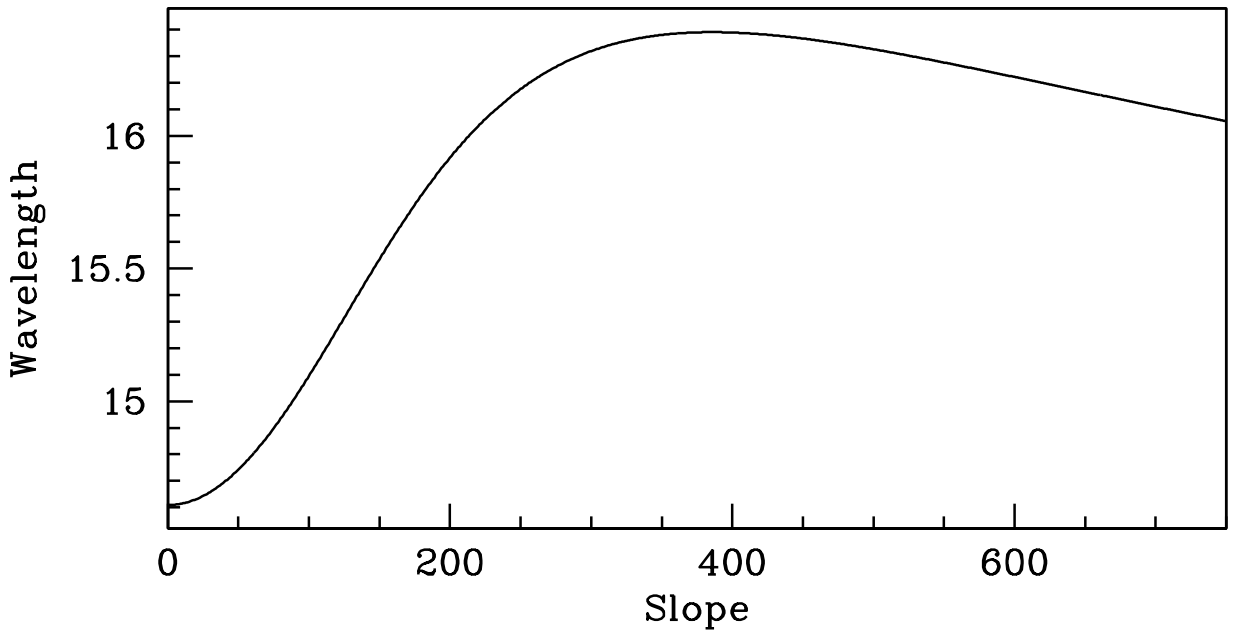
Degradation of

uniform vegetation

Colonisation

of bare ground

Wavelength vs Slope for Degradation of Uniform Vegetation



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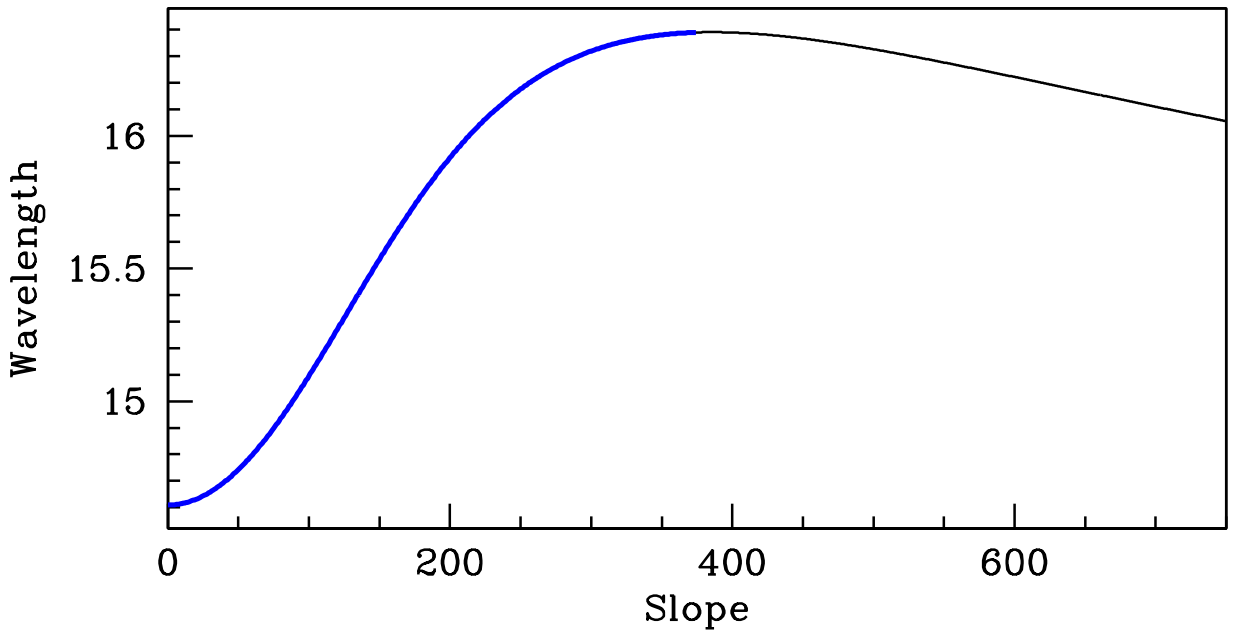


For realistic parameters, wavelength increases with slope, contrary to data

When Does Vegetation Colonise Bare Ground?

Downhill $\longleftrightarrow$ Uphill

Time ↓

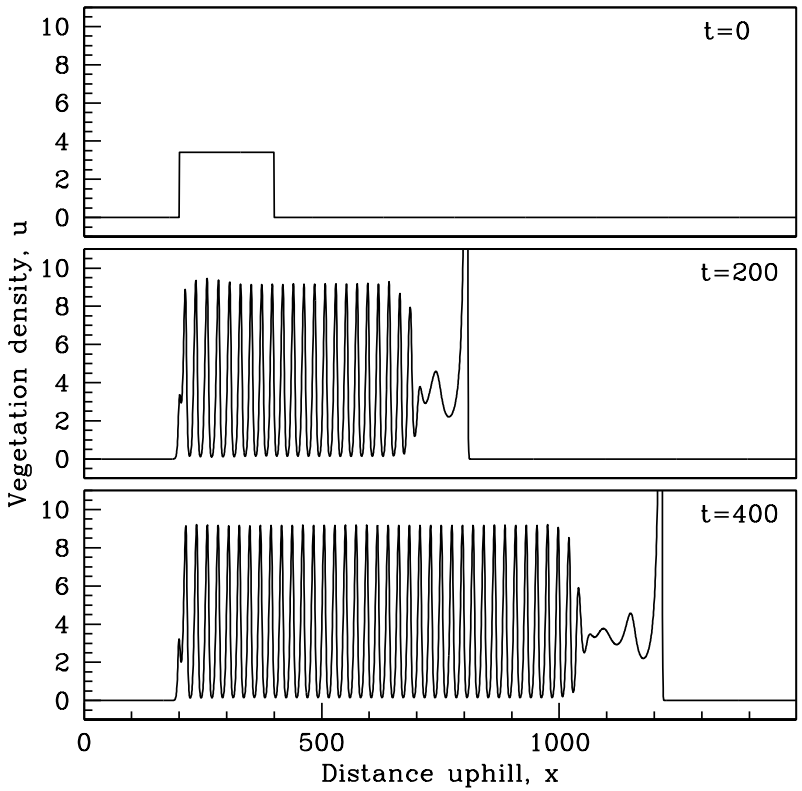
Very low rainfall: an isolated vegetation patch dies out

