

Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns in Semi-Deserts

Jonathan A. Sherratt

Department of Mathematics
and Maxwell Institute for Mathematical Sciences
Heriot-Watt University

Universiteit Leiden, 8 February 2016

This talk can be downloaded from my web site

www.ma.hw.ac.uk/~jas

Outline

- 1 Ecological Background
- 2 Pattern Formation in a Mathematical Model
- 3 Pattern Existence and Stability
- 4 Predictions of Pattern Wavelength vs Slope
- 5 Conclusions and References

Outline

- 1 Ecological Background
- 2 Pattern Formation in a Mathematical Model
- 3 Pattern Existence and Stability
- 4 Predictions of Pattern Wavelength vs Slope
- 5 Conclusions and References

Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.

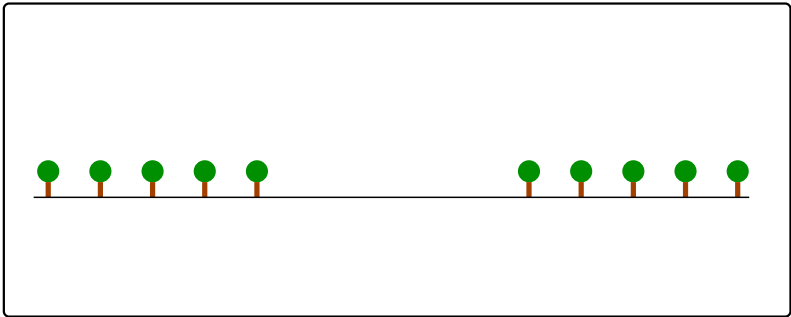


W National Park, Niger

Average patch width is 50 m

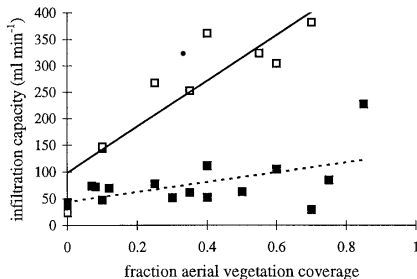
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



Data from Burkina Faso

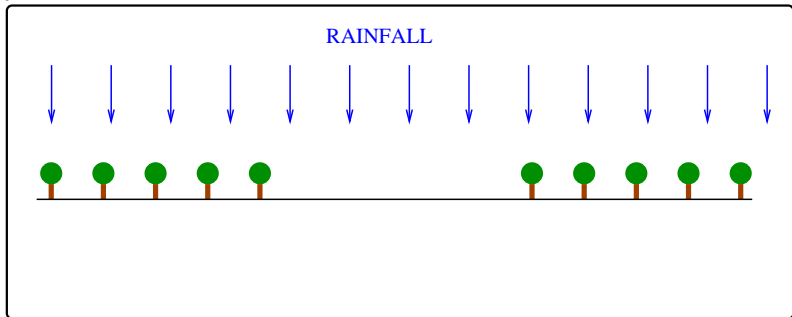
Rietkerk et al

Plant Ecology 148: 207-224, 2000

More plants \Rightarrow more roots and organic matter in soil
 \Rightarrow more infiltration of rainwater

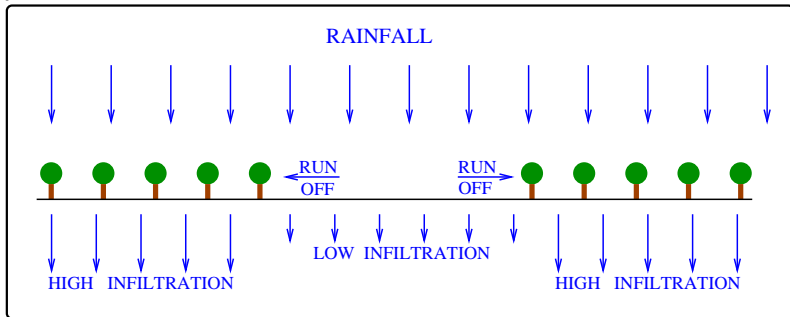
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



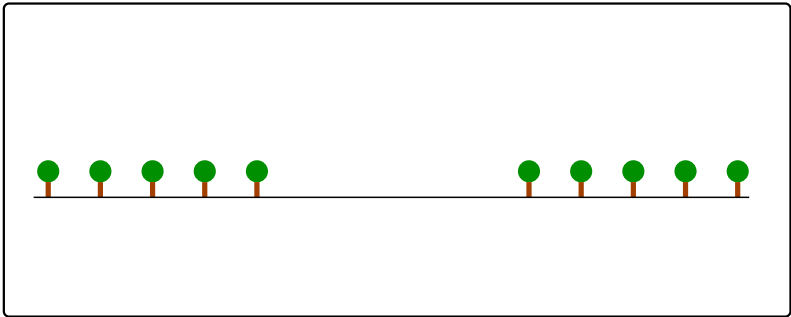
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



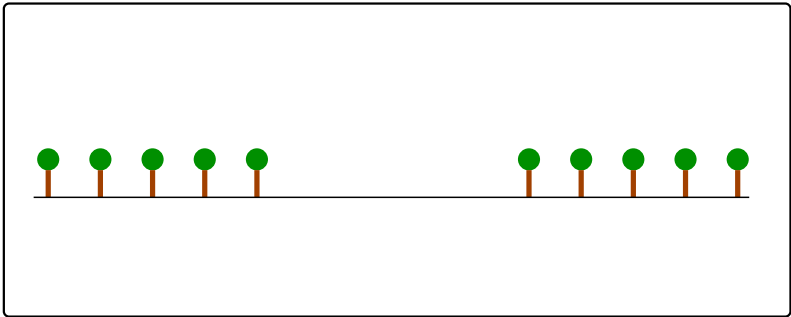
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



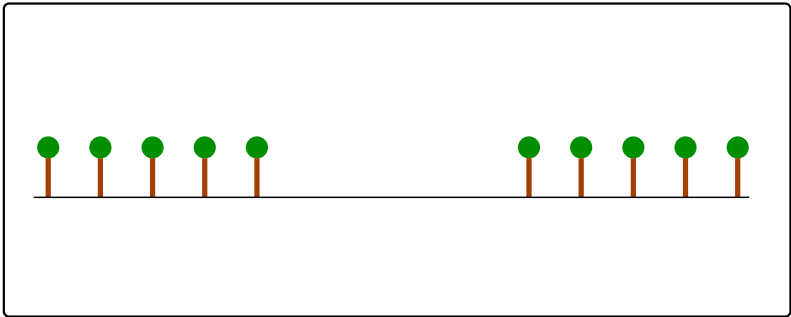
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



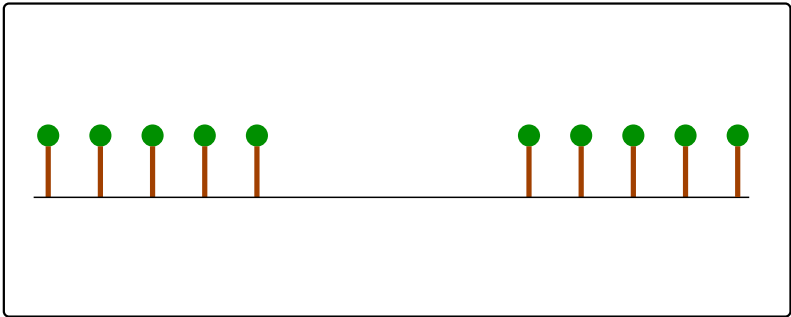
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



