Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns in Semi-Deserts

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Heriot-Watt University

Universiteit Leiden, 8 February 2016

This talk can be downloaded from my web site

www.ma.hw.ac.uk/~jas

Outline

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Vegetation Patterns

Desert ecosystems provide a classic example of self-organised pattern formation.

W National Park, Niger

Average patch width is 50 m

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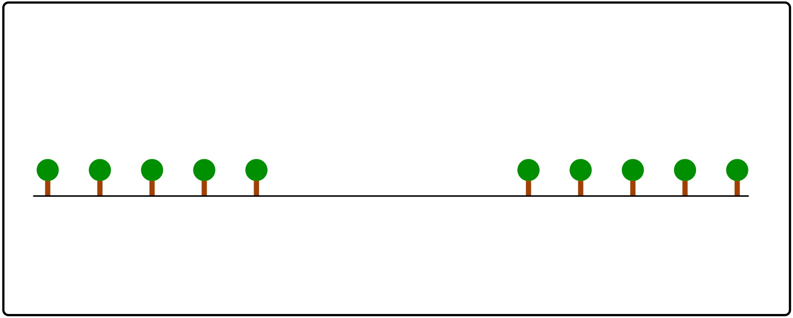

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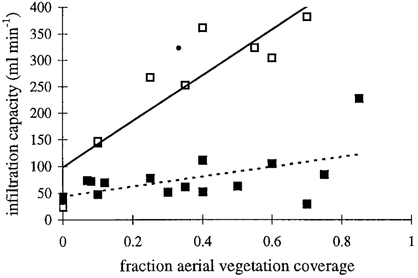

Data from Burkina Faso

Rietkerk et al
Plant Ecology 148: 207-224, 2000

More plants $\Rightarrow$ more roots and organic matter in soil
$\Rightarrow$ more infiltration of rainwater

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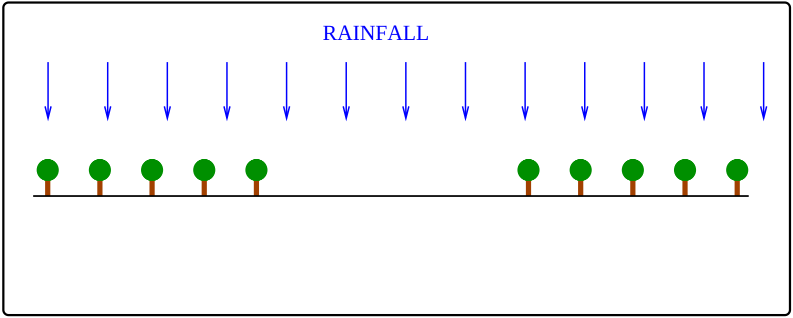

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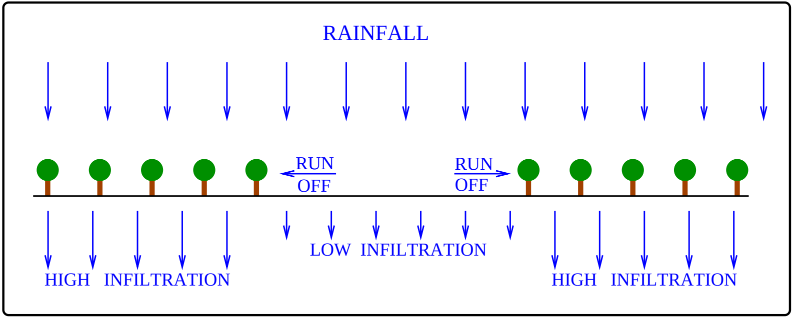

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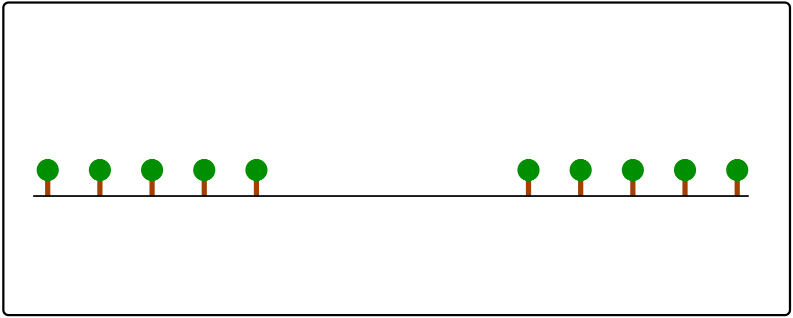

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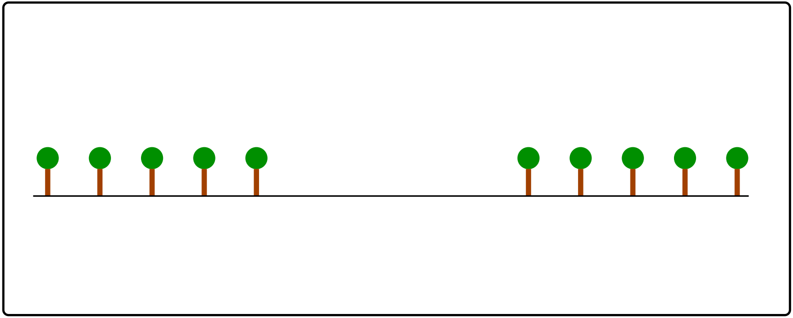

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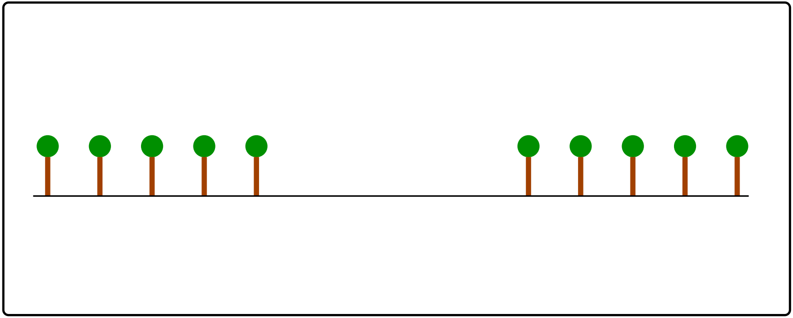

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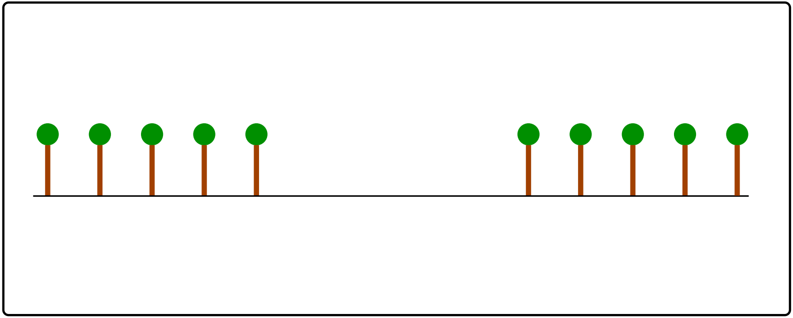

