Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns in Semi-Deserts

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This talk can be downloaded from my web site
www.ma.hw.ac.uk/~jas
Outline

1. Ecological Background
2. Pattern Formation in a Mathematical Model
3. Pattern Existence and Stability
4. Predictions of Pattern Wavelength vs Slope
5. Conclusions and References
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Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns

Banded Vegetation on Slopes
Data on Wavelength vs Slope

Vegetation Patterns

Pattern Formation in a Mathematical Model
Pattern Existence and Stability
Predictions of Pattern Wavelength vs Slope
Conclusions and References

Outline
Desert ecosystems provide a classic example of self-organised pattern formation.

W National Park, Niger
Average patch width is 50 m
Desert ecosystems provide a classic example of self-organised pattern formation.
Desert ecosystems provide a classic example of self-organised pattern formation.

Data from Burkina Faso
Rietkerk et al
Plant Ecology 148: 207-224, 2000

More plants $\Rightarrow$ more roots and organic matter in soil
$\Rightarrow$ more infiltration of rainwater
Desert ecosystems provide a classic example of self-organised pattern formation.
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Desert ecosystems provide a classic example of self-organised pattern formation.
On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia
(Western New South Wales)

Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia, Mexico and S-W USA.
On slopes, run-off occurs in one direction only, giving striped patterns parallel to the contours.

Bushy vegetation in Niger

Mitchell grass in Australia

(Western New South Wales)

Wavelength can be measured via remote sensing.
Data on Wavelength vs Slope

Data from sub-Saharan Africa and S-W USA shows that the wavelength of banded vegetation patterns is negatively correlated with slope.

Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)
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Data from Nevada, USA (Pelletier et al, J. Geophys. Res. 117: F04026, 2012)

How does this compare with predictions of mathematical models?
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Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns
Mathematical Model of Klausmeier

\[
\frac{\partial u}{\partial t} = wu^2 - Bu + \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial w}{\partial t} = A - w - wu^2 + \nu \frac{\partial w}{\partial x} + D \frac{\partial^2 w}{\partial x^2}
\]

(Klausmeier, Science 284: 1826-8, 1999)
Mathematical Model of Klausmeier

\[
\frac{\partial u}{\partial t} = \text{plant growth} \left( wu^2 \right) - \text{plant loss} \left( Bu \right) + \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial w}{\partial t} = \text{average rainfall} \left( A \right) - \text{evaporation} \left( w \right) - \text{uptake by plants} \left( wu^2 \right) + \nu \frac{\partial w}{\partial x} + D \frac{\partial^2 w}{\partial x^2}
\]

The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.
The nonlinearity in water uptake occurs because the presence of plants increases water infiltration into the soil.
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water profiles](image_url)

**Vegetation, \( u \)**

**Water, \( w \)**

**Distance uphill, \( x \)**
Typical Solution of the Model

Graph showing the typical solution of the model with two curves: one for vegetation density (u) and one for water content (w) over distance uphill (x).
Typical Solution of the Model

The typical solution of the mathematical model of Klausmeier shows oscillatory patterns of vegetation density and water content along the distance uphill. The vegetation density, $u$, and water content, $w$, are plotted against distance uphill, $x$.
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, u

Water, w

Distance uphill, x
Typical Solution of the Model
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water profiles against distance uphill.]

- Vegetation, $u$
- Water, $w$

Distance uphill, $x$
Typical Solution of the Model

- Vegetation, u
- Water, w
- Distance uphill, x

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Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill.](image-url)
Typical Solution of the Model

![Graph showing vegetation and water patterns](image-url)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

![Graph showing the typical solution of the model with two plots: one for vegetation and one for water content vs distance uphill. The vegetation plot shows oscillations with a wavelength, while the water content plot shows a more complex pattern.](image-url)
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill.](image-url)
Typical Solution of the Model

The typical solution of the model for vegetation $u$ and water $w$ as functions of distance uphill $x$. The plots illustrate oscillatory patterns, with peaks and troughs representing varying vegetation and water levels across the landscape.

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Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns
Typical Solution of the Model

The graph shows the typical solution of the model with two subplots:

1. **Vegetation, u**: The top subplot represents the vegetation distribution over distance uphill, x. The graph fluctuates periodically, indicating the presence of vegetation patterns.