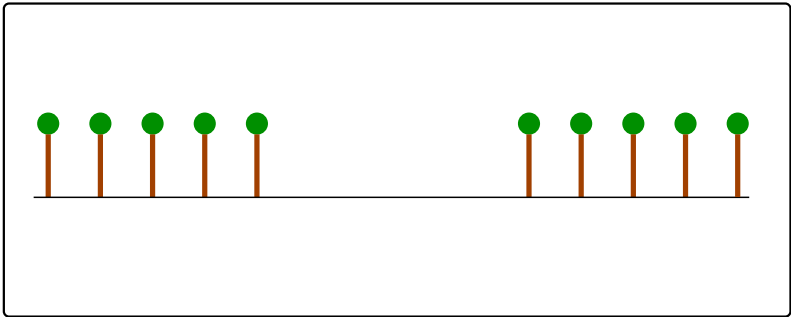
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



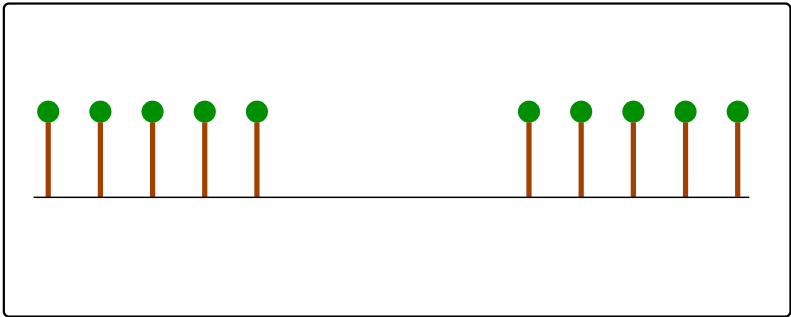
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



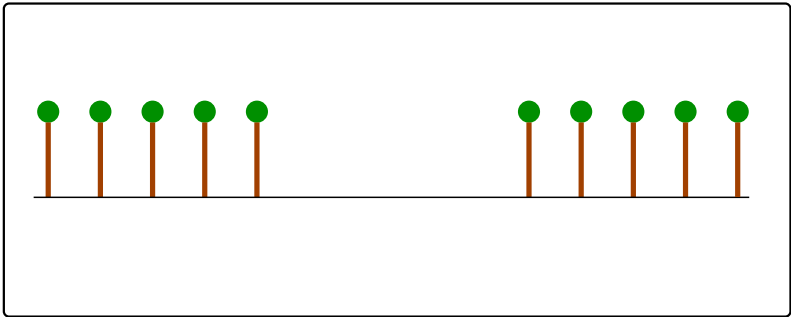
Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.



Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.

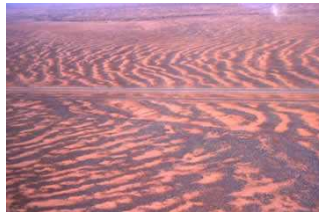


Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.



Bushy vegetation in Niger



Mitchell grass in Australia
(Western New South Wales)

Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia, Mexico and S-W USA.

Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.



Bushy vegetation in Niger

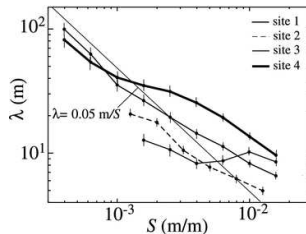


Mitchell grass in Australia
(Western New South Wales)

Wavelength can be measured via remote sensing.

Data on Wavelength vs Slope

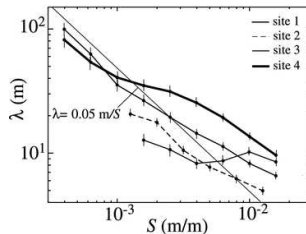
Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.



Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

Data on Wavelength vs Slope

Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.



Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

How does this compare with predictions of mathematical models?

Outline

- 1 Ecological Background
- 2 **Pattern Formation in a Mathematical Model**
- 3 Pattern Existence and Stability
- 4 Predictions of Pattern Wavelength vs Slope
- 5 Conclusions and References

Mathematical Model of Klausmeier

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= \overbrace{wu^2}^{\text{plant growth}} - \overbrace{Bu}^{\text{plant loss}} + \overbrace{\partial^2 u / \partial x^2}^{\text{plant dispersal}} \\
 \frac{\partial w}{\partial t} &= \underbrace{A}_{\text{average rainfall}} - \underbrace{w}_{\text{evaporation \& drainage}} - \underbrace{wu^2}_{\text{uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{flow downhill}} + \underbrace{D \frac{\partial^2 w}{\partial x^2}}_{\text{diffusion of water}}
 \end{aligned}$$

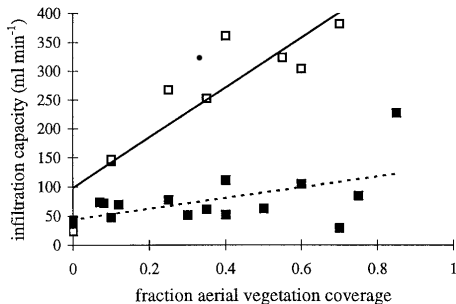
(Klausmeier, Science 284: 1826-8, 1999)

Mathematical Model of Klausmeier

$$\begin{aligned}
 \partial u / \partial t &= \underbrace{wu^2}_{\text{plant growth}} - \underbrace{Bu}_{\text{plant loss}} + \underbrace{\partial^2 u / \partial x^2}_{\text{plant dispersal}} \\
 \partial w / \partial t &= \underbrace{A}_{\text{average rainfall}} - \underbrace{w}_{\text{evaporation \& drainage}} - \underbrace{wu^2}_{\text{uptake by plants}} + \underbrace{\nu \partial w / \partial x}_{\text{flow downhill}} + \underbrace{D \partial^2 w / \partial x^2}_{\text{diffusion of water}}
 \end{aligned}$$

The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.

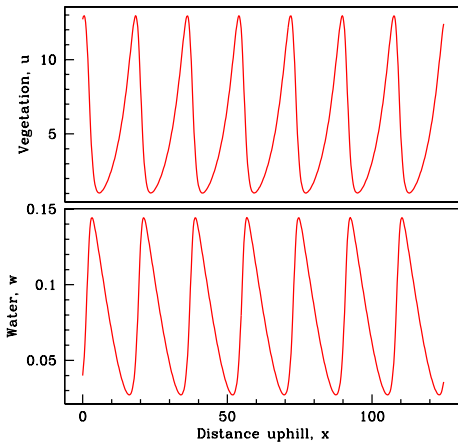
Mathematical Model of Klausmeier



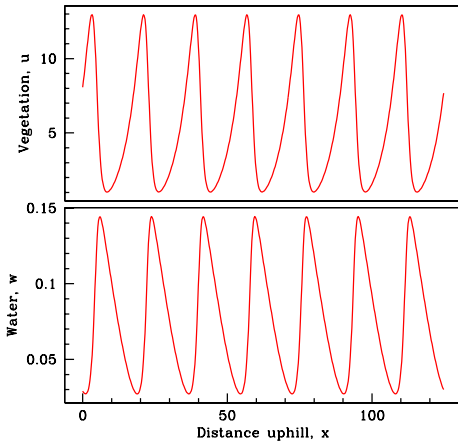
$$\begin{aligned} \text{Water uptake} = & \\ & \text{Water density} \\ & \times \text{Plant density} \\ & \times \left(\text{infiltration rate} \right) \end{aligned}$$

The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.

