Patterns of Sources and Sinks in Oscillatory Systems

Jonathan A. Sherratt

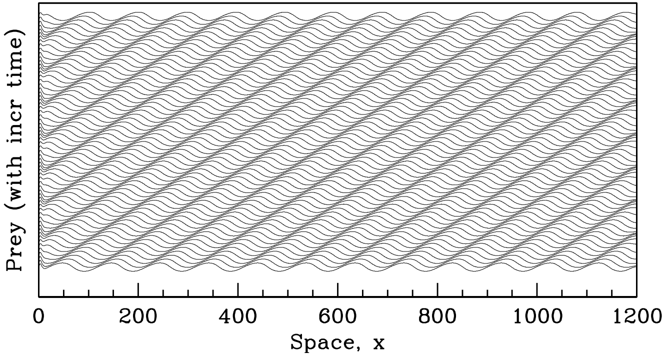
Department of Mathematics
& Maxwell Institute for Mathematical Sciences

Heriot-Watt University

Pattern Formation: The Inspiration of Alan Turing
St John's College, Oxford, 14-16 March, 2012

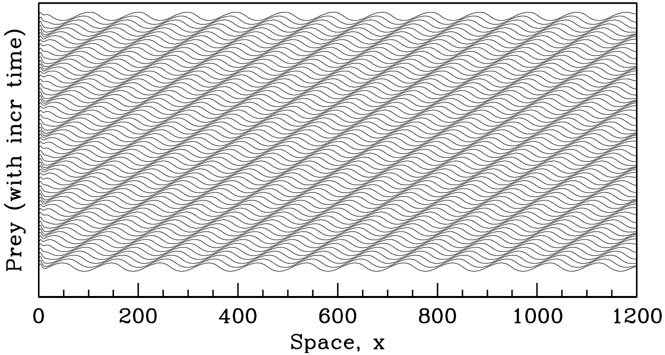
This talk can be downloaded from my web site
`www.ma.hw.ac.uk/∼jas`

Applications of Wavetrains



This pattern is a wavetrain: a periodic function of $x - at$

Applications of Wavetrains



This pattern is a wavetrain: a periodic function of $x - at$

Wavetrains are a generic feature of oscillatory systems

Applications of Wavetrains

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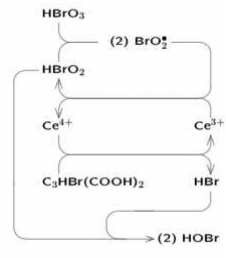
Science **Prev | Table of Co**

www.sciencemag.org
Science 15 June 1973:
Vol. 180 no. 4091 pp. 1171-1173
DOI: 10.1126/science.180.4091.1171

Horizontal Bands in the Belousov Reaction

N. Kopell and L. N. Howard



Applications of Wavetrains

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Physica D 200 (2005) 303–324
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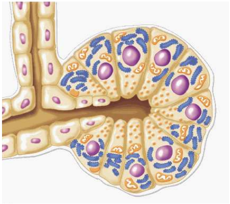
Complex oscillations and waves of calcium in pancreatic acinar cells

David Simpson, Vivien Kirk*, James Sneyd

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand

Received 28 January 2004; received in revised form 9 November 2004; accepted 15 November 2004



Applications of Wavetrains

The Royal Society

Spatial asynchrony and periodic travelling waves in cyclic populations of field voles

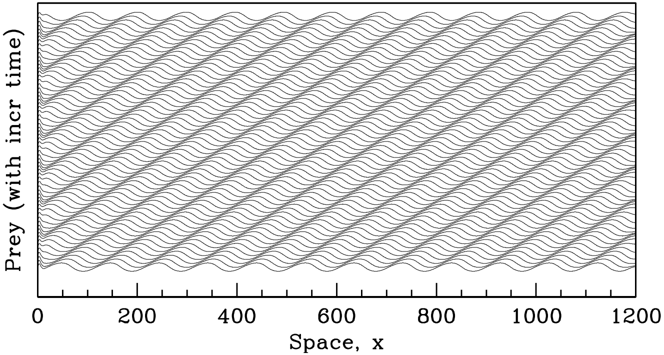
Xavier Lambin[1*], David A. Elston[2], Steve J. Petty[3] and James L. MacKinnon[1]