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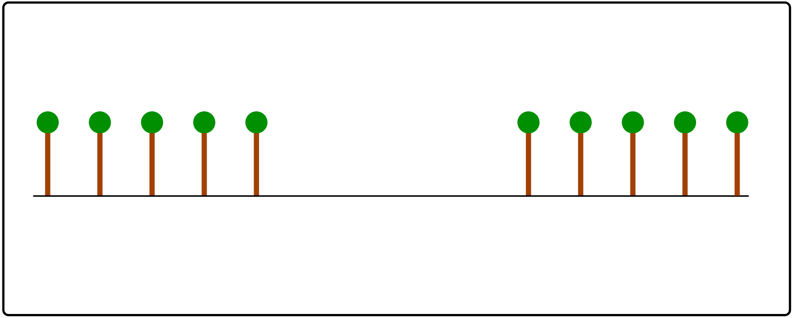

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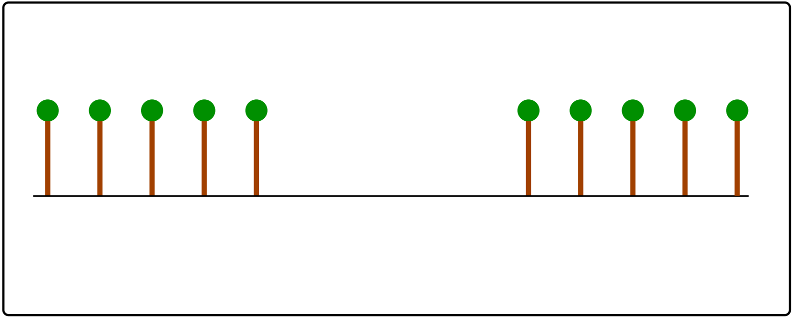

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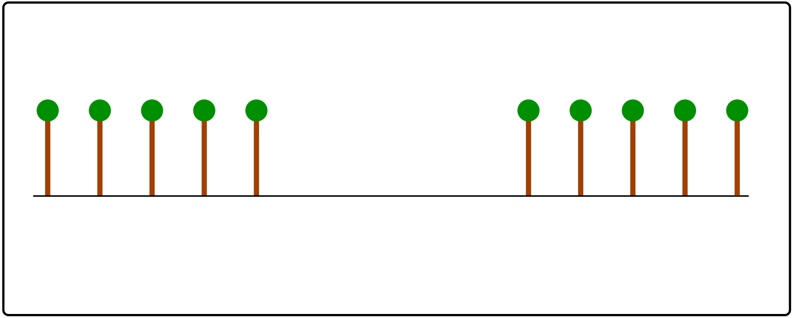

Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia

(Western New South Wales)

Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia, Mexico and S-W USA.

Banded Vegetation on Slopes

On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia

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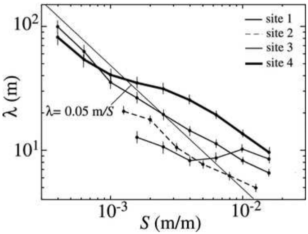
Wavelength can be measured via remote sensing.

Data on Wavelength vs Slope

Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.



Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

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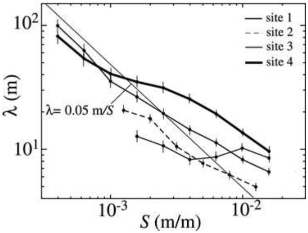

Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

How does this compare with predictions of mathematical models?

Outline

Mathematical Model of Klausmeier

$$
\begin{aligned}
\partial u/\partial t &= \overbrace{wu^2}^{\text{plant growth}} - \overbrace{Bu}^{\text{plant loss}} + \overbrace{\partial^2 u/\partial x^2}^{\text{plant dispersal}} \\
\partial w/\partial t &= \underbrace{A}_{\text{average rainfall}} - \underbrace{w}_{\text{evaporation \& drainage}} - \underbrace{wu^2}_{\text{uptake by plants}} + \underbrace{\nu\,\partial w/\partial x}_{\text{flow downhill}} + \underbrace{D\,\partial^2 w/\partial x^2}_{\text{diffusion of water}}
\end{aligned}
$$

(Klausmeier, Science 284: 1826-8, 1999)

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The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.

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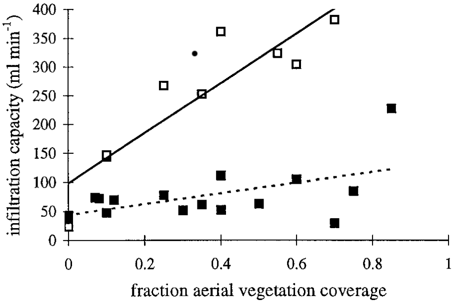

Water uptake=
Water density
$\times$ Plant density
$\times \begin{pmatrix}\text{infiltration}\\ \text{rate}\end{pmatrix}$

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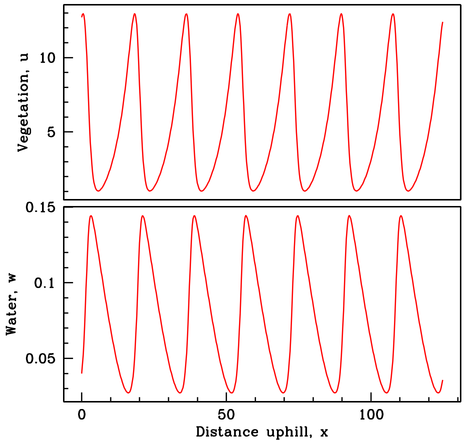
Typical Solution of the Model



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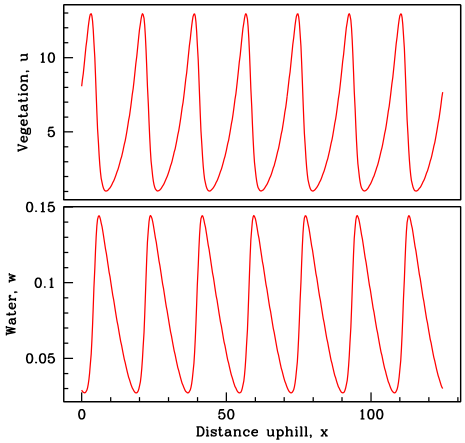