Slightly larger rainfall: both edges move uphill

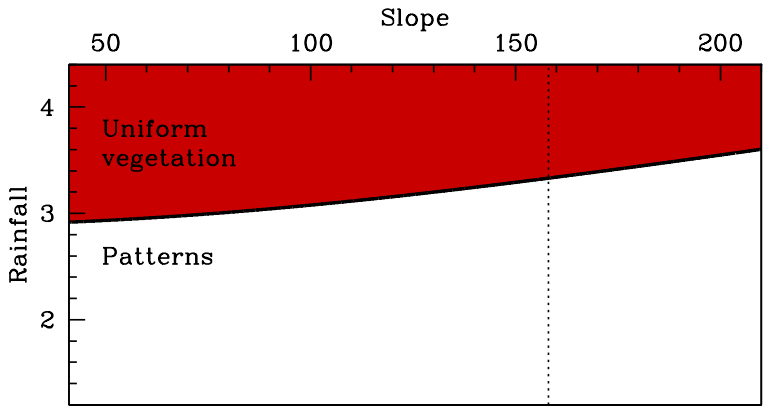
Larger rainfall: the patch expands both uphill and downhill

When Does Vegetation Colonise Bare Ground?

The key critical case is when the downhill edge is stationary



Wavelength vs Slope for Colonisation



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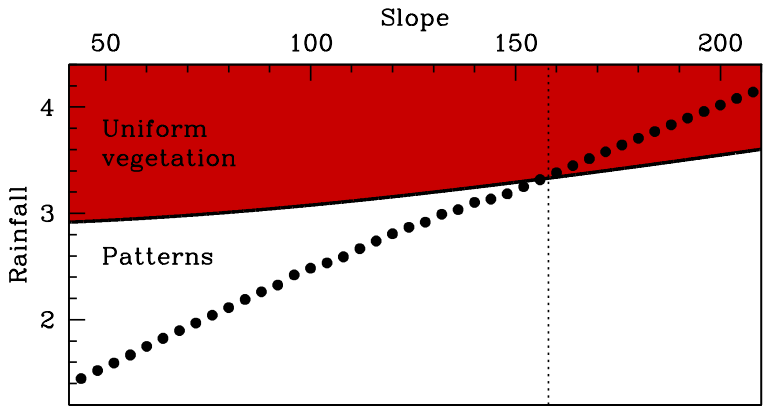

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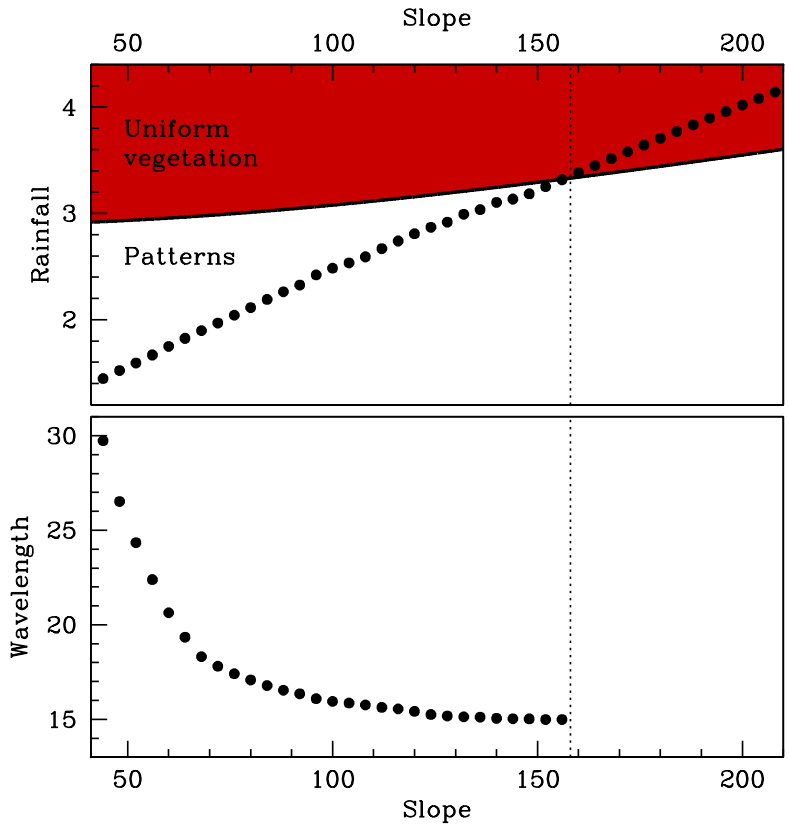

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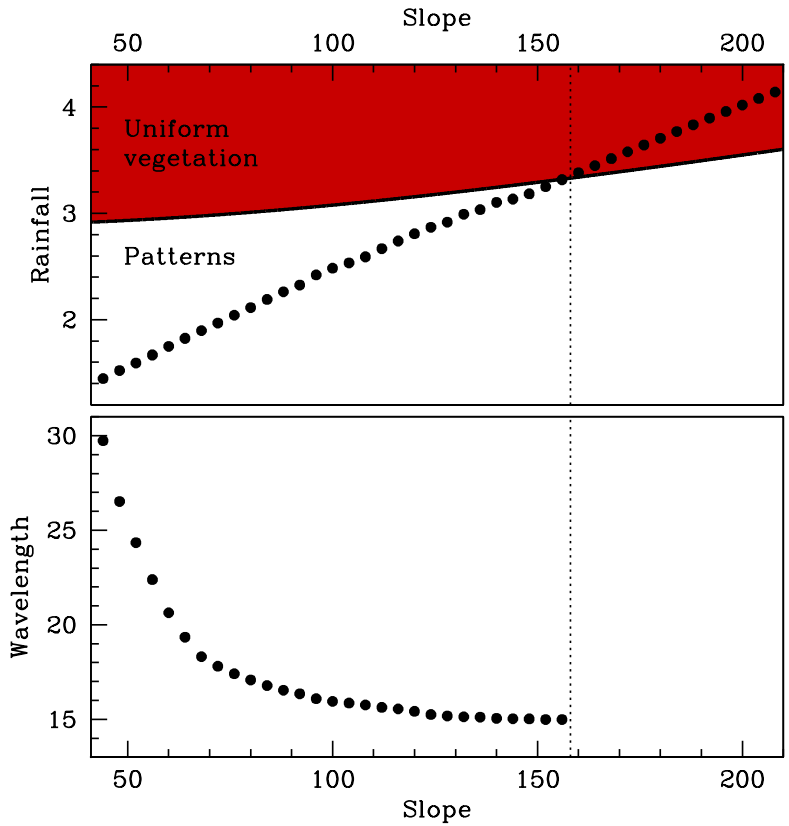


Wavelength decreases with slope, in agreement with data

Outline

Conclusions

Wavelength is positively correlated with slope $\Rightarrow$ vegetation pattern originated by degradation of uniform vegetation

Wavelength is negatively correlated with slope $\Rightarrow$ vegetation pattern originated by colonisation of bare ground

References

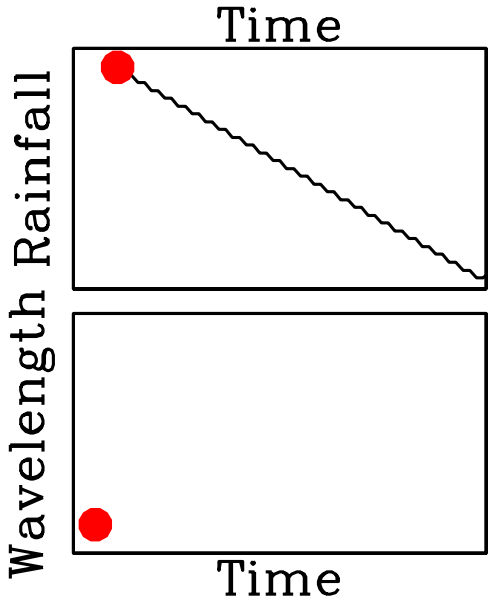
J.A. Sherratt: History-dependent patterns of whole ecosystems. *Ecological Complexity* 14, 8-20 (2013).