[1]*Department of Zoology, University of Aberdeen, Tillydrone Avenue, Aberdeen AB24 2TN, UK*
[2]*Biomathematics and Statistics Scotland, Environmental Modelling Unit, Macaulay Land Use Research Institute, Craigiebuckler, Aberdeen AB15 8QH, UK*
[3]*Forest Research, Woodland Ecology Branch, Northern Research Station, Roslin, Midlothian EH25 9SY, UK*

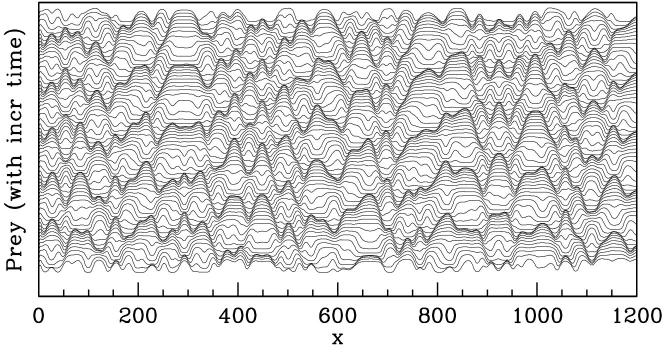
vole

weasel

Applications of Wavetrains

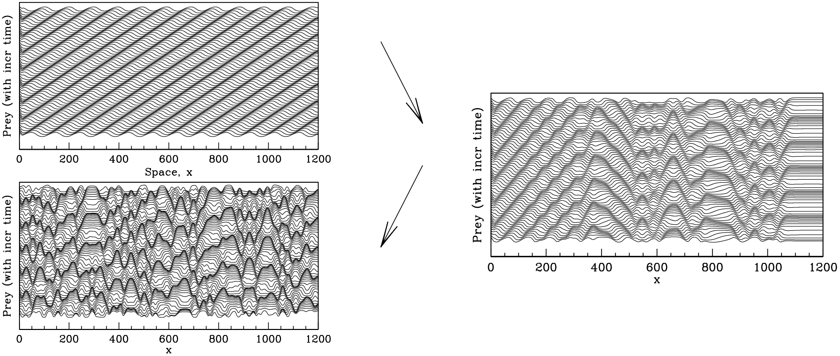


Applications of Wavetrains



For other parameters: more disordered dynamics

Transition from Pattern to Disorder



Outline

The Complex Ginzburg-Landau Equation

I consider a generic oscillator model, the complex Ginzburg-Landau equation:

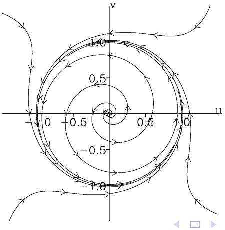
$$A_t = (1+\mathrm{i}b)A_{xx} + A - (1+\mathrm{i}c)|A|^2A.$$

I will look exclusively at $b = 0$. Then writing

$$A(x,t) = e^{-\mathrm{i}at}[u(x,t) + \mathrm{i}v(x,t)]$$

gives a reaction-diffusion system of "λ–ω" type:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (1-r^2)u - (a+cr^2)v \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (a+cr^2)u + (1-r^2)v \\ \text{where } r &= \sqrt{u^2+v^2} \end{aligned}$$



Amplitude and Phase Equations

To study these equations, it is helpful to use the variables $r(x,t) = \sqrt{u^2 + v^2}$ and $\theta(x,t) = \tan^{-1}(v/u)$, giving

$$\begin{aligned} r_t &= r_{xx} - r\theta_x^2 + r(1 - r^2) \\ \theta_t &= \theta_{xx} + \frac{2r_x\theta_x}{r} + a - cr^2 \end{aligned}$$

There is a family of wavetrain solutions ($0 < r^* < 1$):

$$\left\{ \begin{array}{l} r = r^* \\ \theta = \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \end{array} \right\}$$

$$\leftrightarrow \left\{ \begin{array}{l} u = r^* \cos \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \\ v = r^* \sin \left[(a + c{r^*}^2)t \pm \sqrt{(1 - {r^*}^2)}x \right] \end{array} \right\}$$

Wavetrain Generation by Dirichlet Bndy Conditions

I consider these equations
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Dirichlet boundary conditions are very natural in ecology, at transitions between different types of habitat.