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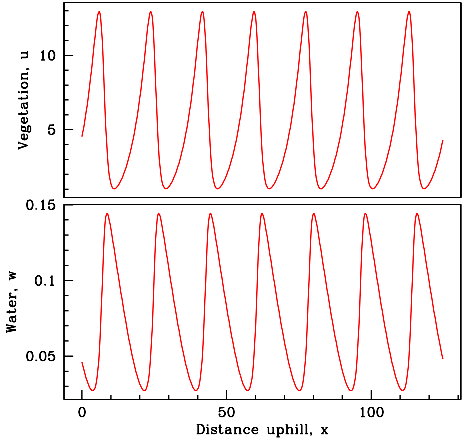

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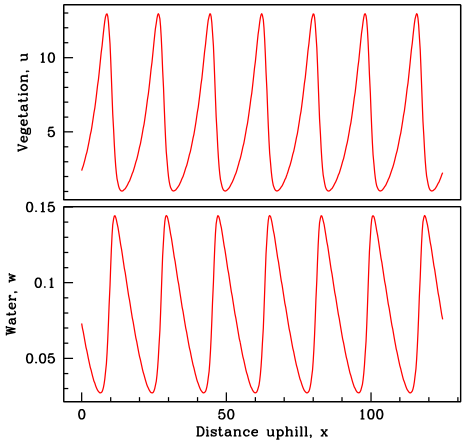

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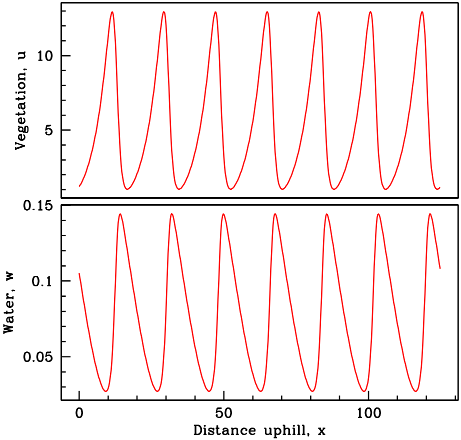

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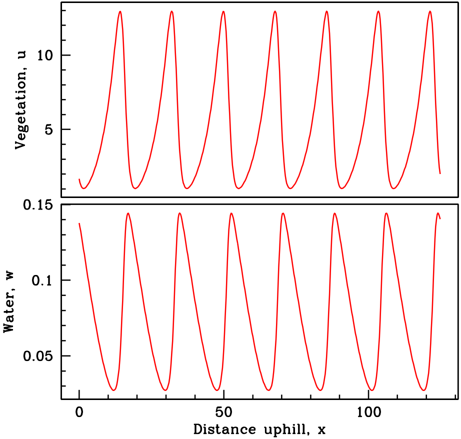

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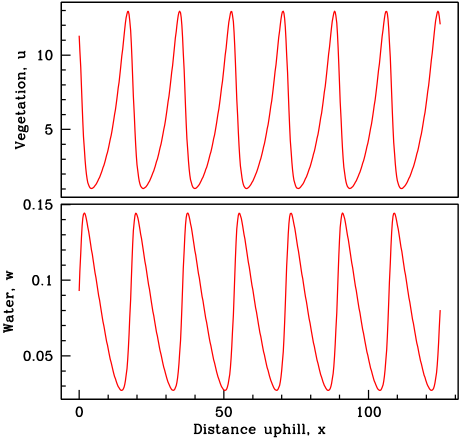

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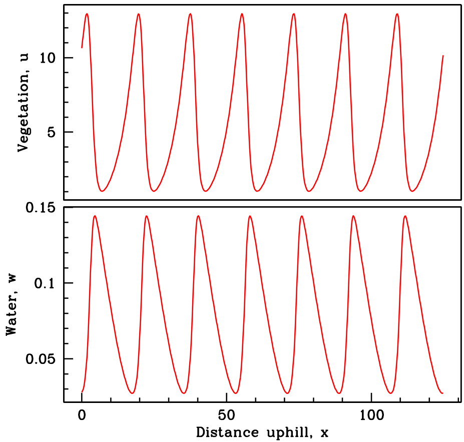

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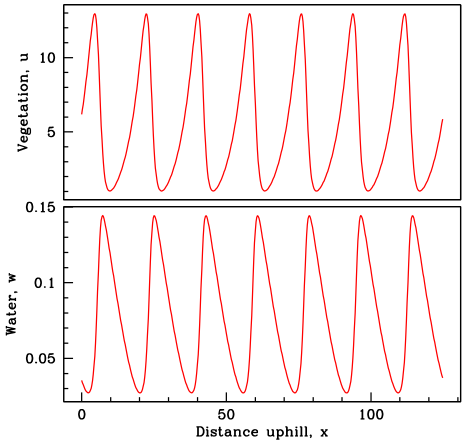

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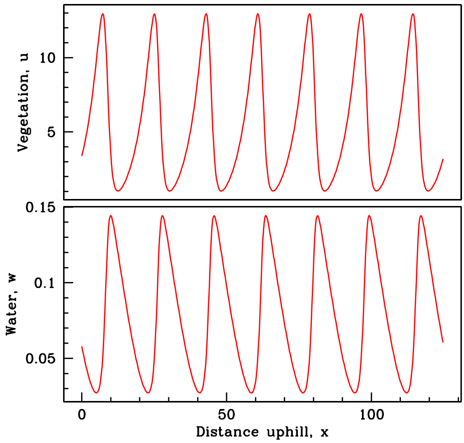

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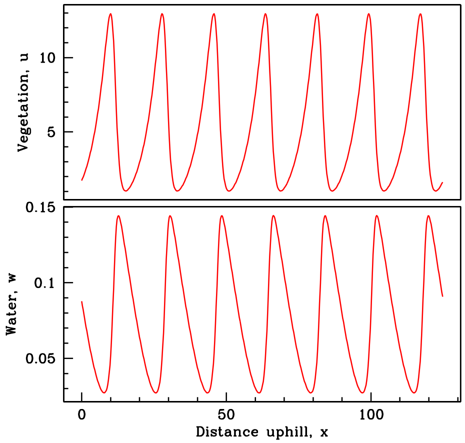

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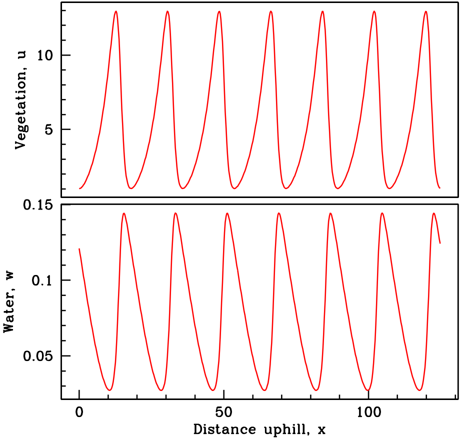