A.S. Dagbovie, J.A. Sherratt: Pattern selection and hysteresis in the Rietkerk model for banded vegetation in semi-arid environments. *J. R. Soc. Interface* 11: 20140465 (2014).

List of Frames

History-Dependent Patterns

Model prediction: as rainfall is varied within the range giving patterns, abrupt changes in pattern wavelength occur.



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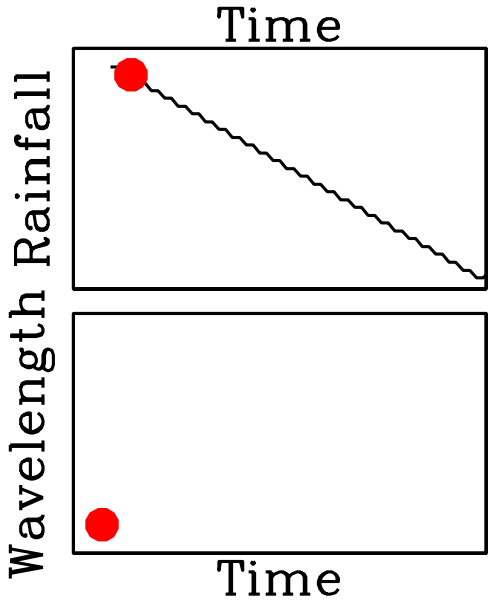

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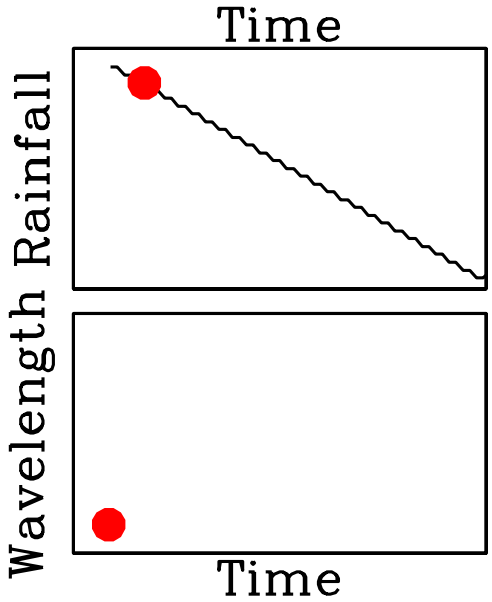

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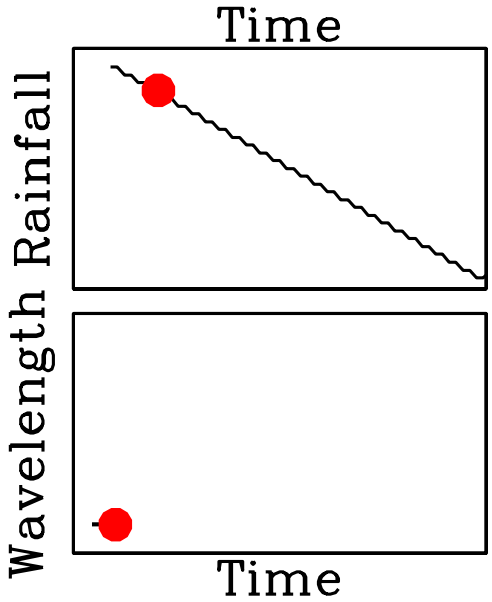

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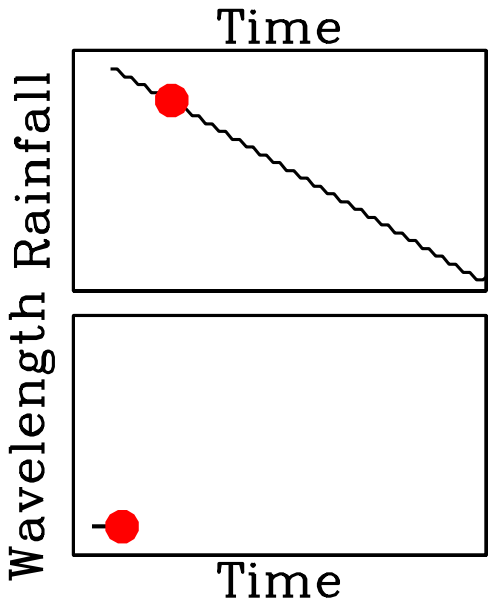

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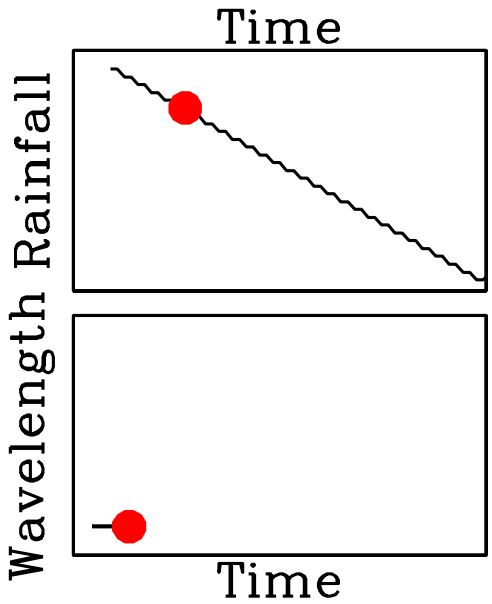

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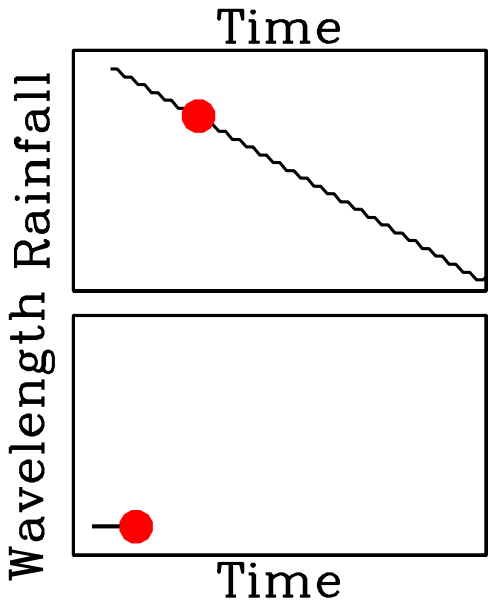

