Predicting the Wavelength of Vegetation Patterns using Mathematical Models

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This talk can be downloaded from my web site

www.ma.hw.ac.uk/∼jas

Outline

1. Ecological Background
2. Detailed Calculation of Possible Wavelengths
3. Effects of Changing Rainfall Levels
4. Wavelength Selection: Two Examples
5. Further Reading

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Vegetation Patterns

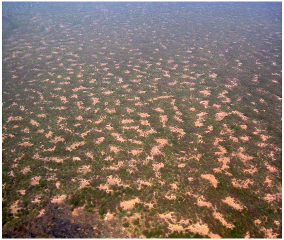
High rainfall:
uniform vegetation

Very low rainfall:
no vegetation

Vegetation Patterns

High rainfall:
uniform vegetation

Low rainfall:
patterned vegetation



W National Park, Niger
Average patch width 50 m

Very low rainfall:
no vegetation

Vegetation Patterns

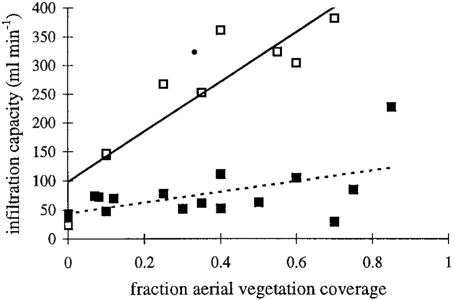
Bushy vegetation in Niger

Mitchell grass in Australia

(Western New South Wales)

- Banded vegetation patterns are found on gentle slopes in semi-arid areas of Africa, Australia and Mexico
- Plants vary from grasses to shrubs and trees

Why Do Plants Form Patterns?



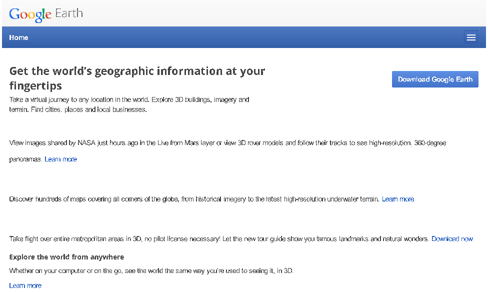
Data from Burkina Faso

Rietkerk et al
Plant Ecology 148: 207-224, 2000

More plants $\Rightarrow$ more roots and organic matter in soil
$\Rightarrow$ more infiltration of rainwater

Pattern Wavelength: A Quantitative Statistic

- The wavelength of vegetation bands is probably the most accessible quantitative statistic for vegetation patterns.



- Our topic: how to predict pattern wavelength using mathematical models

Mathematical Model of Klausmeier

Rate of change of water = Rainfall – Evaporation – Uptake by plants + Flow downhill

Rate of change plant biomass = Growth, proportional to water uptake – Mortality + Random dispersal

$$\partial w/\partial t = A - w - wu^2 + \nu \partial w/\partial x$$

$$\partial u/\partial t = wu^2 - Bu + \partial^2 u/\partial x^2$$

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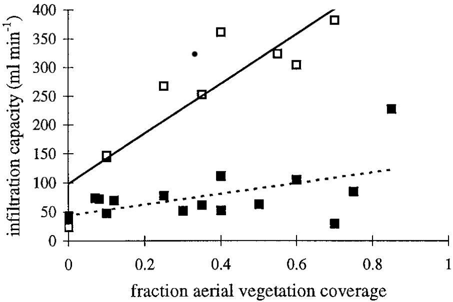

$$wu^2 = w \cdot u \cdot \left(\begin{array}{c}\text{infiltration}\\ \text{rate}\end{array}\right)$$

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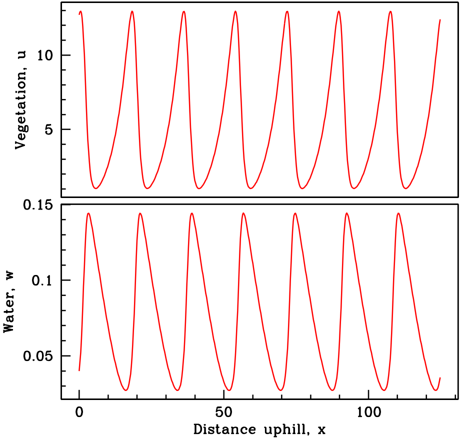
Parameters: A: rainfall B: plant loss ν: slope