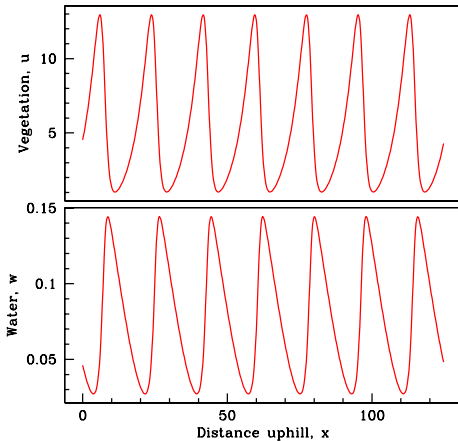
Typical Solution of the Model



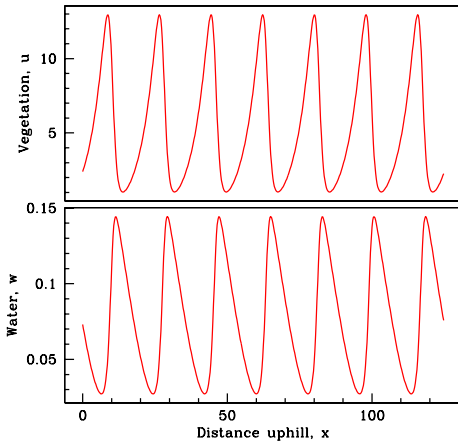
Typical Solution of the Model



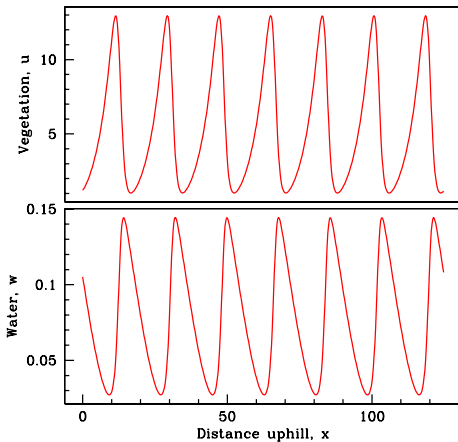
Typical Solution of the Model



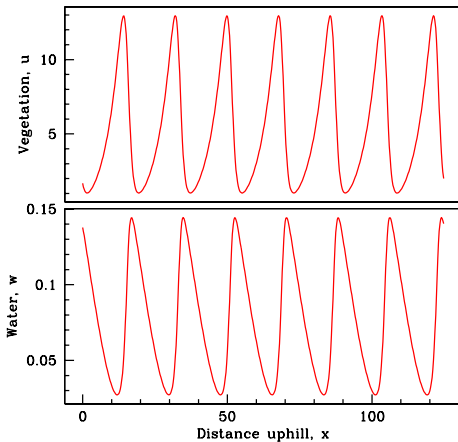
Typical Solution of the Model



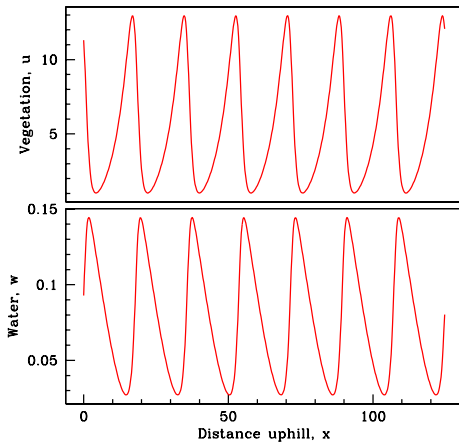
Typical Solution of the Model



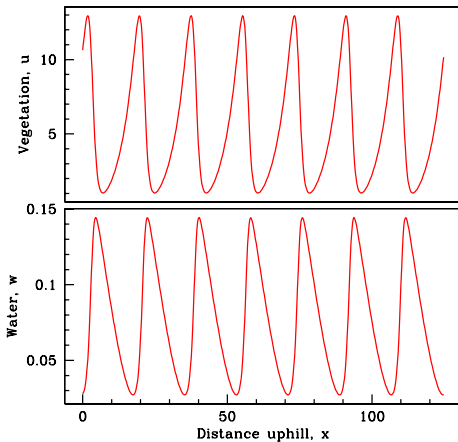
Typical Solution of the Model



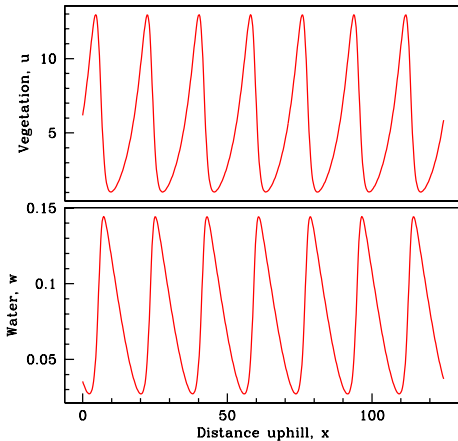
Typical Solution of the Model



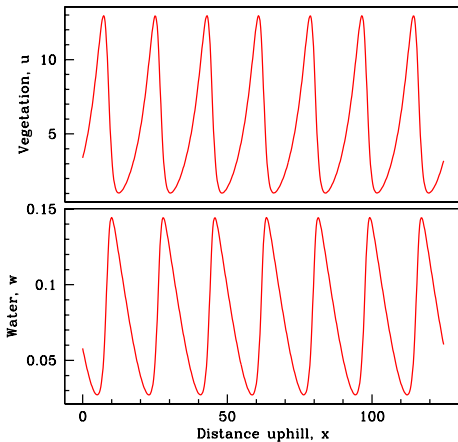
Typical Solution of the Model



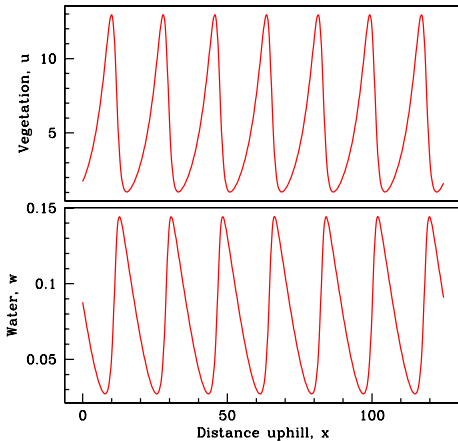
Typical Solution of the Model



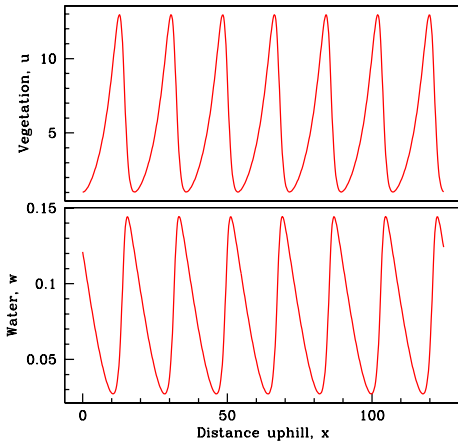
Typical Solution of the Model



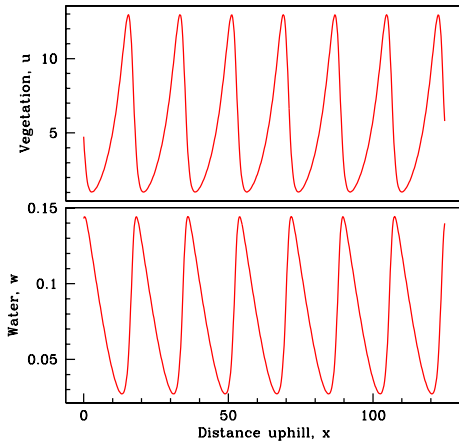
Typical Solution of the Model



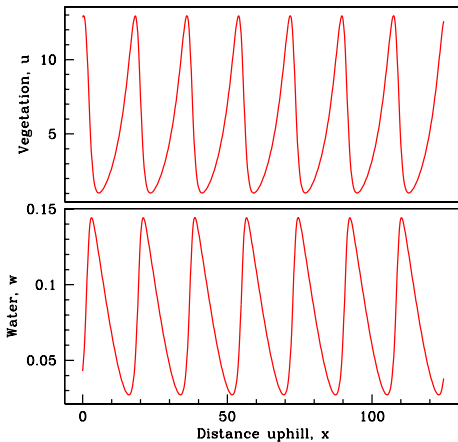
Typical Solution of the Model



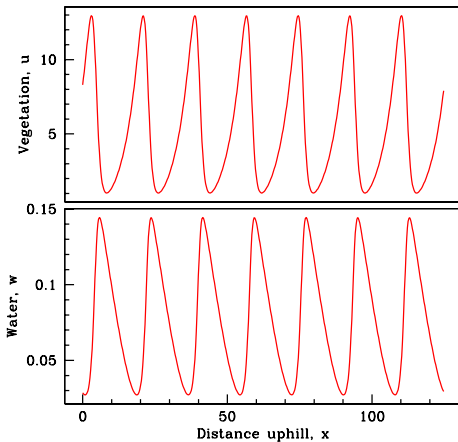
Typical Solution of the Model



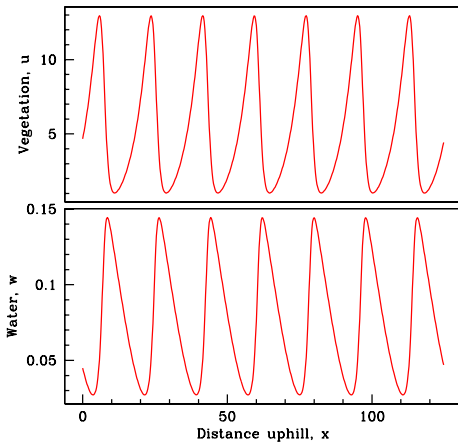
Typical Solution of the Model



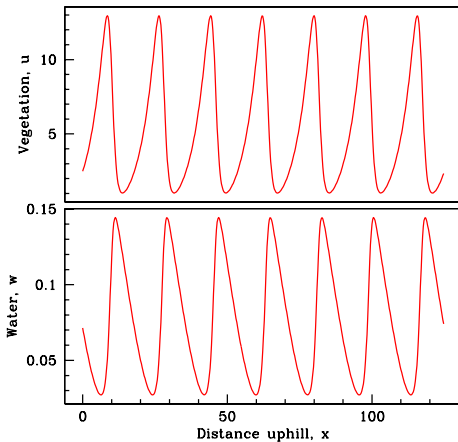
Typical Solution of the Model



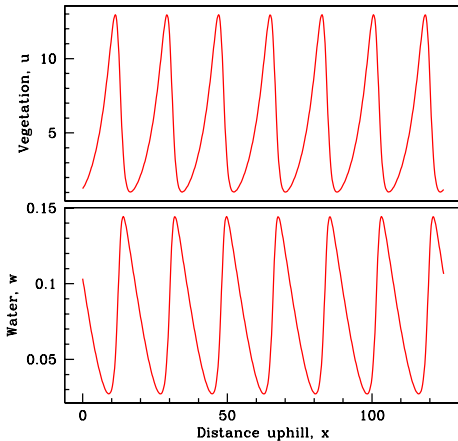
Typical Solution of the Model



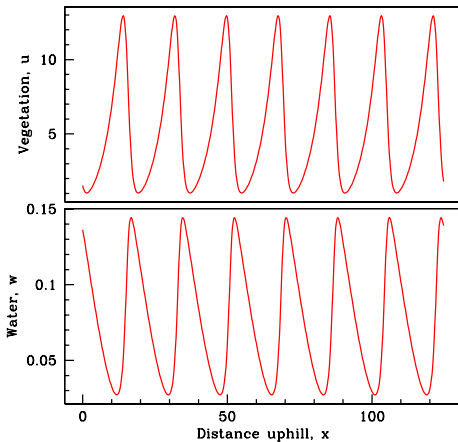
Typical Solution of the Model