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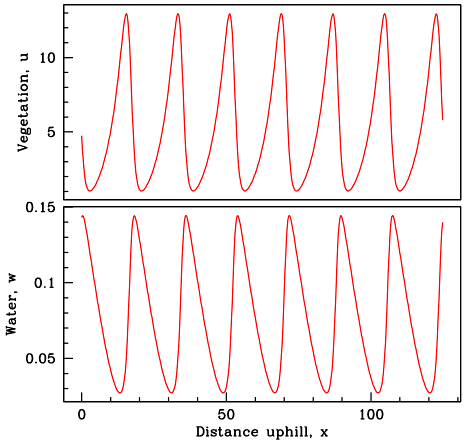

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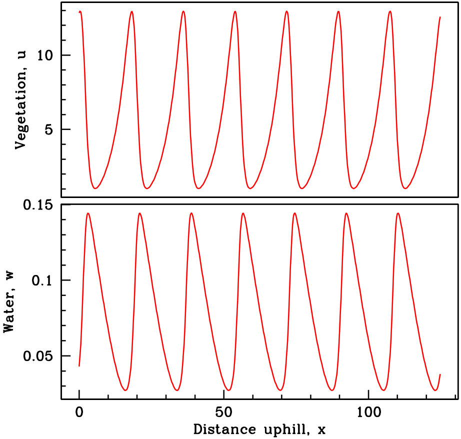

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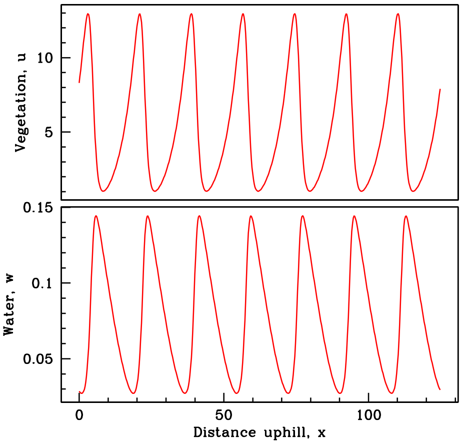

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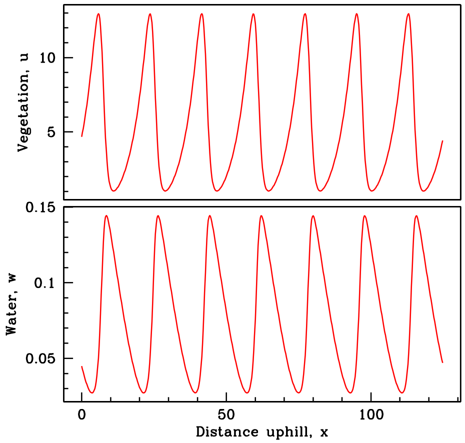

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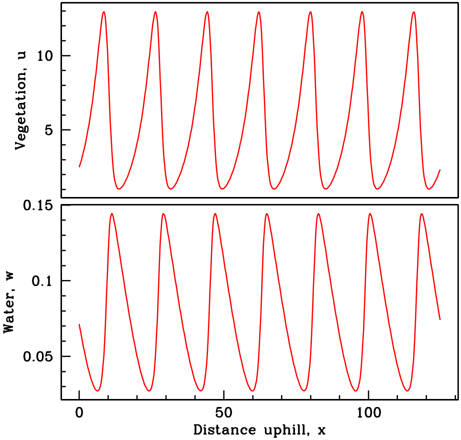

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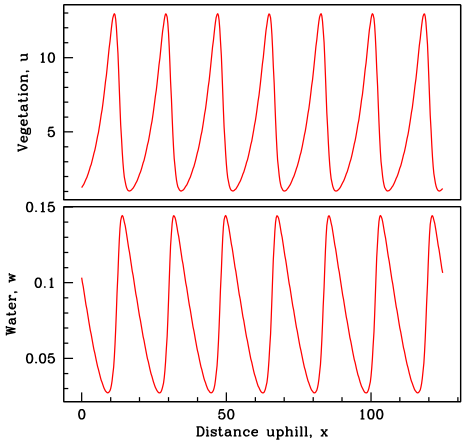

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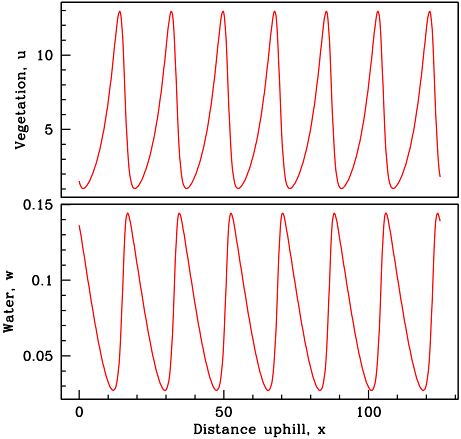

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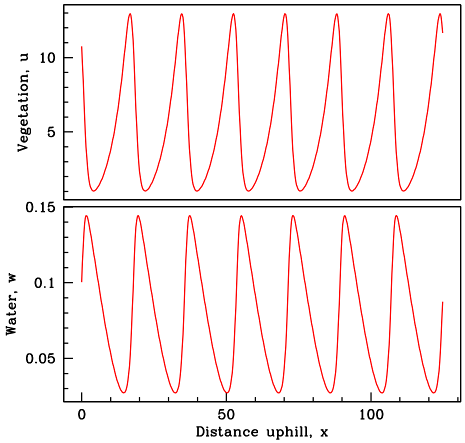

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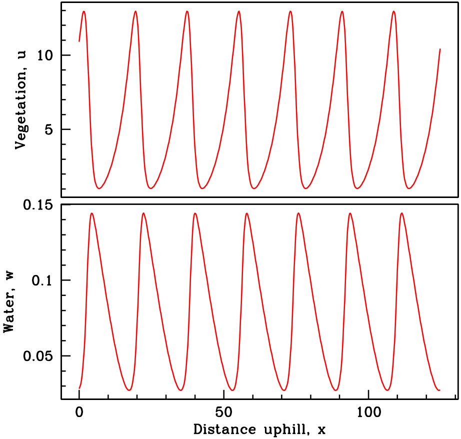

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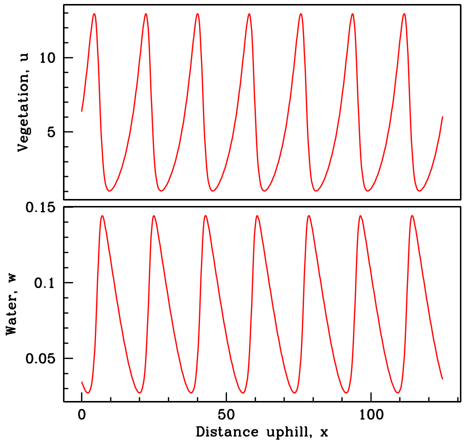