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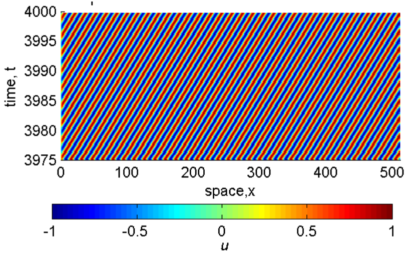
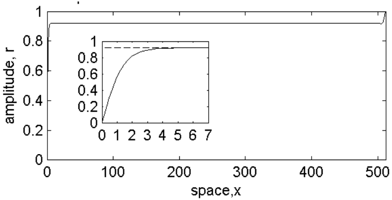


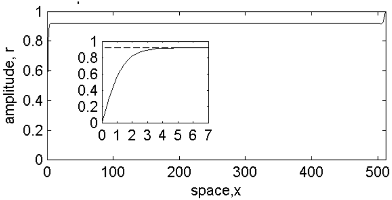


Wavetrain Generation by Dirichlet Bndy Conditions

Conclusion

Dirichlet boundary conditions generate a wavetrain

$$
\begin{aligned}
r &= R^* \tanh\left(x/\sqrt{2}\right) \\
\theta_x &= \Psi^* \tanh\left(x/\sqrt{2}\right) \\
R^* &= \left\{\tfrac{1}{2}\left[1+\sqrt{1+\tfrac{8}{9}c^2}\right]\right\}^{-1/2} \\
\Psi^* &= -\mathrm{sign}(c)\left(1-R^{*\,2}\right)^{1/2}
\end{aligned}
$$



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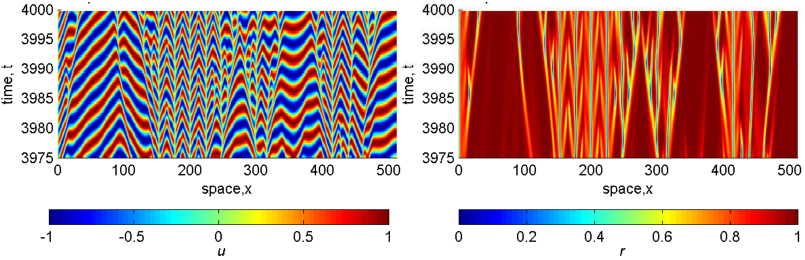
The wavetrain of amplitude R^* is stable $\Leftrightarrow |c| < 1.110468\ldots$

What happens when $|c| > 1.110468\ldots$?

Outline

Two Types of Solution

There are two types of solution for $|c| > 1.110468\ldots$



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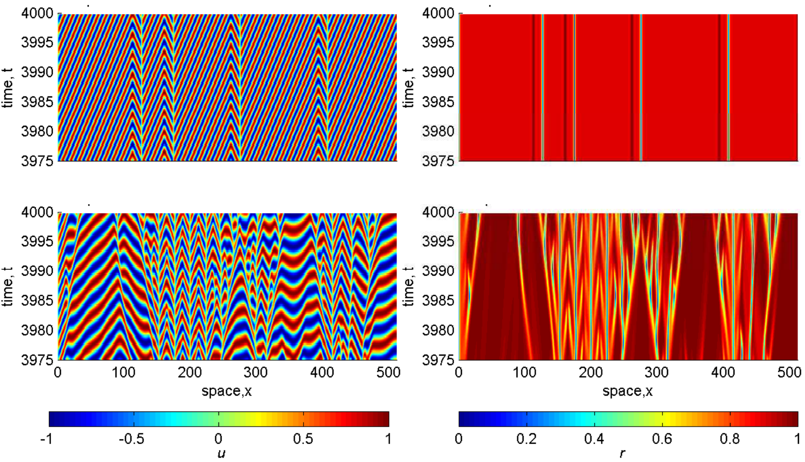