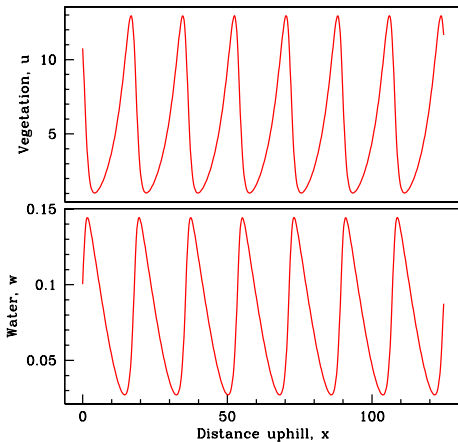
Typical Solution of the Model



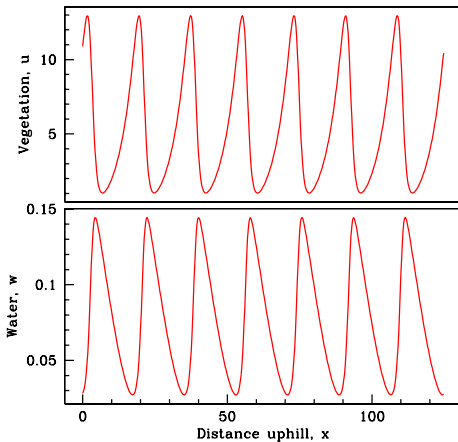
Typical Solution of the Model



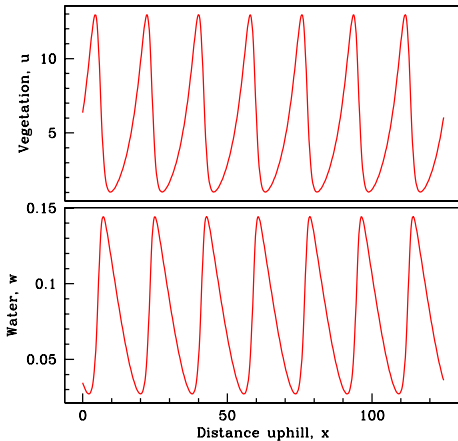
Typical Solution of the Model



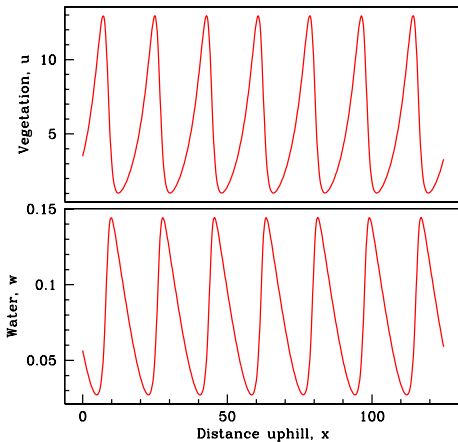
Typical Solution of the Model



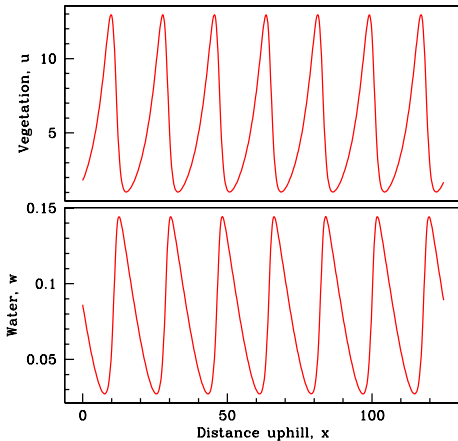
Typical Solution of the Model



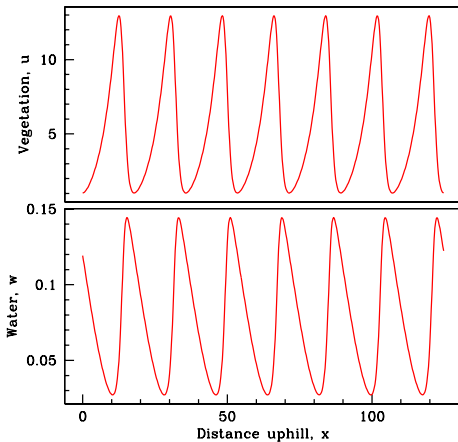
Typical Solution of the Model



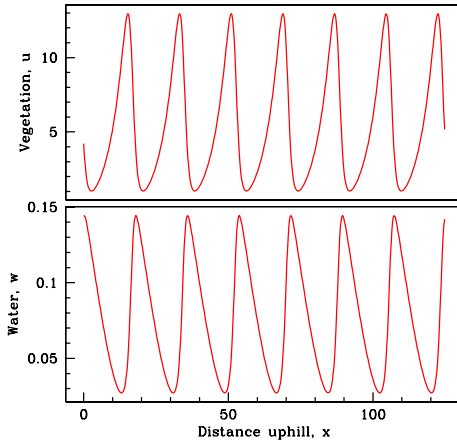
Typical Solution of the Model



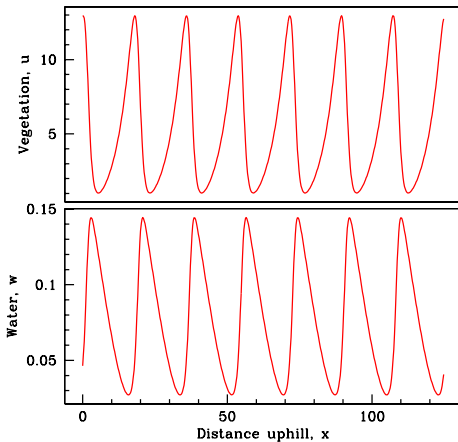
Typical Solution of the Model



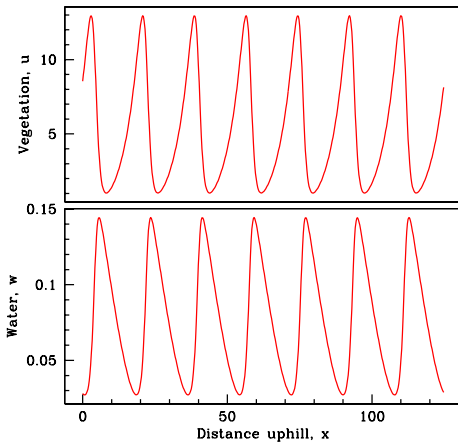
Typical Solution of the Model



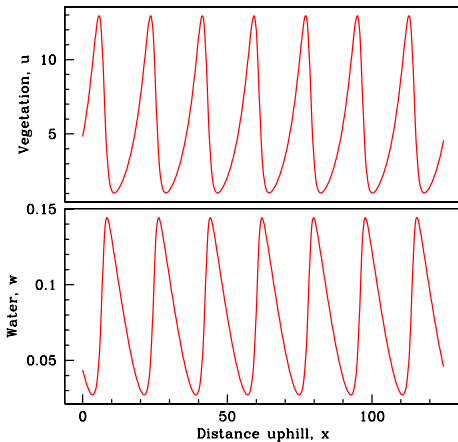
Typical Solution of the Model



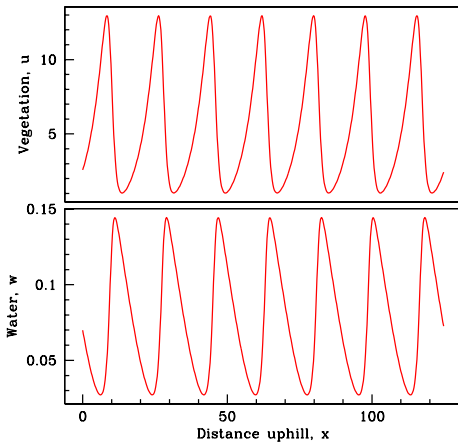
Typical Solution of the Model



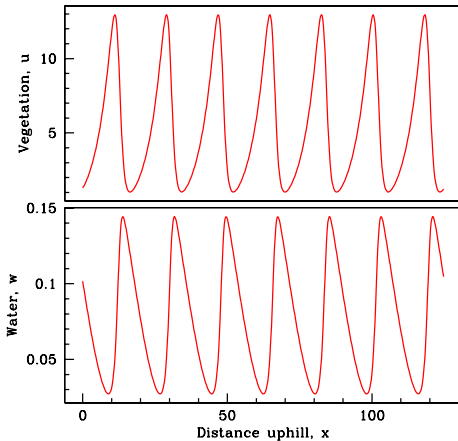
Typical Solution of the Model



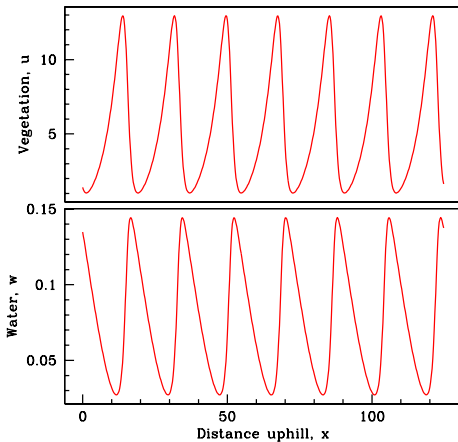
Typical Solution of the Model



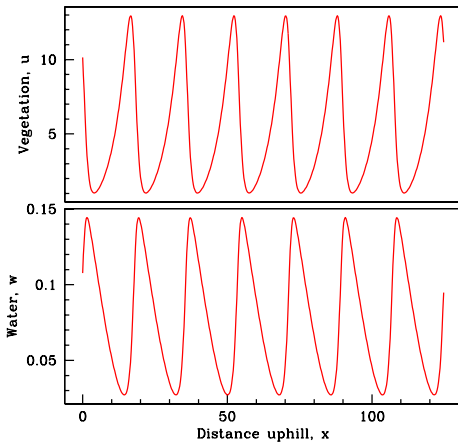
Typical Solution of the Model



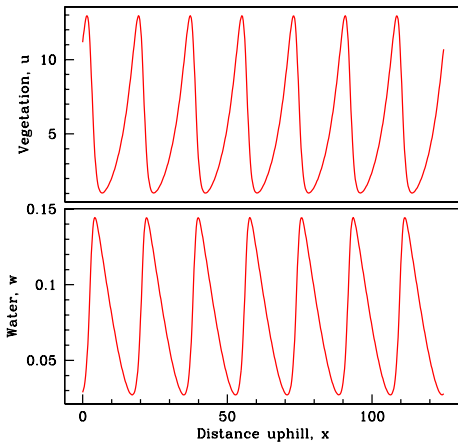
Typical Solution of the Model



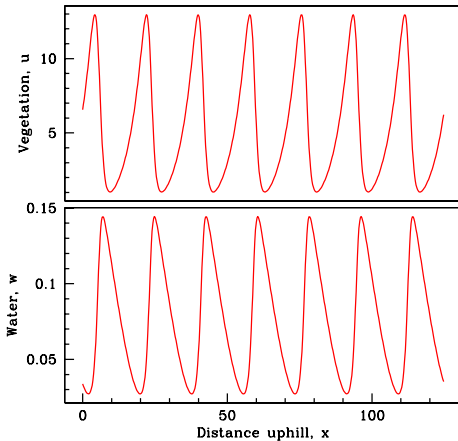
Typical Solution of the Model



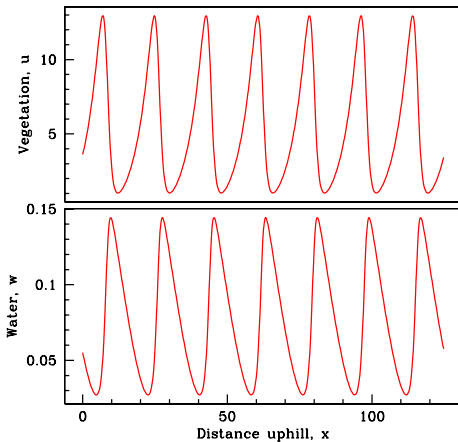
Typical Solution of the Model