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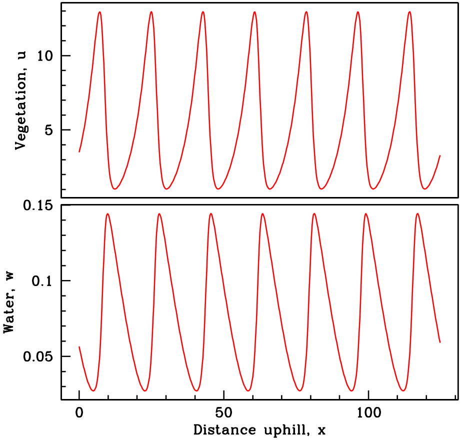

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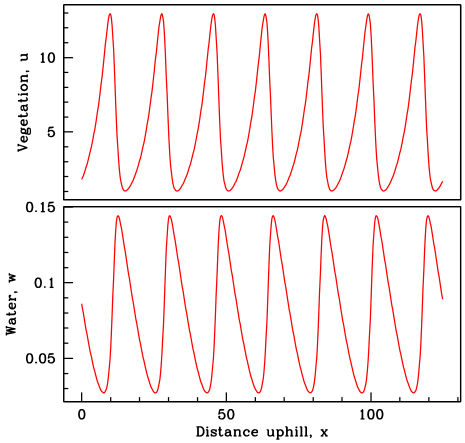

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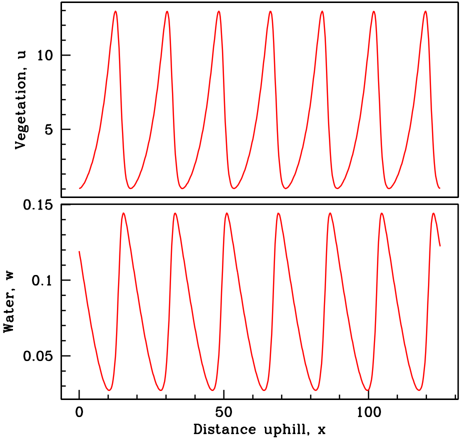

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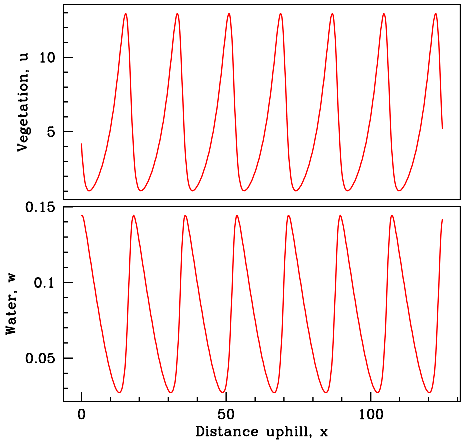

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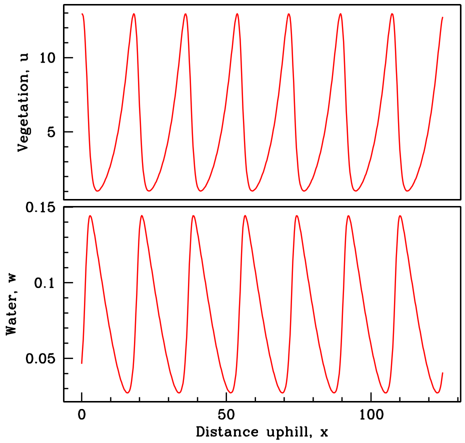

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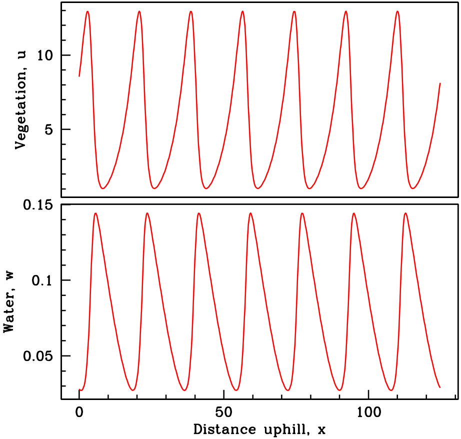

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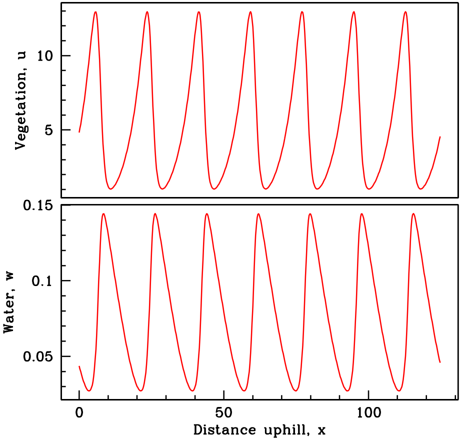

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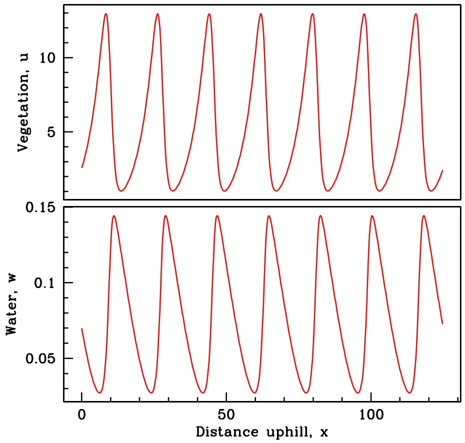

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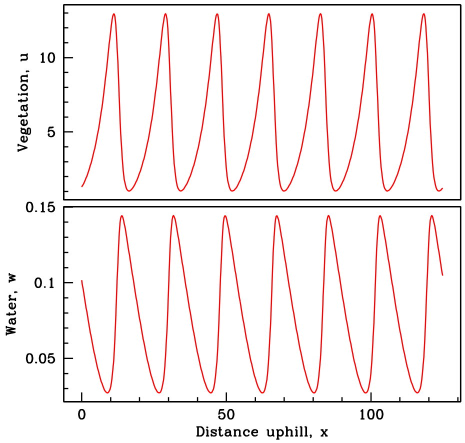

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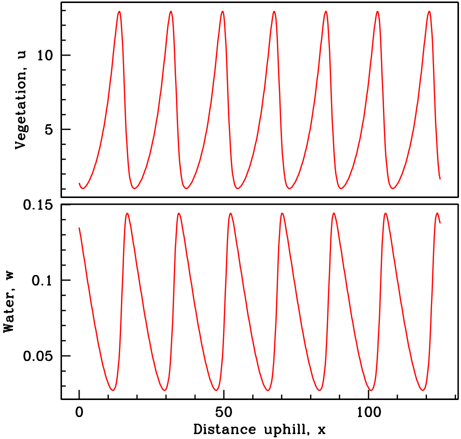