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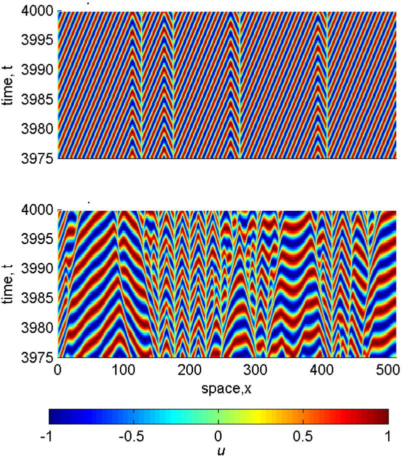

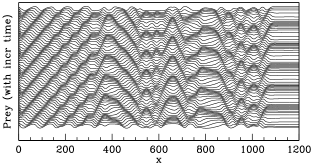

Convective and Absolute Stability

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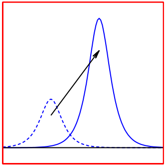
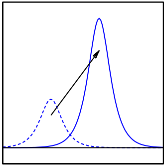
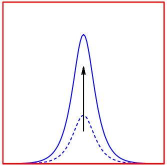
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".



Convective and Absolute Stability

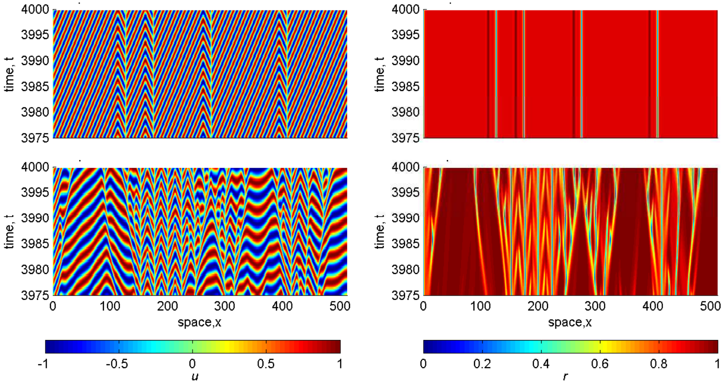
- There are two types of solution for $|c| > 1.110468\ldots$
- The key concept for distinguishing these is "absolute stability".
- In spatially extended systems, a solution can be unstable, but with any perturbation that grows also moving. This is "convective instability".
- Alternatively, a solution can be unstable with perturbations growing without moving. This is "absolute instability".





Generation of Absolutely Stable and Unstable Wavetrains by Dirichlet Boundary Conditions

Numerical simulations show distinct behaviours in the absolutely stable and unstable parameter regimes



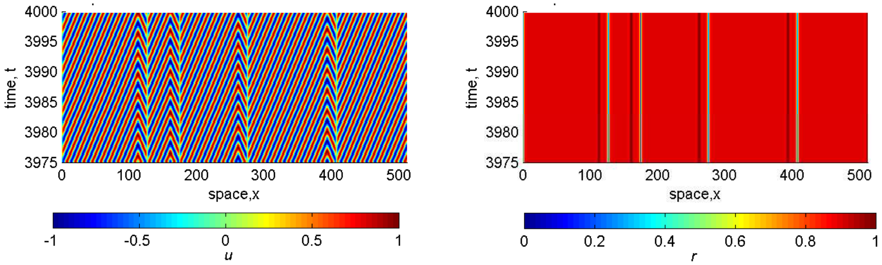
Convectively unstable, absolutely stable

Absolutely unstable

Outline

Sources, Sinks, and Convective Instability

The solution in the convectively unstable but absolutely stable case is a pattern of “sources and sinks”.



Note: sources and sinks are defined in terms of group velocity.

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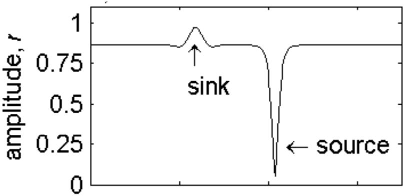

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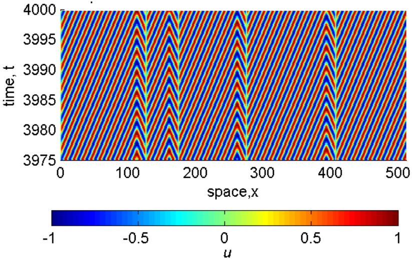

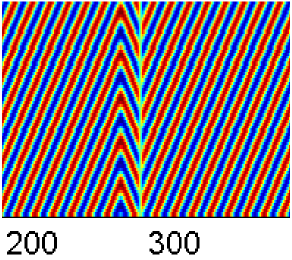

Question: How can an unstable wavetrain persist between the sources and sinks?