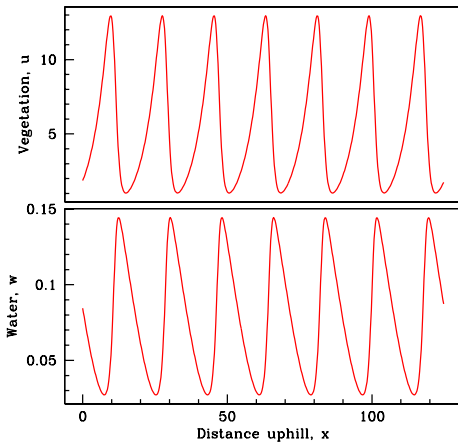
Typical Solution of the Model



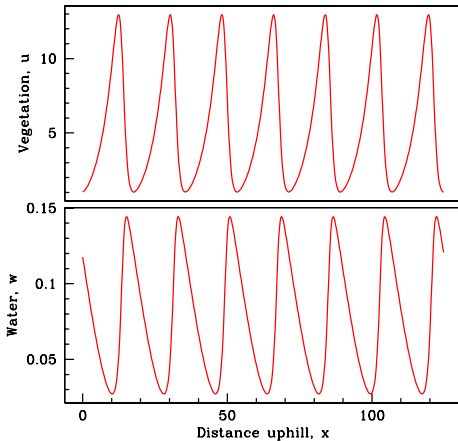
Typical Solution of the Model



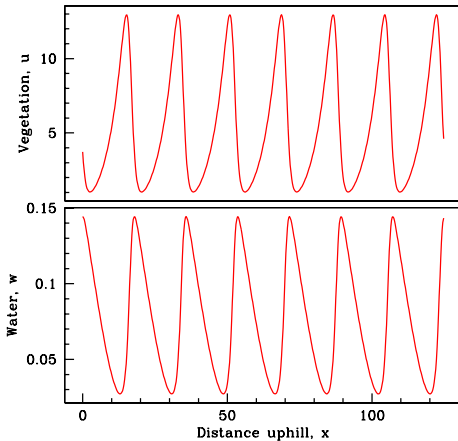
Typical Solution of the Model



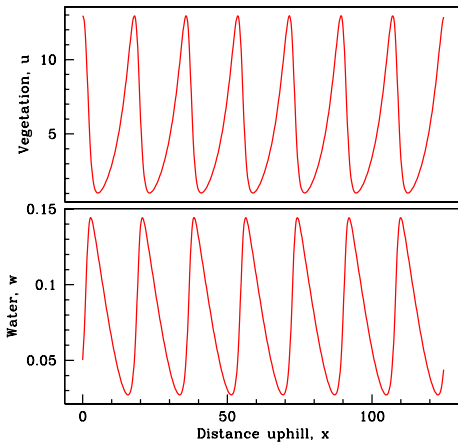
Typical Solution of the Model



Typical Solution of the Model



Typical Solution of the Model



Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$

Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$
- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations

Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0, w = A$
- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations
- The other steady state (u_s, w_s) is stable to homogeneous perturbations but can be unstable to inhomogeneous perturbations \Rightarrow pattern formation

Predicting Pattern Wavelength: Textbook Approach

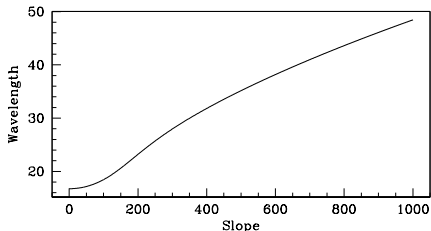
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state (u_s, w_s).



The expected wavelength \leftrightarrow the frequency of noise giving the fastest growth rate.

Predicting Pattern Wavelength: Textbook Approach

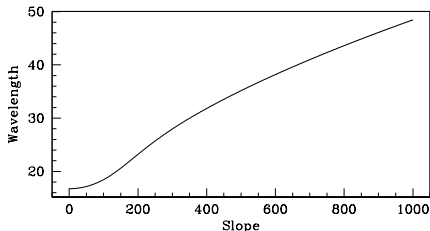
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state (u_s, w_s).