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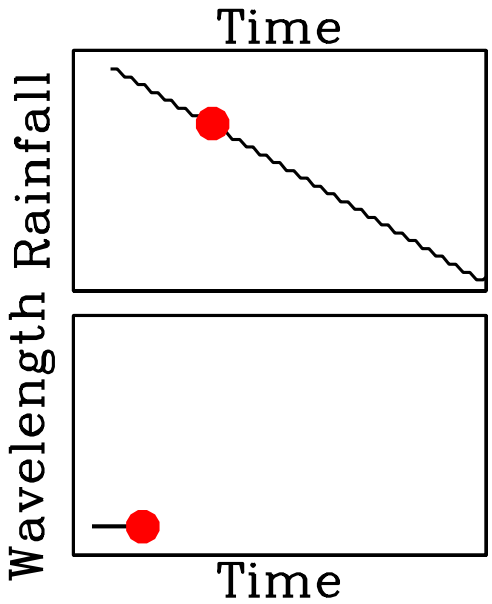

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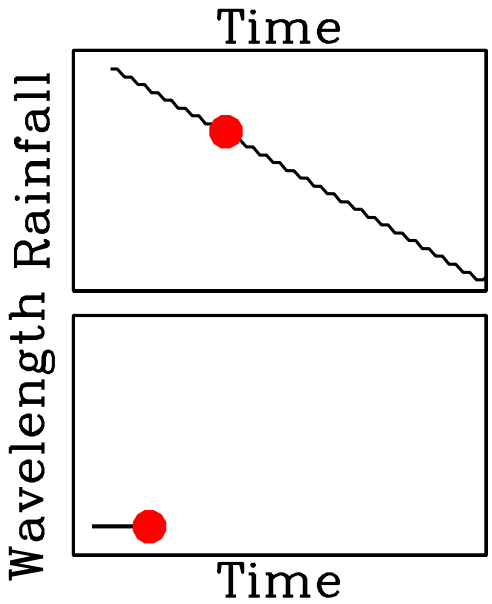

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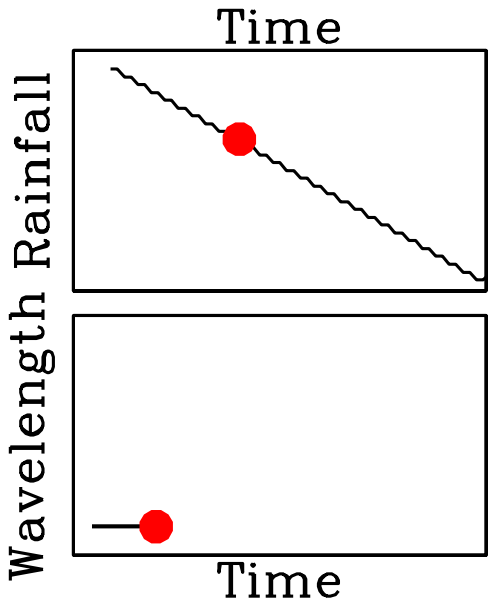

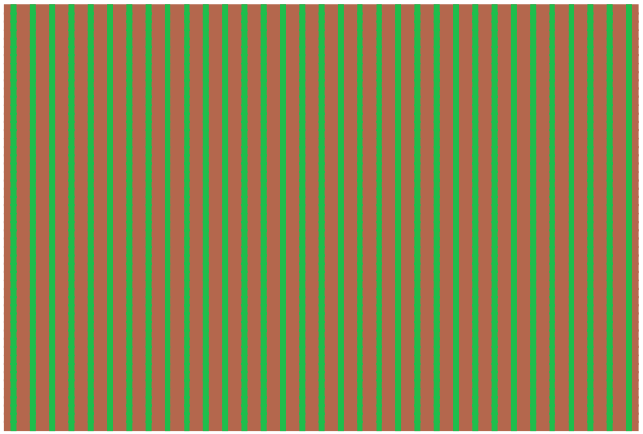

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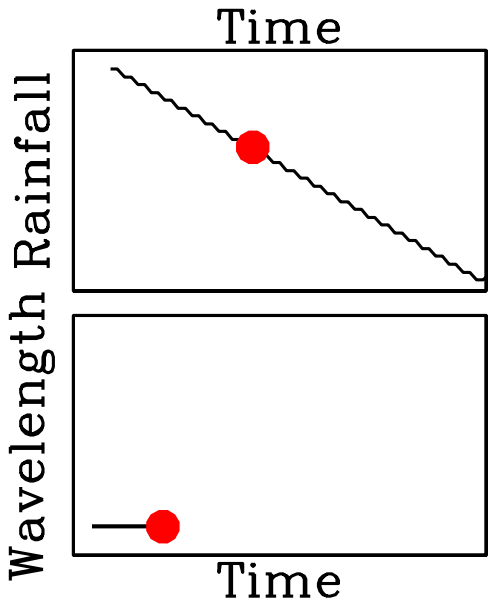

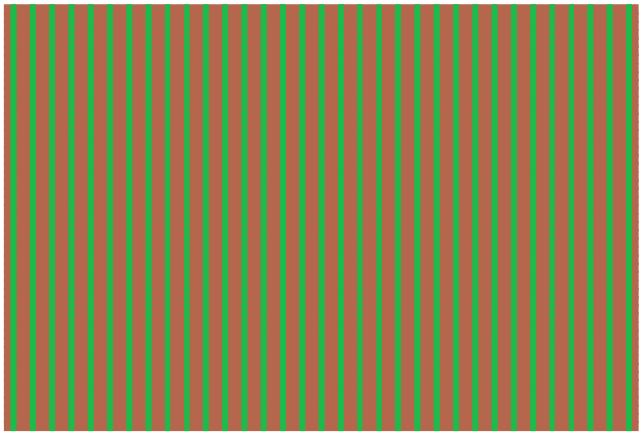

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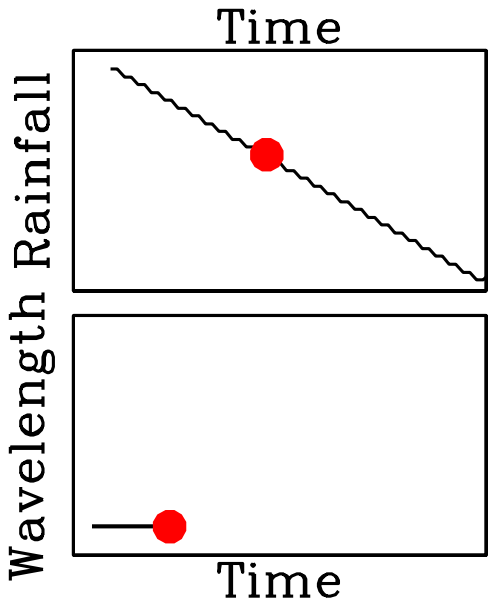