Sources, Sinks, and Convective Instability

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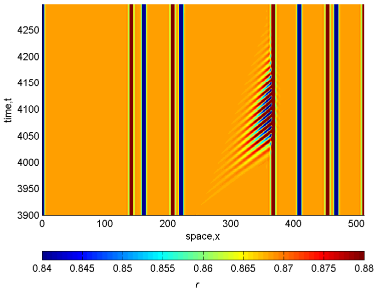
Answer: Any growing perturbations moves, and is absorbed when it reaches a sink.



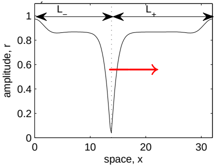
Previous Mathematical Work on Sources and Sinks

- Sources are "Nozaki–Bekki" holes (Nozaki & Bekki, Phys. Lett. A 110: 133-135, 1985), on which the literature is extensive ($>$ 100 citations).
- Sinks are also well studied, though only numerically.
- Important work on classification of "defects" has been done by Sandstede & Scheel (SIAM J. Appl. Dyn. Syst. 3: 1-68, 2004).
- But patterns of sources and sinks have received almost no attention.

Question: are there constraints on the distances separating sources and sinks?

Numerical Study of Source-Sink Separations

What is the effect of translating a source in between two fixed sinks?



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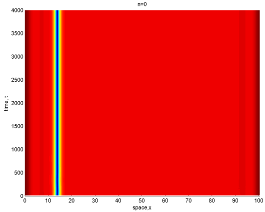
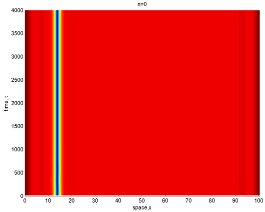


Original solution

Numerical Study of Source-Sink Separations



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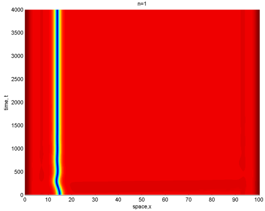
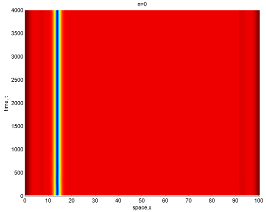


Solution with translated source

Numerical Study of Source-Sink Separations



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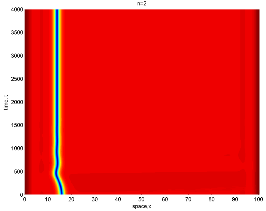

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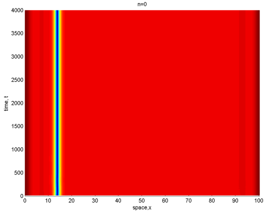

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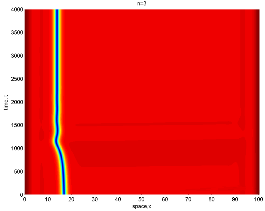

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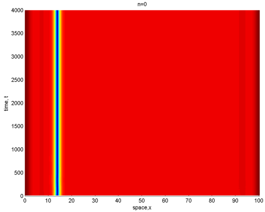

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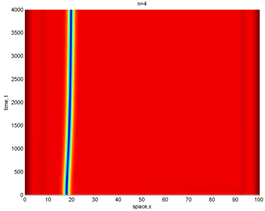
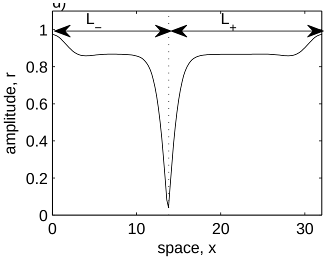


Solution with translated source

Conclusions on Source-Sink Separations

Numerical results suggest: source-sink separations appear to be constrained to a discrete set of possible values.