This implies a positive correlation between wavelength and slope, contrary to data.

Predicting Pattern Wavelength: Textbook Approach

The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state (u_s, w_s).



“To date, no model of vegetation band formation has been shown to reproduce this inverse relationship between spacing and slope.” (Pelletier et al, J. Geophys. Res. 117, F04026, 2012)

The Origin of Vegetation Patterns

“Most unstable frequency” assumes that patterns develop from a pre-existing unstable uniform state.

Vegetation patterns develop via
either degradation of uniform vegetation
or colonisation of bare ground

Outline

- 1 Ecological Background
- 2 Pattern Formation in a Mathematical Model
- 3 Pattern Existence and Stability**
- 4 Predictions of Pattern Wavelength vs Slope
- 5 Conclusions and References

Travelling Wave Equations

The patterns move at constant shape and speed

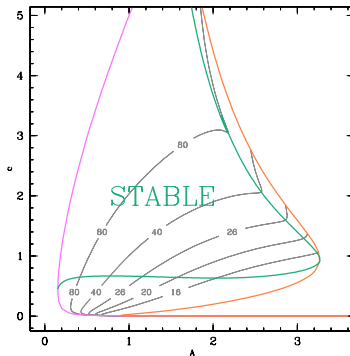
$$\Rightarrow u(x, t) = U(z), w(x, t) = W(z), z = x - ct$$

$$\begin{aligned} d^2 U/dz^2 + c dU/dz + WU^2 - BU &= 0 \\ D d^2 W/dz^2 + (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these equations

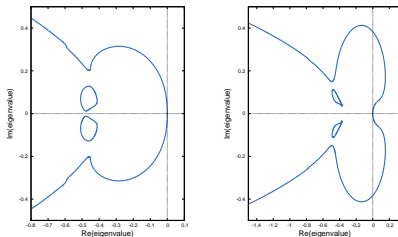
Pattern Stability

Not all of the possible patterns are stable as solutions of the model equations.