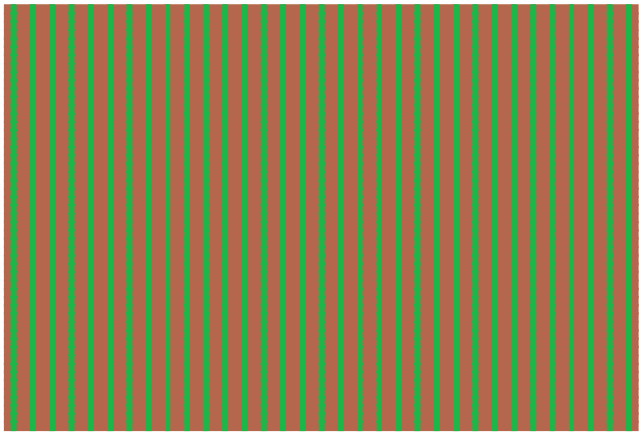

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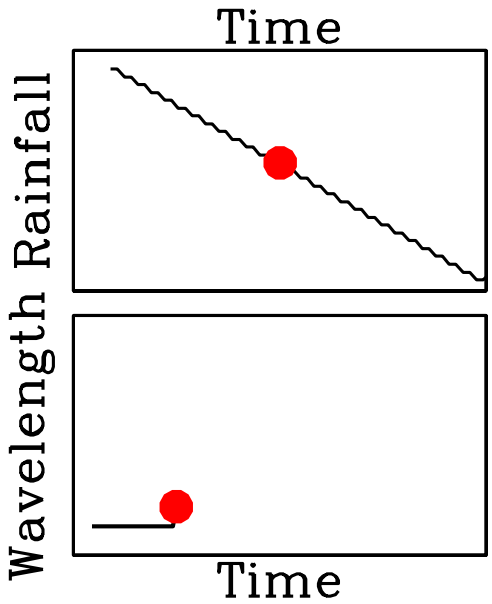

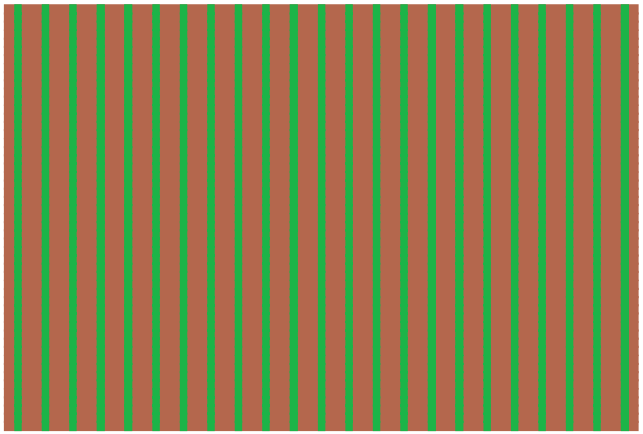

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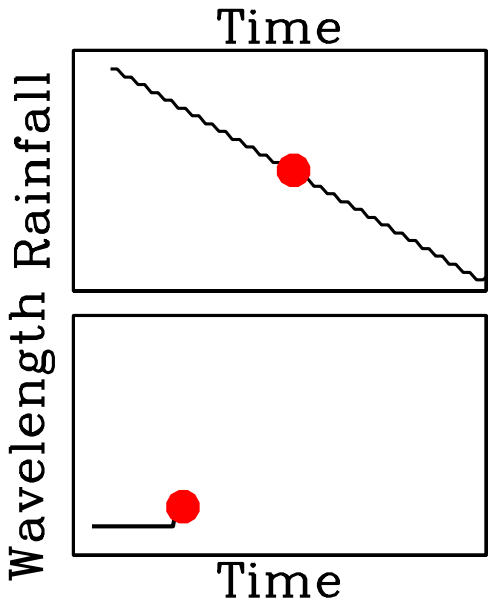

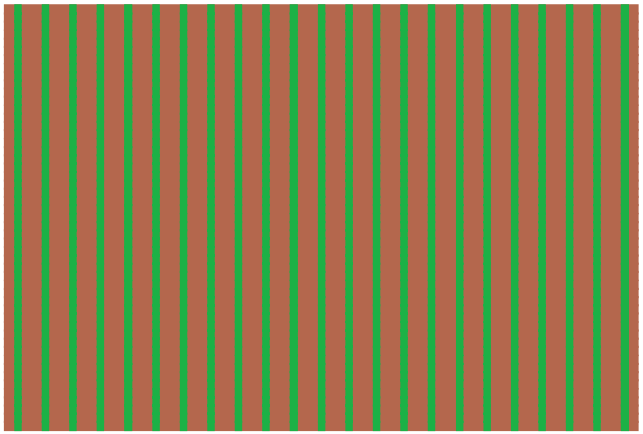

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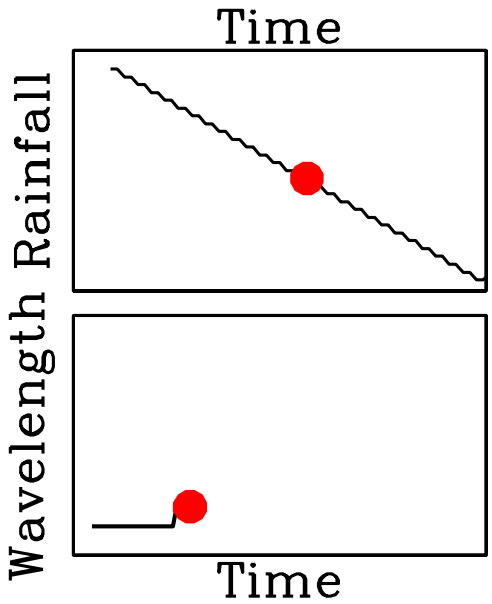

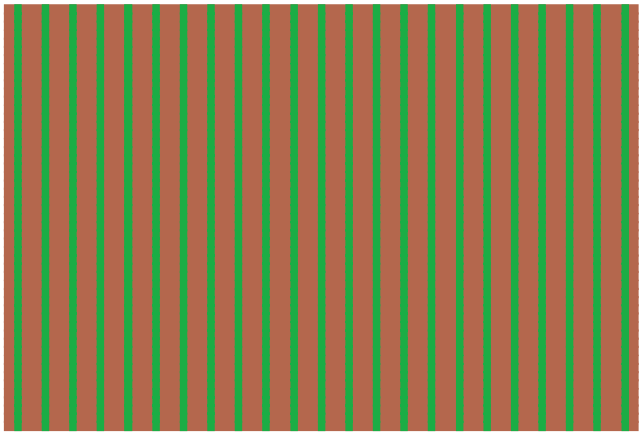

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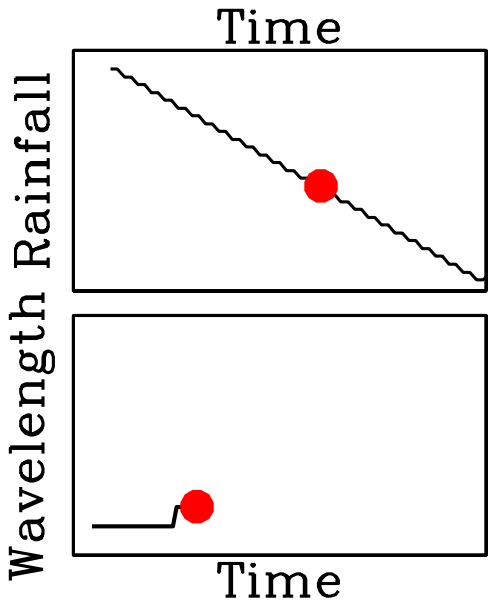