Analytical study shows:



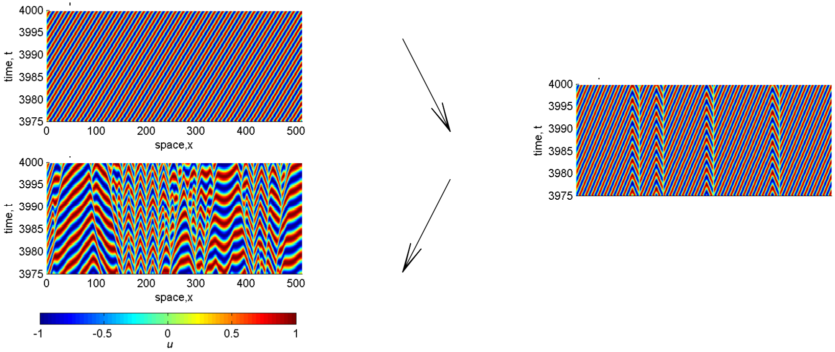
$$\arg\left[\exp\left(-L_-(1+i\delta)/\sqrt{2}\right) + \exp\left(-L_+(1+i\delta)/\sqrt{2}\right)\right] = \text{constant}$$

to leading order for large L_-, L_+ where

$$\delta^2 = 11 - 24/\left(\left[1+\sqrt{1+\tfrac{8}{9}c^2}\right]\right)$$

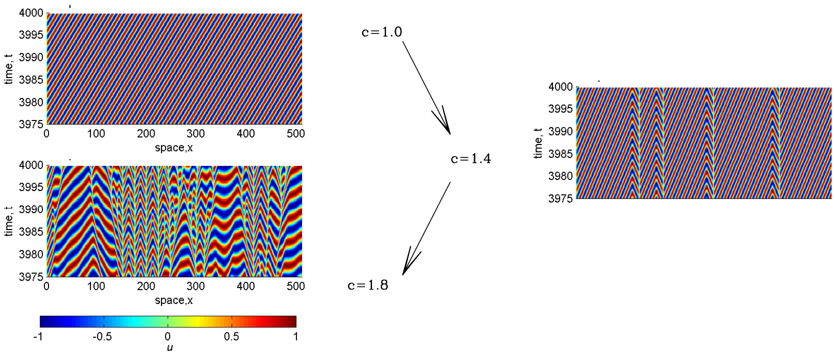
Outline

Methodology



Objective: a more detailed understanding of the transition from pattern to chaos

Methodology



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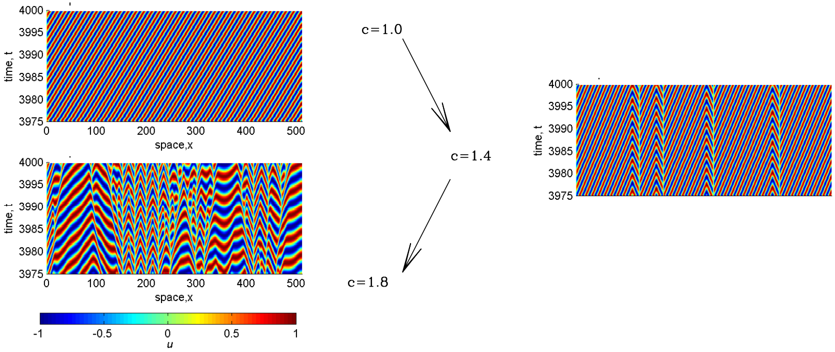
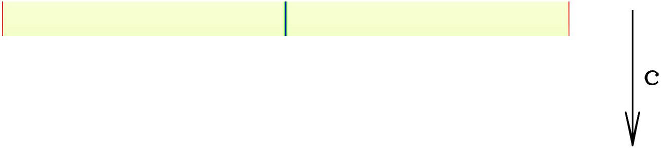


Approach: increase *c* very slowly: by 0.001 every 3000 time units

Slowly Increasing the Parameter c



$|A| = 0.0$  $|A| = 1.0$

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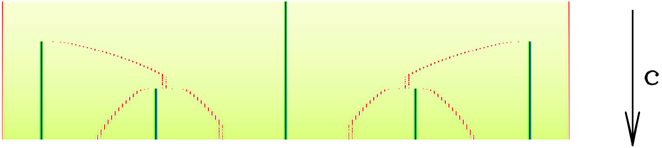