2. **Water, w**: The bottom subplot represents the water distribution over distance uphill, x. Similar to the vegetation subplot, it shows periodic fluctuations, suggesting the interaction between vegetation and water dynamics.

These visualizations are key to understanding the ecological background and the mathematical model's predictions on pattern formation and stability.
Typical Solution of the Model

![Graph showing pattern formation in a mathematical model. The graph plots vegetation and water content against distance uphill.]
Typical Solution of the Model

![Graph showing vegetation and water distribution over distance uphill.](image)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model
Typical Solution of the Model

![Graph showing vegetation and water profiles vs distance uphill]

- Vegetation, $u$
- Water, $w$
- Distance uphill, $x$

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Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns
Typical Solution of the Model

![Graph showing typical solution of the model with vegetation and water levels as functions of distance uphill.](Image)
Typical Solution of the Model

Vegetation, \( u \)

Water, \( w \)

Distance uphill, \( x \)
Typical Solution of the Model
Typical Solution of the Model
Typical Solution of the Model

![Graph showing vegetation and water distribution along a gradient](image-url)
Typical Solution of the Model

![Graph showing the typical solution of the model with two plots: one for vegetation (u) and another for water (w) vs distance uphill (x).]
Typical Solution of the Model

![Graph showing vegetation and water profiles over distance uphill, x.](image-url)
Typical Solution of the Model

![Graph showing vegetation and water profiles against distance uphill.](image-url)
Typical Solution of the Model

![Graph showing typical solution of the model](image-url)
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

**Vegetation, u**

![Graph showing vegetation pattern over distance](image)

**Water, w**

![Graph showing water pattern over distance](image)

Distance uphill, x
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, $u$

Water, $w$

Distance uphill, $x$
Typical Solution of the Model

Vegetation, \( u \)

Water, \( w \)

Distance uphill, \( x \)
Using Mathematical Models to Infer the Historical Origin of Vegetation Patterns

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Homogeneous Steady States

For all parameter values, there is a stable “desert” steady state $u = 0$, $w = A$
Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0, \ w = A$
- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations
Homogeneous Steady States

- For all parameter values, there is a stable “desert” steady state $u = 0, w = A$

- When $A \geq 2B$, there are also two non-trivial steady states, one of which is unstable to homogeneous perturbations

- The other steady state $(u_s, w_s)$ is stable to homogeneous perturbations but can be unstable to inhomogeneous perturbations $\Rightarrow$ pattern formation
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

The expected wavelength \(\leftrightarrow\) the frequency of noise giving the fastest growth rate.
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

This implies a positive correlation between wavelength and slope, contrary to data.
The standard approach to predicting pattern wavelength is to apply a small perturbation to the steady state \((u_s, w_s)\).

“To date, no model of vegetation band formation has been shown to reproduce this inverse relationship between spacing and slope.” (Pelletier et al, J. Geophys. Res. 117, F04026, 2012)
“Most unstable frequency” assumes that patterns develop from a pre-existing unstable uniform state.

Vegetation patterns develop via

- degradation of uniform vegetation
- colonisation of bare ground
The patterns move at constant shape and speed
\[ u(x, t) = U(z), \quad w(x, t) = W(z), \quad z = x - ct \]

\[
\begin{align*}
& d^2 U/dz^2 + c \frac{dU}{dz} + WU^2 - BU = 0 \\
& D \frac{d^2 W}{dz^2} + (\nu + c) \frac{dW}{dz} + A - W - WU^2 = 0
\end{align*}
\]

The patterns are periodic (limit cycle) solutions of these equations
Bifurcation Diagram for Travelling Wave Equations

- Travelling Wave Equations
- Bifurcation Diagram for Travelling Wave Equations
- When do Patterns Form?
- Pattern Stability
- Variations in Rainfall: Hysteresis

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Using Mathematical Models to Infer the Historical Origin of Vegetation
Bifurcation Diagram for Travelling Wave Equations

- Traveling Wave Equations
- Bifurcation Diagram for Travelling Wave Equations
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Using Mathematical Models to Infer the Historical Origin of Vegetation
When do Patterns Form?

- Min rainfall for patterns
- Turing bifurcation
- Locus of homoclinic solns
- Parameter region giving patterns
- Locus of Hopf bifurcation points

Min rainfall for uniform veg

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Using Mathematical Models to Infer the Historical Origin of Vegetation
Not all of the possible patterns are stable as solutions of the model equations.
Pattern Stability: The Key Result

**Key Result**

Many of the possible patterns are unstable and thus will never be seen.