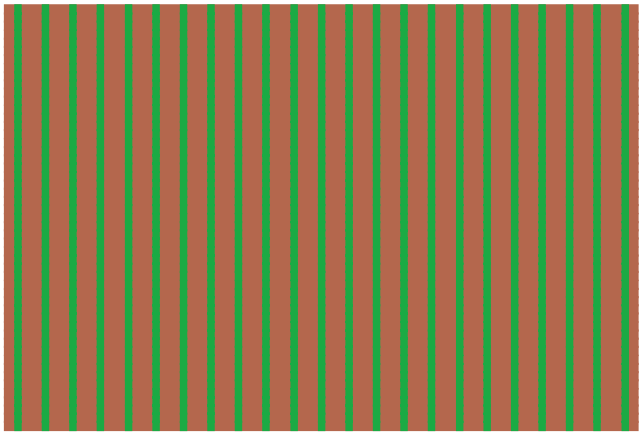

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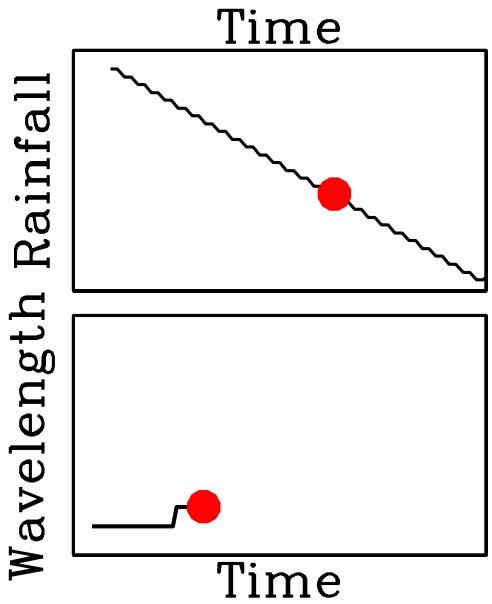

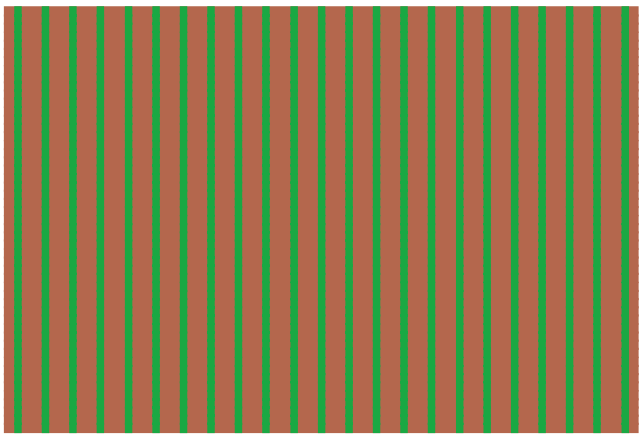

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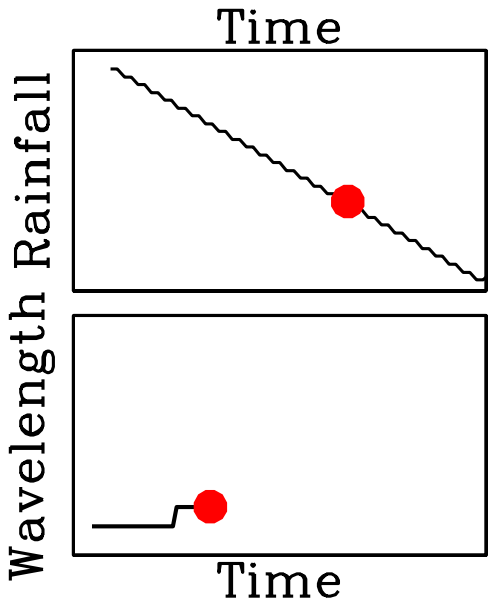

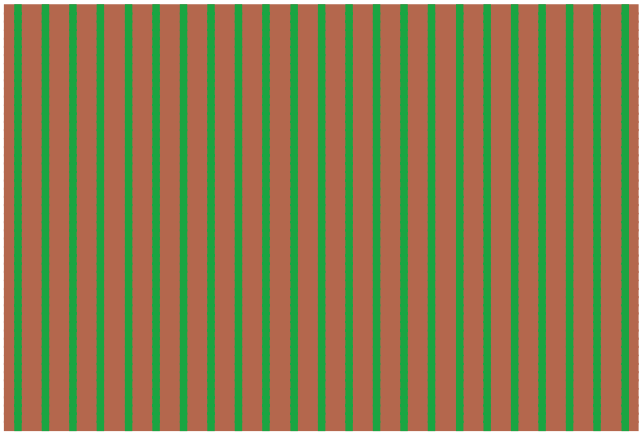

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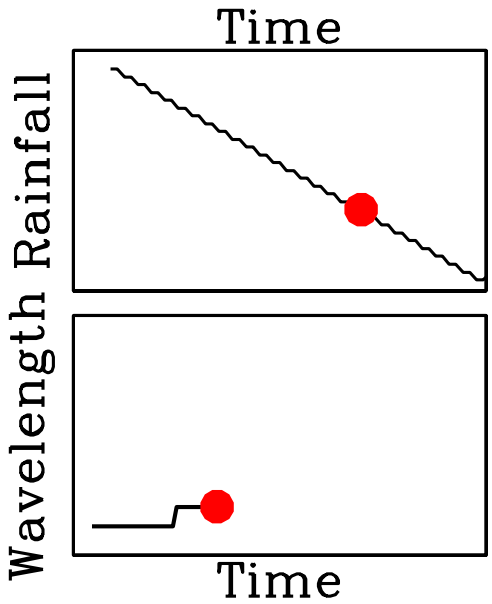

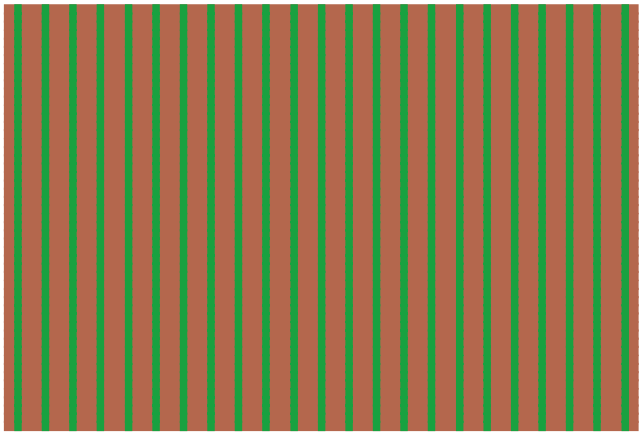

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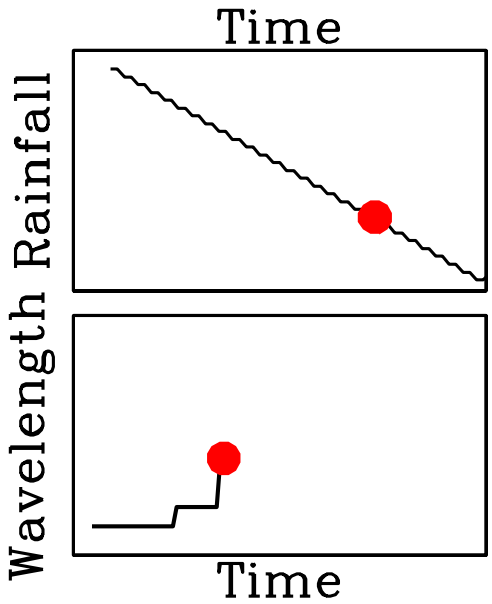