Typical Solution of the Model



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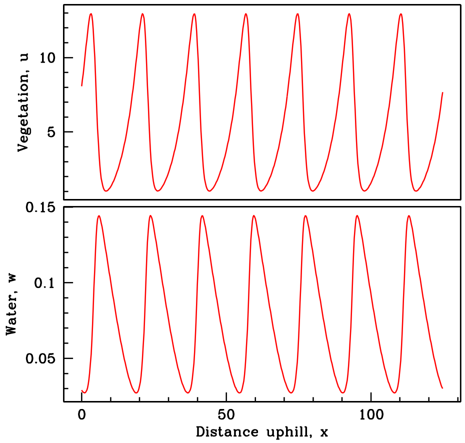

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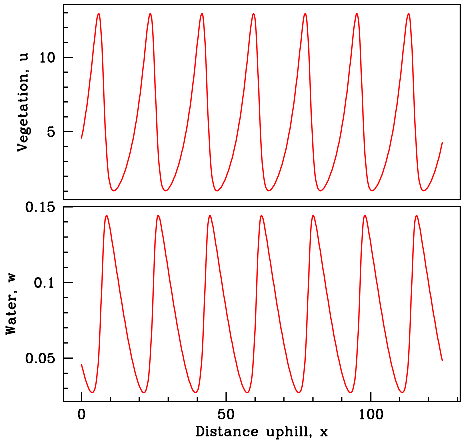

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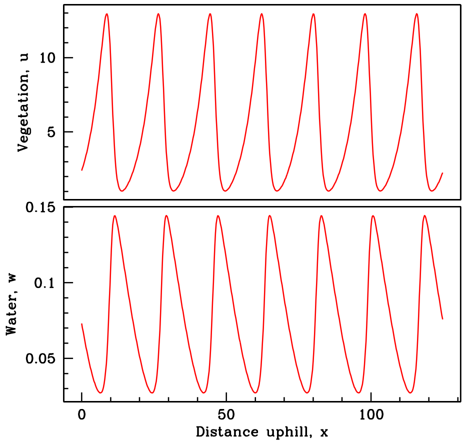

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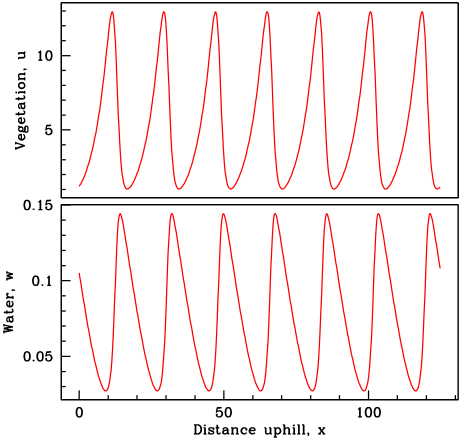

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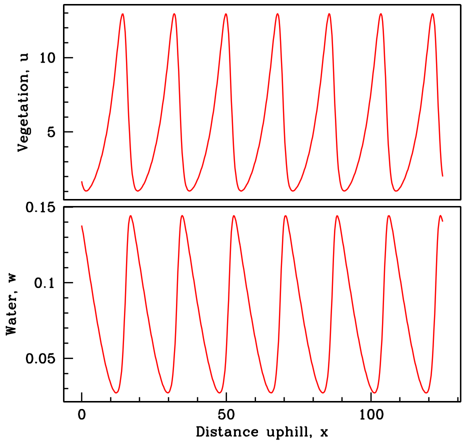

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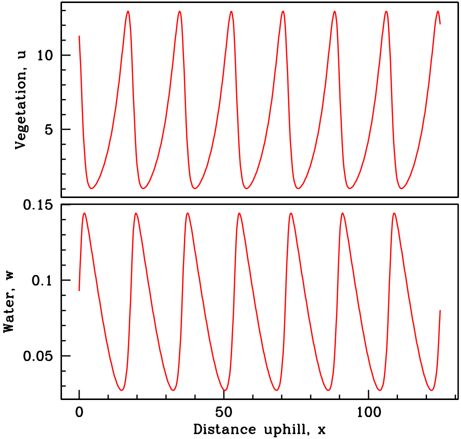