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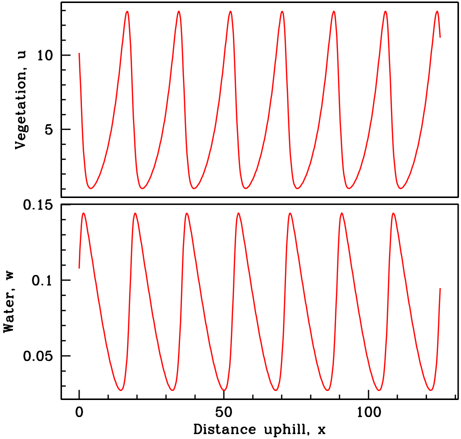

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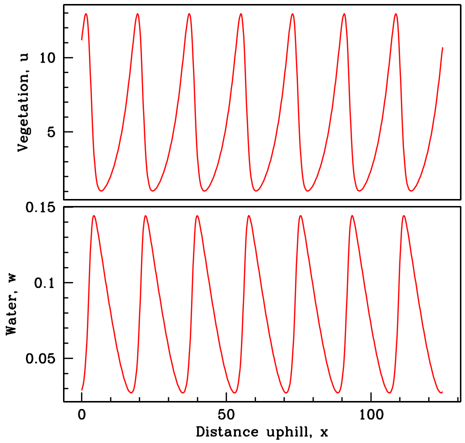

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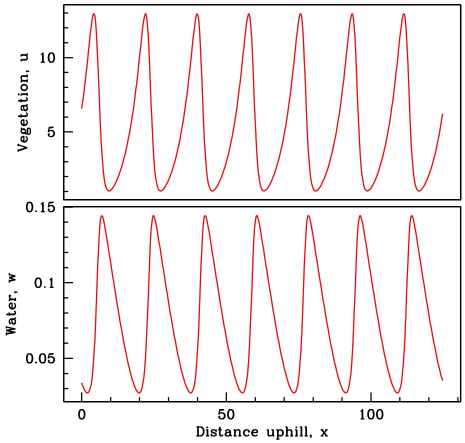

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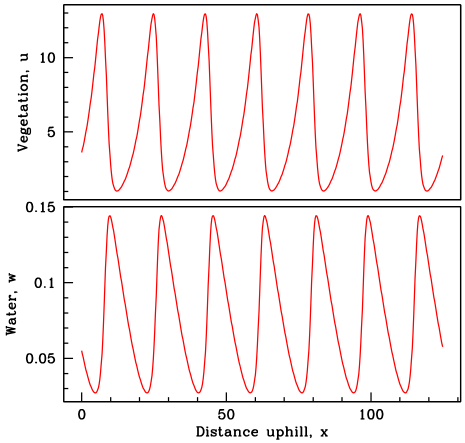

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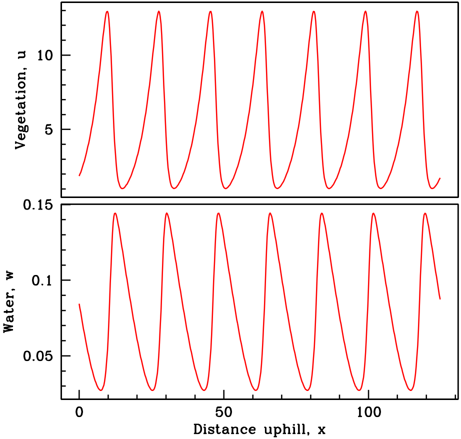

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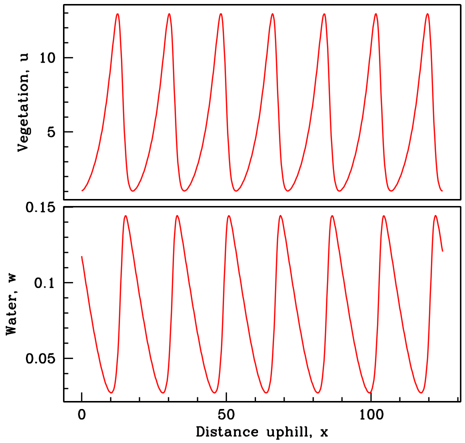

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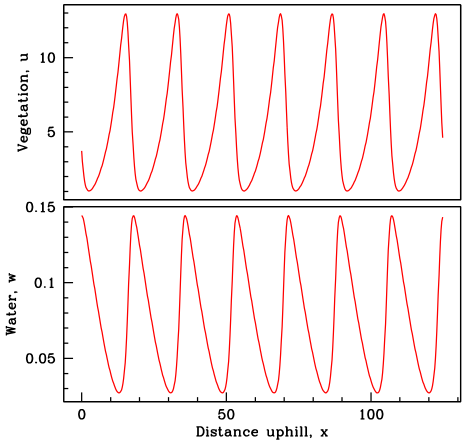

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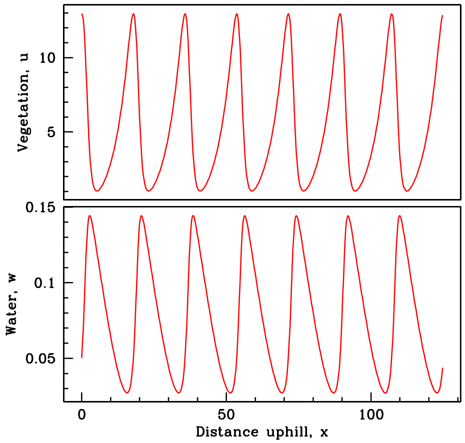

Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$

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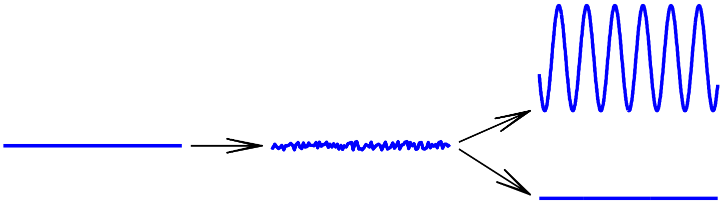
- For all parameter values, there is a stable "desert" steady state $u = 0$, $w = A$
- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations
- The other steady state (u_s, w_s) is stable to homogeneous perturbations but can be unstable to inhomogeneous perturbations $\Rightarrow$ pattern formation

Predicting Pattern Wavelength: Textbook Approach

The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state (u_s, w_s).



The expected wavelength $\leftrightarrow$ the frequency of noise giving the fastest growth rate.