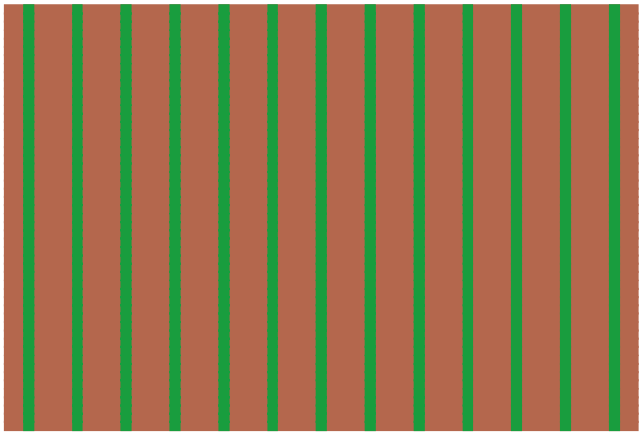

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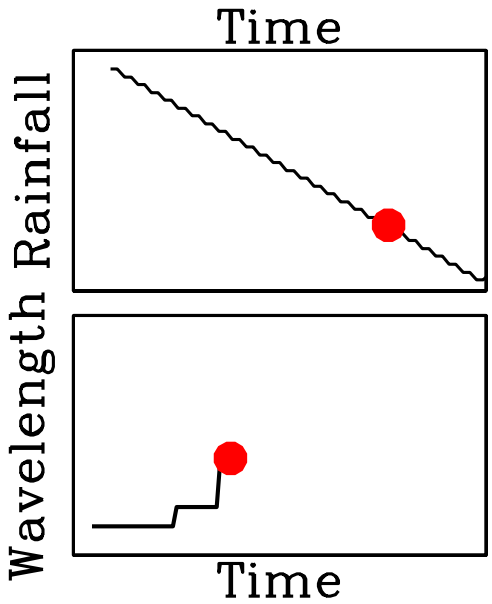

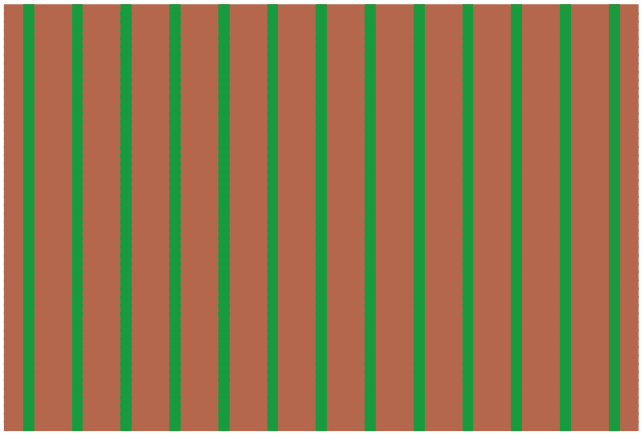

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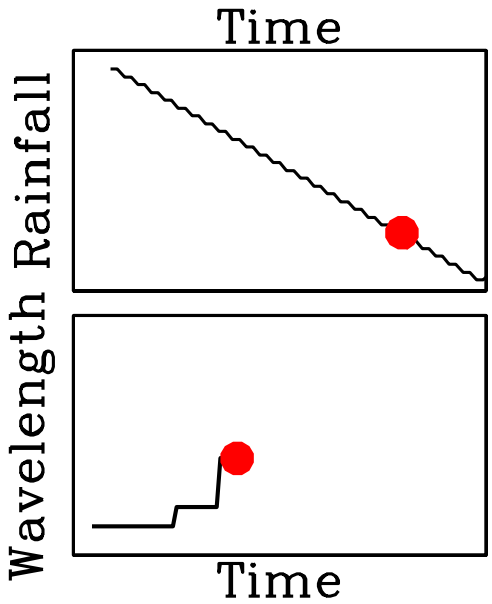

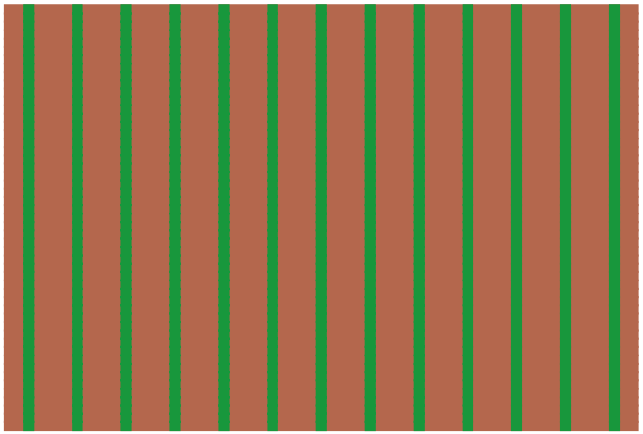

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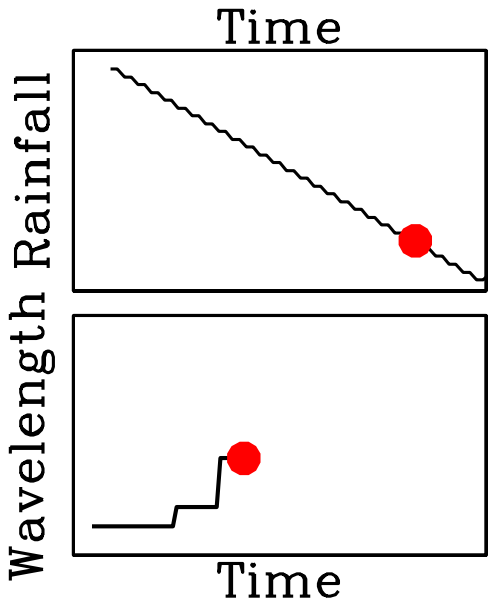

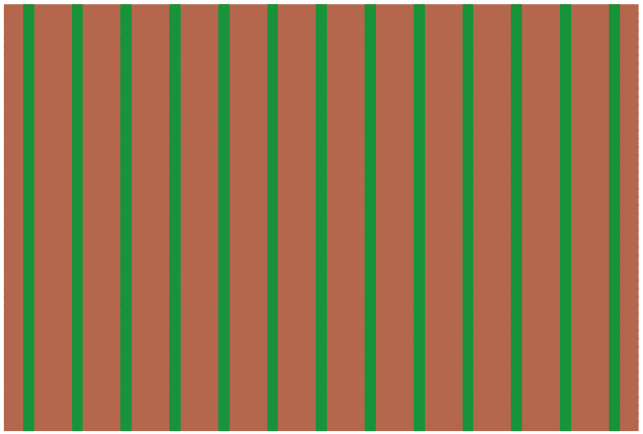

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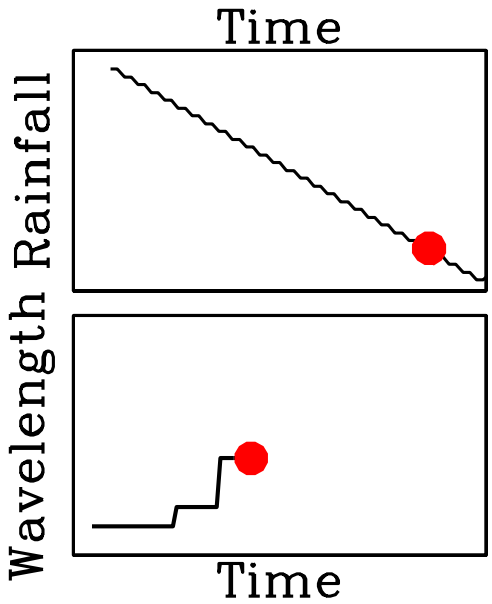