Pattern Stability: Numerical Approach

The boundary between stable and unstable patterns can be calculated by numerical continuation of the essential spectrum.
(J.D.M. Rademacher, B. Sandstede, A. Scheel, Computing absolute and essential spectra using continuation, Physica D 229 166-183, 2007)



Calculations of this type can be performed using the software package WAVETRAIN (www.ma.hw.ac.uk/wavetrain).

Pattern Stability: The Key Result

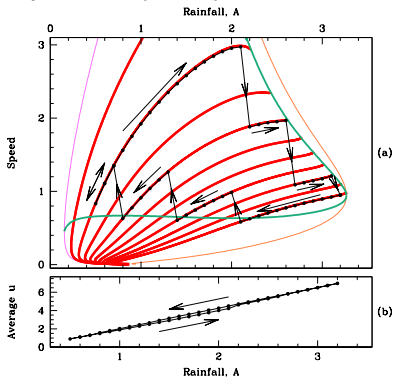
Key Result

Some of the possible patterns are unstable and thus will never be seen.

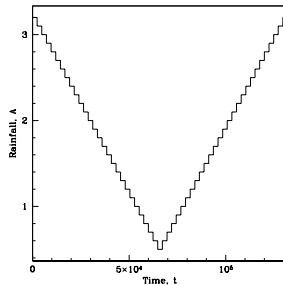
However, for a wide range of rainfall levels, there are multiple stable patterns.

Variations in Rainfall: Hysteresis

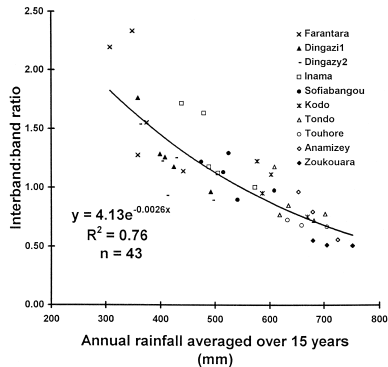
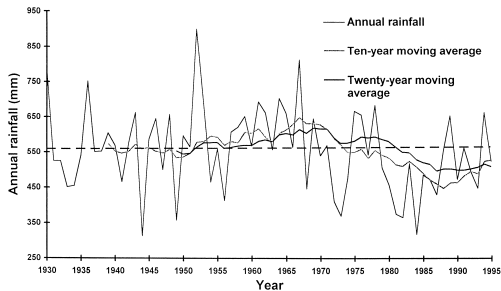
The existence of multiple stable patterns suggests the possibility of hysteresis.



Domain length 150, periodic bc's



Data on the Effects of Changing Rainfall

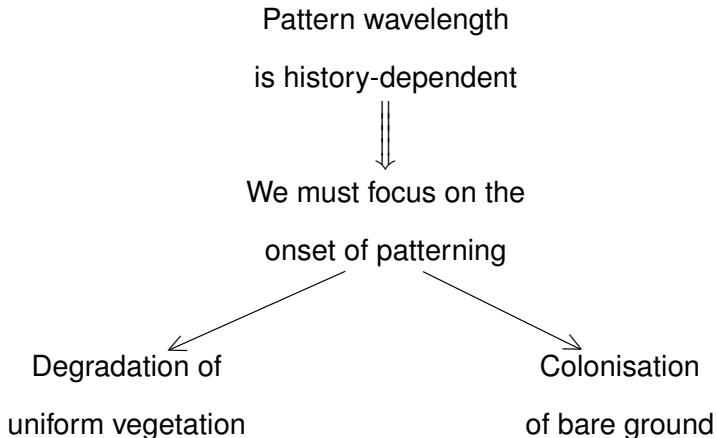


Data from 1950-1995 (C. Valentin & J.M. d'Herbès, Catena 37:231, 1999)

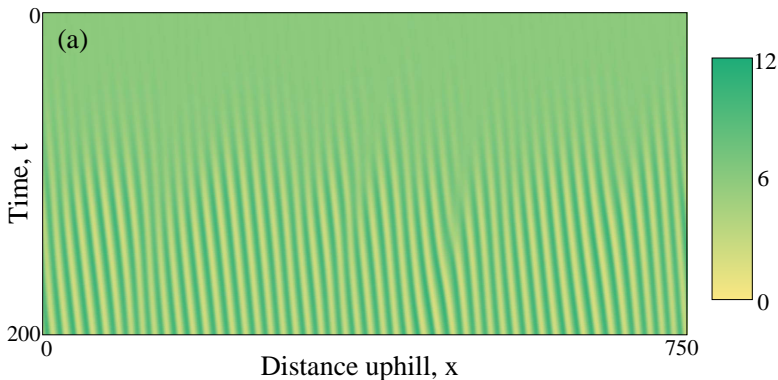
Outline

- 1 Ecological Background
- 2 Pattern Formation in a Mathematical Model
- 3 Pattern Existence and Stability
- 4 **Predictions of Pattern Wavelength vs Slope**
- 5 Conclusions and References

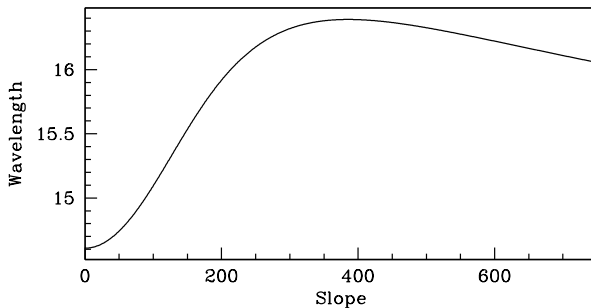
How to Predict Pattern Wavelength



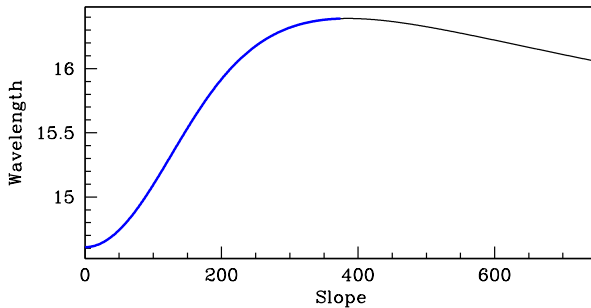
Wavelength vs Slope for Degradation of Uniform Vegetation



Wavelength vs Slope for Degradation of Uniform Vegetation



Wavelength vs Slope for Degradation of Uniform Vegetation



For realistic parameters, wavelength increases with slope,
contrary to data

When Does Vegetation Colonise Bare Ground?

Downhill \longleftrightarrow Uphill



Time

Very low rainfall: an isolated vegetation patch dies out



Slightly larger rainfall: both edges move uphill

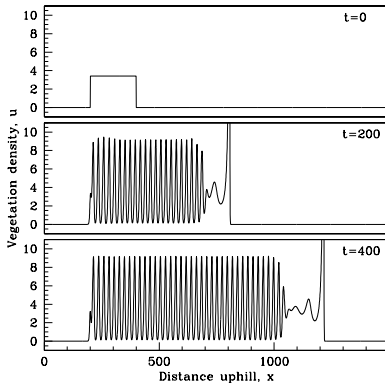


Larger rainfall: the patch expands both uphill and downhill

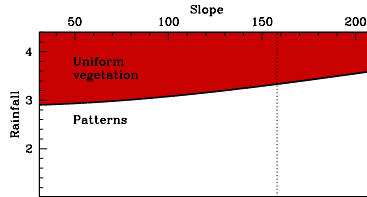


When Does Vegetation Colonise Bare Ground?