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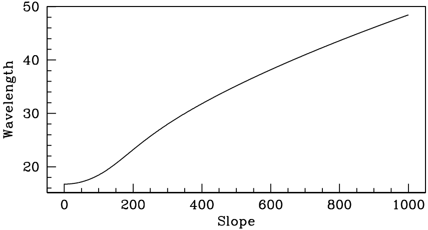

This implies a positive correlation between wavelength and slope, contrary to data.

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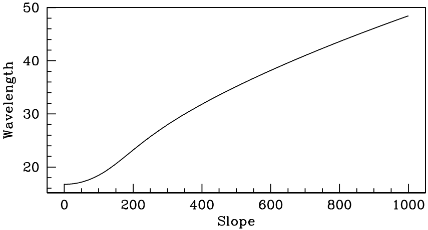


"To date, no model of vegetation band formation has been shown to reproduce this inverse relationship between spacing and slope." (Pelletier et al, J. Geophys. Res. 117, F04026, 2012)

The Origin of Vegetation Patterns

"Most unstable frequency" assumes that patterns develop from a pre-existing unstable uniform state.

Vegetation patterns develop via

either degradation of uniform vegetation

or colonisation of bare ground

Outline

Travelling Wave Equations

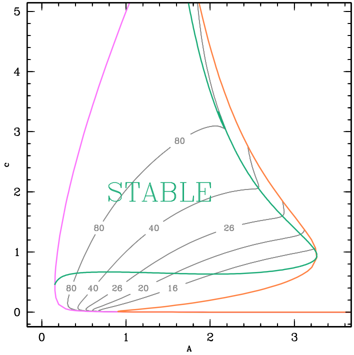
The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z)$, $w(x,t) = W(z)$, $z = x - ct$

$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ D\,d^2W/dz^2 + (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

The patterns are periodic (limit cycle) solutions of these equations

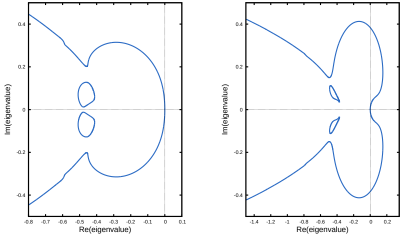
Pattern Stability

Not all of the possible patterns are stable as solutions of the model equations.



Pattern Stability: Numerical Approach

The boundary between stable and unstable patterns can be calculated by numerical continuation of the essential spectrum.
(J.D.M. Rademacher, B. Sandstede, A. Scheel, Computing absolute and essential spectra using continuation, Physica D 229 166-183, 2007)



Calculations of this type can be performed using the software package WAVETRAIN (`www.ma.hw.ac.uk/wavetrain`).

Pattern Stability: The Key Result

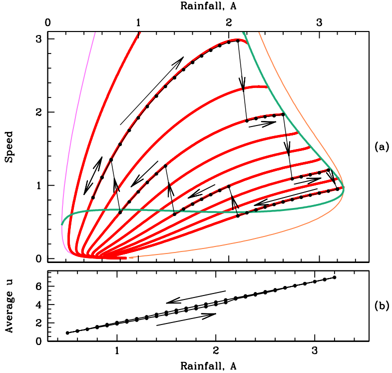
Key Result

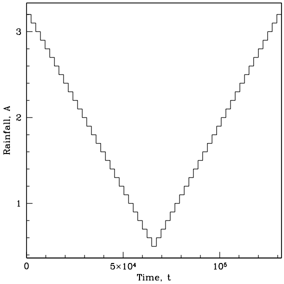
Some of the possible patterns are unstable and thus will never be seen.

However, for a wide range of rainfall levels, there are multiple stable patterns.

Variations in Rainfall: Hysteresis

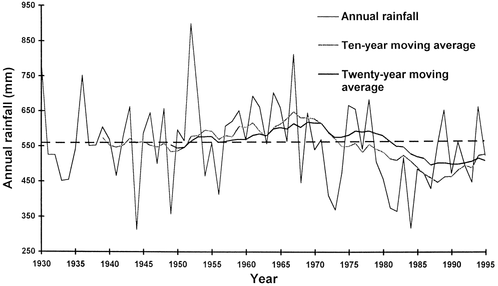
The existence of multiple stable patterns suggests the possibility of hysteresis.

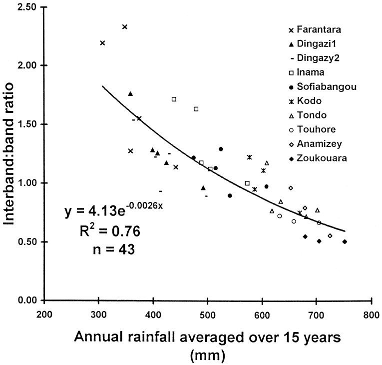




Domain length 150, periodic bc's

Data on the Effects of Changing Rainfall





Data from 1950-1995 (C. Valentin & J.M. d'Herbès, Catena 37:231, 1999)

Outline

How to Predict Pattern Wavelength

Pattern wavelength

is history-dependent

$\Downarrow$

We must focus on the

onset of patterning

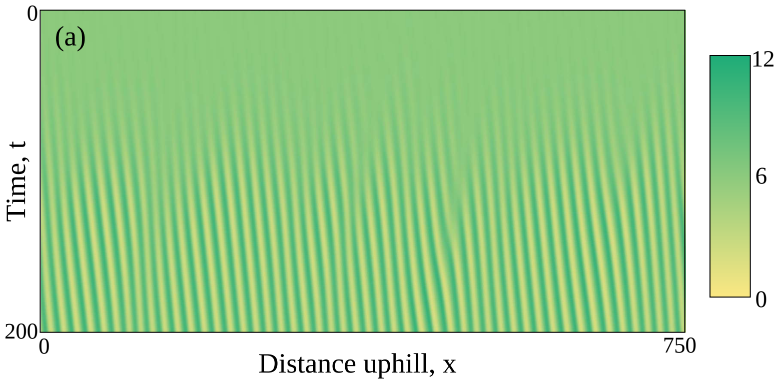
Degradation of

uniform vegetation

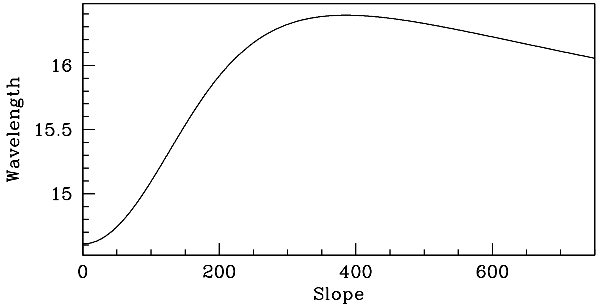
Colonisation

of bare ground

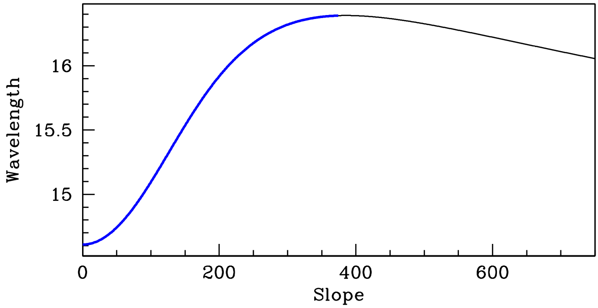
Wavelength vs Slope for Degradation of Uniform Vegetation



Wavelength vs Slope for Degradation of Uniform Vegetation



Wavelength vs Slope for Degradation of Uniform Vegetation



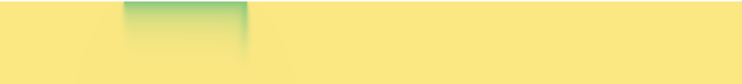
For realistic parameters, wavelength increases with slope, contrary to data

When Does Vegetation Colonise Bare Ground?