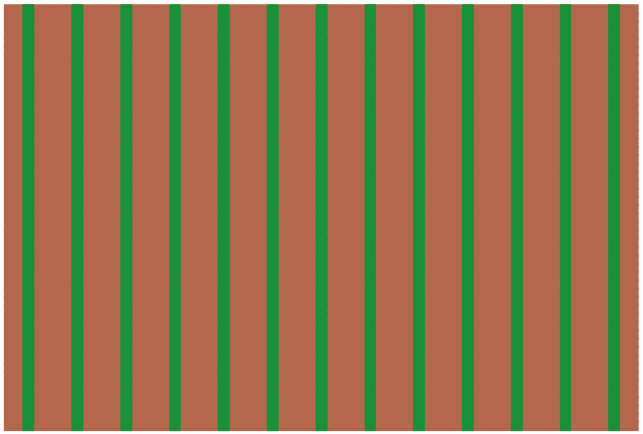

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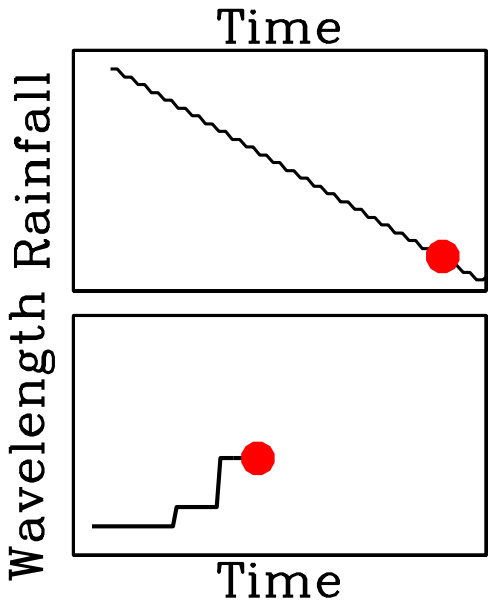

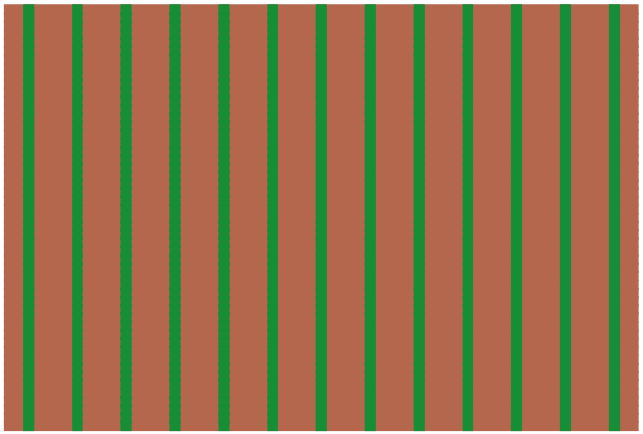

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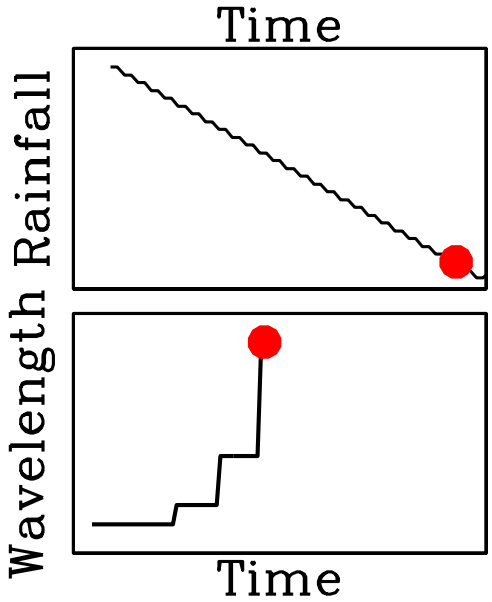

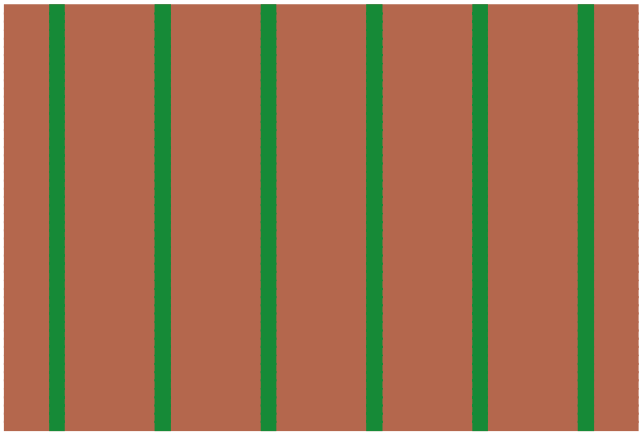

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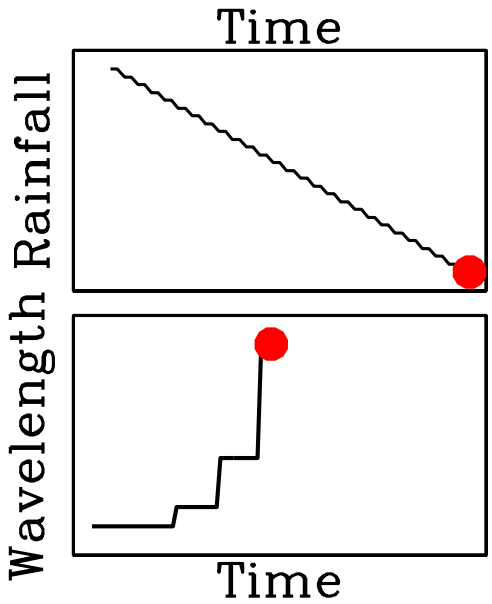

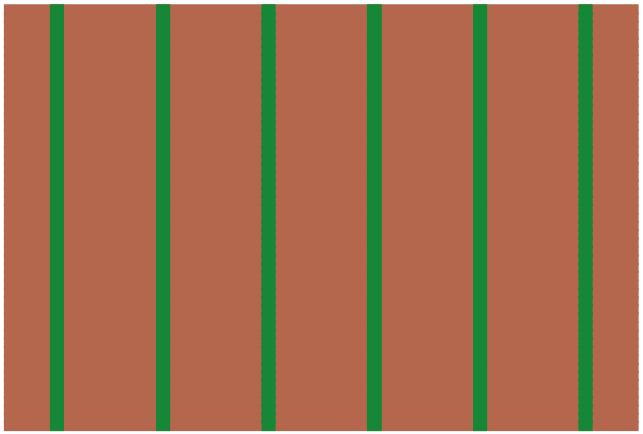

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