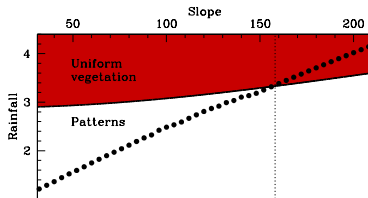
The key critical case is when the downhill edge is stationary



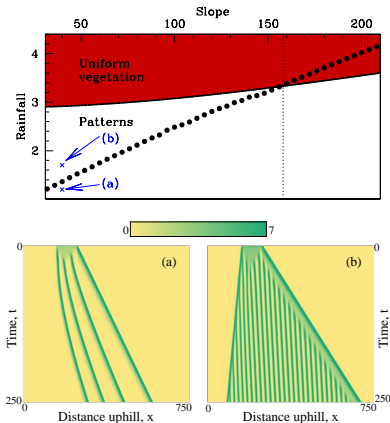
Wavelength vs Slope for Colonisation



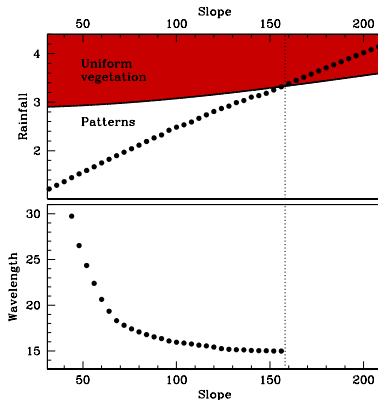
Wavelength vs Slope for Colonisation



Wavelength vs Slope for Colonisation



Wavelength vs Slope for Colonisation



Wavelength decreases with slope, in agreement with data

Outline

- 1 Ecological Background
- 2 Pattern Formation in a Mathematical Model
- 3 Pattern Existence and Stability
- 4 Predictions of Pattern Wavelength vs Slope
- 5 Conclusions and References

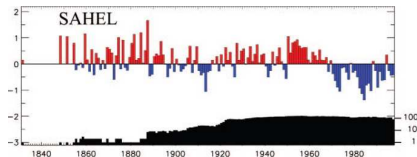
Example: The African Sahel



- Patterned vegetation is widespread in the Sahel
- Several studies of banded vegetation show wavelength \downarrow as slope \uparrow

Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.



- There is no direct data on rainfall before c. 1850
- Proxy data: (i) lake levels, esp. Lake Chad; (ii) historical chronologies, e.g. Bornu Empire; (iii) memories of local peoples.

Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.
- Reasonable assumption: areas with vegetation patterns today had uniform vegetation at the end of the 17th century.
- Since wavelength decreases with slope, my results imply that vegetation must have died out and then recolonised since the end of the 17th century.
- The most severe drought since 1700 was c. 1738-1756. So today's vegetation patterns result from recolonisation since 1760.

Conclusions

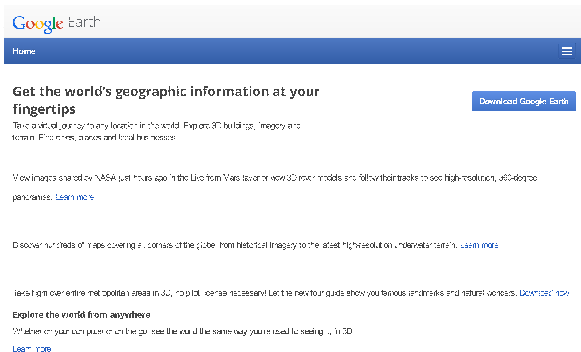
Wavelength is positively correlated with slope \Rightarrow vegetation pattern originated by degradation of uniform vegetation

Wavelength is negatively correlated with slope \Rightarrow vegetation pattern originated by colonisation of bare ground

Main message: combined wavelength–slope data is much more valuable than wavelength data alone.

Remote Sensing of Wavelength and Elevation

Google Earth: online satellite images, min. 15 m resolution



The image is a screenshot of the Google Earth website's homepage. At the top, the Google Earth logo is displayed in its characteristic multi-colored font. Below the logo is a dark blue navigation bar with the word "Home" on the left and a hamburger menu icon on the right. The main content area has a white background. It begins with the heading "Get the world's geographic information at your fingertips" in bold black text. To the right of this heading is a blue button with the text "Download Google Earth". Below the heading is a paragraph of text: "Take a virtual journey to any location in the world. Explore 3D buildings, imagery and terrain. Find cities, places and local businesses." This is followed by a horizontal line. Below the line is another paragraph: "View images shared by NASA just hours ago in the Live from Mars feature or view 3D model models and follow their tracks to see high-resolution, 360-degree panoramas." This is followed by a link "Learn more". Another horizontal line follows. Below it is a paragraph: "Discover hundreds of maps covering all corners of the globe, from historical imagery to the latest high-resolution, understanding terrain." This is followed by a link "Learn more". Another horizontal line follows. Below it is a paragraph: "Journey from entire metropolitan areas in 3D, to global terrain necessary! Let the new tour guide show you various landmarks and natural wonders." This is followed by a link "Download now". Another horizontal line follows. Below it is the heading "Explore the world from anywhere" in bold. This is followed by a paragraph: "Whether on your laptop or on the go, see the world the same way you're used to seeing it, in 3D." This is followed by a link "Learn more".

Remote Sensing of Wavelength and Elevation

WorldDEM: online elevation data, 12 m resolution



The screenshot shows the WorldDEM website interface. At the top, there's a navigation bar with links like 'Products & Services', 'Markets', 'Ordering', 'Get Data', 'Contact us', 'Commercial network', and 'Satellite Image Gallery'. Below this is a large 3D visualization of a mountain range with a color gradient from green at the base to red at the peaks. The main content area features the 'WorldDEM™' logo and the tagline 'The New Standard of Global Elevation Models'. A paragraph describes the service as a complete and reliable coverage of the world's elevation data, highlighting its accuracy and global reach. To the right, there's a sidebar with a 'WorldDEM™' section containing links to 'Products', 'Sample Data', 'Order WorldDEM™ Now', and 'Related pages'. At the bottom of the sidebar, there's a 'WorldDEM' logo.

References

J.A. Sherratt: History-dependent patterns of whole ecosystems.
Ecological Complexity 14, 8-20 (2013).

A.S. Dagbovie, J.A. Sherratt: Pattern selection and hysteresis in the Rietkerk model for banded vegetation in semi-arid environments.
J. R. Soc. Interface 11: 20140465 (2014).

J.A. Sherratt: Using wavelength and slope to infer the historical origin of semi-arid vegetation bands. *PNAS USA* 112: 4202-4207 (2015).

J.A. Sherratt: When does colonisation of a semi-arid hillslope generate vegetation patterns? *J. Math. Biol.* in press.

List of Frames

- 1 Ecological Background
 - Vegetation Patterns
 - Banded Vegetation on Slopes
 - Data on Wavelength vs Slope
- 2 Pattern Formation in a Mathematical Model
 - Mathematical Model of Klausmeier
 - Typical Solution of the Model
 - Homogeneous Steady States
 - Predicting Pattern Wavelength: Textbook Approach
 - The Origin of Vegetation Patterns
- 3 Pattern Existence and Stability
 - Travelling Wave Equations
 - Pattern Stability
 - Variations in Rainfall: Hysteresis

- 4 Predictions of Pattern Wavelength vs Slope
 - How to Predict Pattern Wavelength
 - Wavelength vs Slope for Degradation of Uniform Vegetation
 - When Does Vegetation Colonise Bare Ground?
 - Wavelength vs Slope for Colonisation
- 5 Conclusions and References
 - Example: The African Sahel
 - Rainfall History in the Sahel
 - Conclusions
 - References