However, for a wide range of rainfall levels, there are multiple stable patterns.
The existence of multiple stable patterns suggests the possibility of hysteresis.

Domain length 150, periodic bc's
Data on the Effects of Changing Rainfall

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Pattern wavelength is history-dependent.

We must focus on the onset of patterning.

Degradation of uniform vegetation

Colonisation of bare ground
Wavelength vs Slope for Degradation of Uniform Vegetation

Distance uphill, x

Time, t

0 200 750

0 12

(a)

Using Mathematical Models to Infer the Historical Origin of Vegetation
Wavelength vs Slope for Degradation of Uniform Vegetation

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For realistic parameters, wavelength increases with slope, contrary to data.
When Does Vegetation Colonise Bare Ground?

Very low rainfall: an isolated vegetation patch dies out

Slightly larger rainfall: both edges move uphill

Larger rainfall: the patch expands both uphill and downhill
When Does Vegetation Colonise Bare Ground?

- **Low rainfall**
- **High rainfall**

```
Distance uphill, x
```
```
Time, t
```

**Graphs:**
- (a) Low rainfall
- (b) High rainfall
When Does Vegetation Colonise Bare Ground?

The key critical case is when the downhill edge is stationary.

![Graph showing vegetation density over distance and time.](image-url)
Wavelength vs Slope for Colonisation

![Graph showing the relationship between wavelength, slope, and rainfall for vegetation colonisation.](image)
Wavelength vs Slope for Colonisation

![Graph showing the relationship between rainfall, slope, and vegetation patterns.](image-url)
Wavelength vs Slope for Colonisation

(b) Patterns

(a) Uniform vegetation

Distance uphill, x

Time, t

Rainfall

Slope

When Does Vegetation Colonise Bare Ground?

How to Predict Pattern Wavelength

Wavelength vs Slope for Colonisation

Predictions of Pattern Wavelength vs Slope

When Does Vegetation Colonise Bare Ground?

Using Mathematical Models to Infer the Historical Origin of Vegetation

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Wavelength decreases with slope, in agreement with data.
How to Predict Pattern Wavelength

Pattern wavelength is history-dependent
How to Predict Pattern Wavelength

Pattern wavelength is history-dependent

![Graph showing pattern wavelength vs rainfall](image-url)
Pattern wavelength is history-dependent

We must focus on the onset of patterning

Degradation of uniform vegetation

Colonisation of bare ground
Outline

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Patterned vegetation is widespread in the Sahel

Several studies of banded vegetation show wavelength as slope
The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.

The Sahel was relatively humid in the 16th and 17th centuries.

There is no direct data on rainfall before c. 1850

Proxy data: (i) lake levels, esp. Lake Chad; (ii) historical chronologies, e.g. Bornu Empire; (iii) memories of local peoples.
Rainfall History in the Sahel

- The Sahara and Sahel have been arid for about 5000 years, but the level of aridity has varied significantly.
- The Sahel was relatively humid in the 16th and 17th centuries.
- Reasonable assumption: areas with vegetation patterns today had uniform vegetation at the end of the 17th century.
- Since wavelength decreases with slope, my results imply that vegetation must have died out and then recolonised since the end of the 17th century.
- The most severe drought since 1700 was c. 1738-1756. So today’s vegetation patterns result from recolonisation since 1760.
Conclusions

Wavelength is positively correlated with slope ⇒ vegetation pattern originated by degradation of uniform vegetation

Wavelength is negatively correlated with slope ⇒ vegetation pattern originated by colonisation of bare ground

Main message: combined wavelength–slope data is much more valuable than wavelength data alone.
Remote Sensing of Wavelength and Elevation

Google Earth: online satellite images, min. 15 m resolution

Get the world’s geographic information at your fingertips

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Explore the world from anywhere.

Whether on your computer or on the go, see the world the same way you’re used to seeing it, in 3D.
Remote Sensing of Wavelength and Elevation

**WorldDEM:** online elevation data, 12 m resolution
References


Ecological Background

Pattern Formation in a Mathematical Model

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Example: The African Sahel

Rainfall History in the Sahel

Conclusions

References

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