Downhill $\longleftrightarrow$ Uphill

Time

Very low rainfall: an isolated vegetation patch dies out

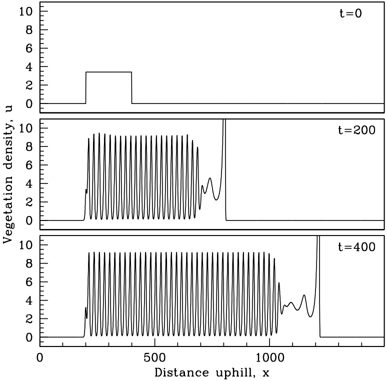


Slightly larger rainfall: both edges move uphill

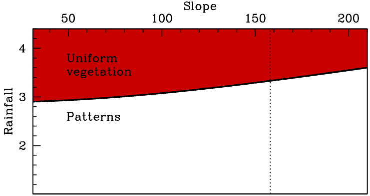
Larger rainfall: the patch expands both uphill and downhill

When Does Vegetation Colonise Bare Ground?

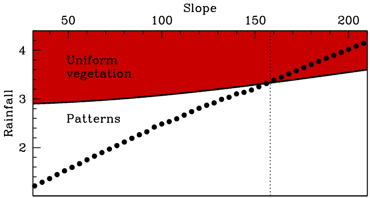
The key critical case is when the downhill edge is stationary



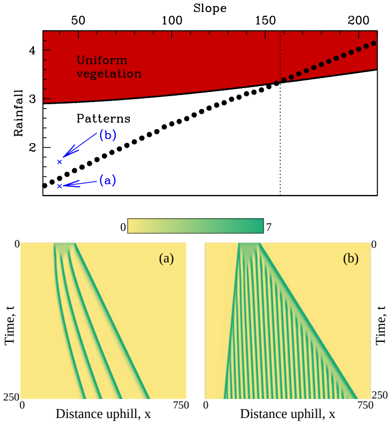
Wavelength vs Slope for Colonisation



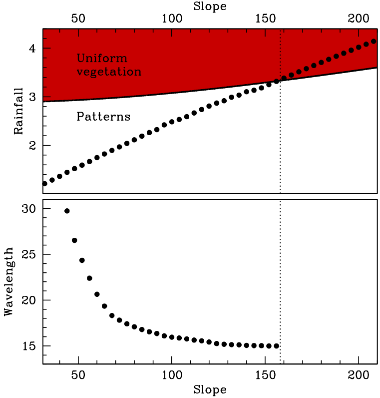
Wavelength vs Slope for Colonisation



Wavelength vs Slope for Colonisation



Wavelength vs Slope for Colonisation



Wavelength decreases with slope, in agreement with data

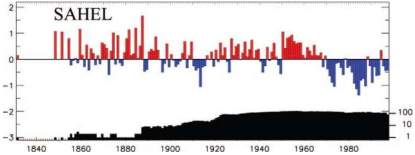
Outline

Example: The African Sahel

- Patterned vegetation is widespread in the Sahel
- Several studies of banded vegetation show wavelength $\downarrow$ as slope $\uparrow$

Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.



- There is no direct data on rainfall before c. 1850
- Proxy data: (i) lake levels, esp. Lake Chad; (ii) historical chronologies, e.g. Bornu Empire; (iii) memories of local peoples.

Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.
- Reasonable assumption: areas with vegetation patterns today had uniform vegetation at the end of the 17th century.
- Since wavelength decreases with slope, my results imply that vegetation must have died out and then recolonised since the end of the 17th century.
- The most severe drought since 1700 was c. 1738-1756. So today's vegetation patterns result from recolonisation since 1760.

Conclusions

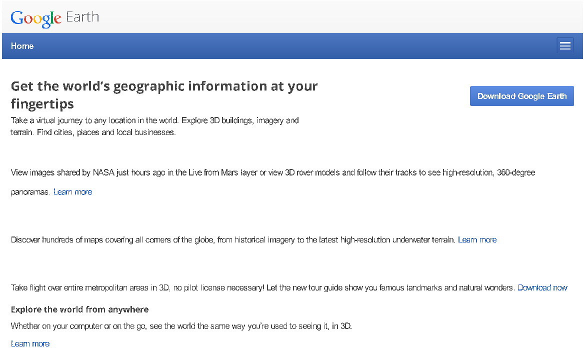
Wavelength is positively correlated with slope $\Rightarrow$ vegetation pattern originated by degradation of uniform vegetation

Wavelength is negatively correlated with slope $\Rightarrow$ vegetation pattern originated by colonisation of bare ground

Main message: combined wavelength–slope data is much more valuable than wavelength data alone.

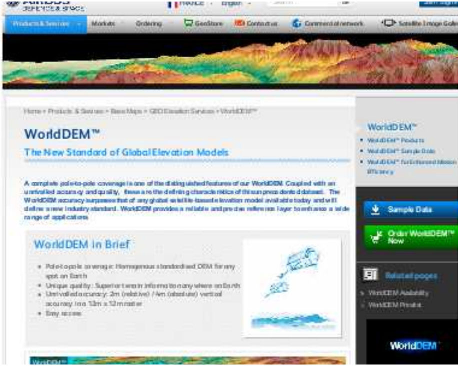
Remote Sensing of Wavelength and Elevation

Google Earth: online satellite images, min. 15 m resolution



Remote Sensing of Wavelength and Elevation

WorldDEM: online elevation data, 12 m resolution



References

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A.S. Dagbovie, J.A. Sherratt: Pattern selection and hysteresis in the Rietkerk model for banded vegetation in semi-arid environments. *J. R. Soc. Interface* 11: 20140465 (2014).

J.A. Sherratt: Using wavelength and slope to infer the historical origin of semi-arid vegetation bands. *PNAS USA* 112: 4202-4207 (2015).

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List of Frames