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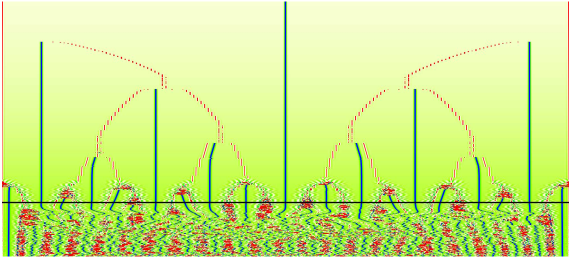

$\downarrow c$

change in
absolute stability

$|A| = 0.0$  $|A| = 1.0$

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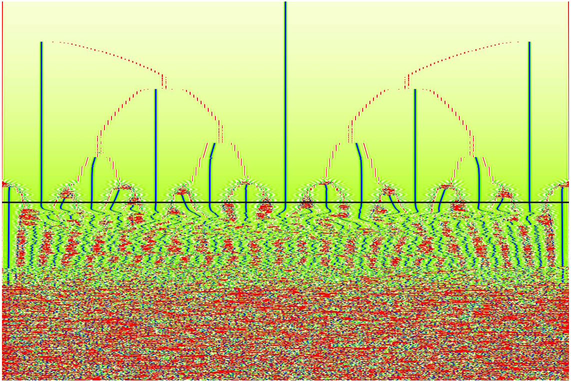

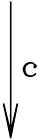

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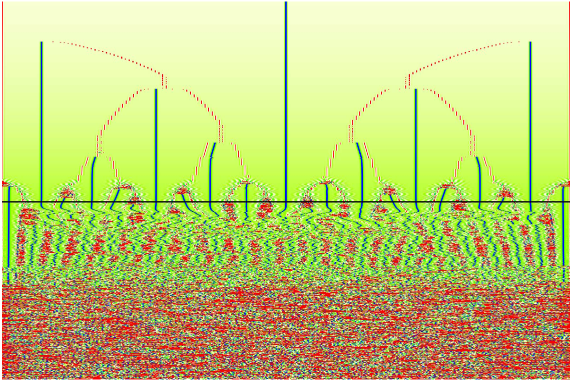
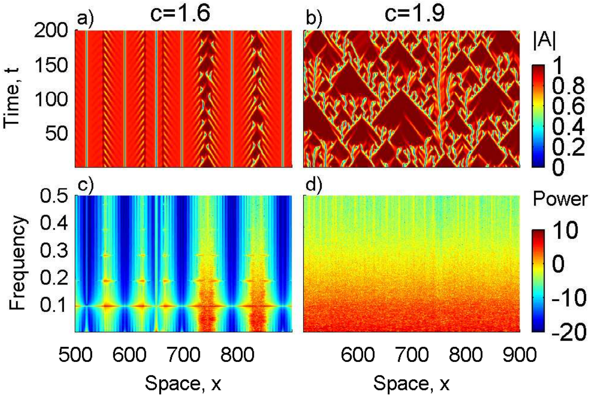
Localised and Global Chaos



Chaos changes from local to global as c is increased above the absolute stability threshold.

Localised and Global Chaos



Outline

Summary and Conclusions

- Wavetrain patterns are a generic feature of oscillatory systems
- The transition from wavetrain patterns to chaos occurs via patterns of sources and sinks
- There is a discrete family of possible source-sink separations
- There is a clear, structured transition in the source-sink pattern, leading to chaos
- The chaos is initially localised, and gradually becomes global

This work is in collaboration with:

Matthew Smith

(Microsoft Research Ltd., Cambridge)

Jens Rademacher

(CWI, Amsterdam)

Publications

M.J. Smith, J.D.M. Rademacher, J.A. Sherratt:
Absolute stability of wavetrains can explain spatiotemporal dynamics in reaction-diffusion systems of lambda-omega type.
SIAM J. Appl. Dyn. Systems 8, 1136-1159 (2009).

J.A. Sherratt, M.J. Smith, J.D.M. Rademacher:
Patterns of sources and sinks in the complex Ginzburg-Landau equation with zero linear dispersion.
SIAM J. Appl. Dyn. Systems 9, 883-918 (2010).

J.A. Sherratt, M.J. Smith:
Transition to spatiotemporal chaos via branching shocks and holes. Submitted.

List of Frames