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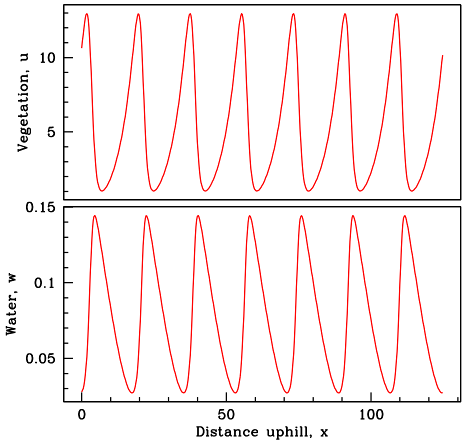

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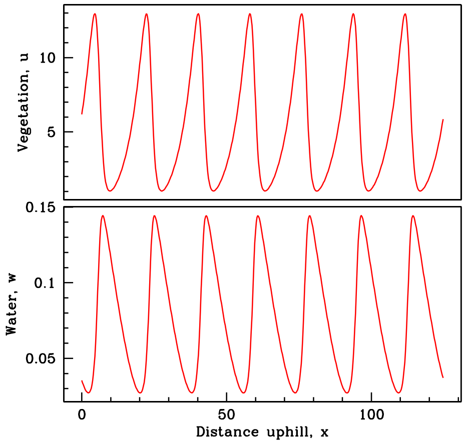

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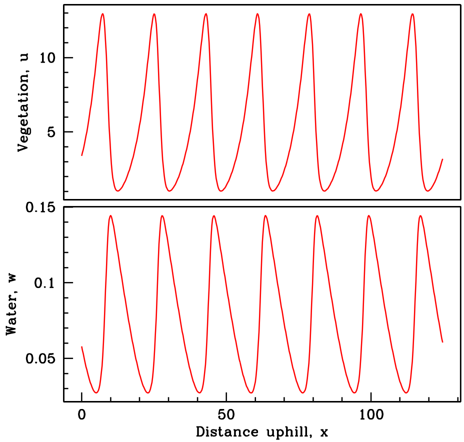

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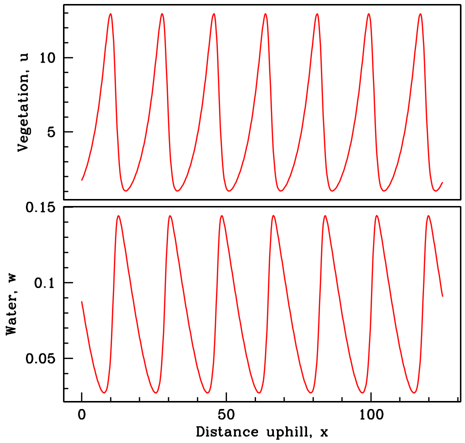

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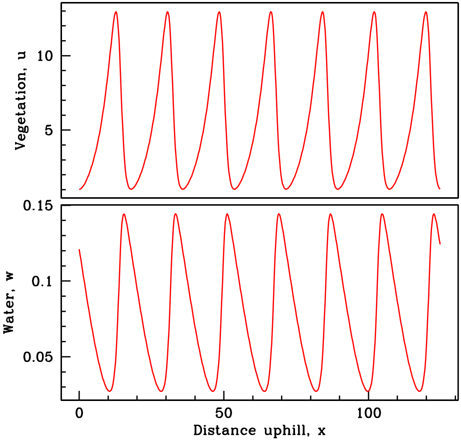

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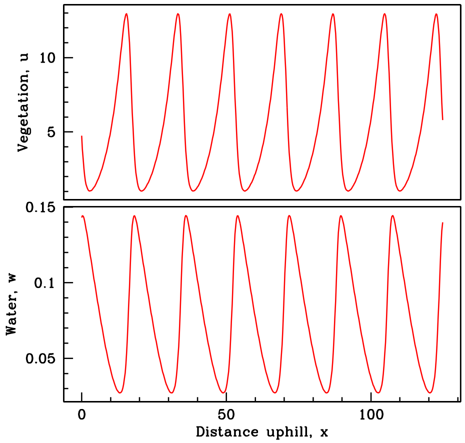

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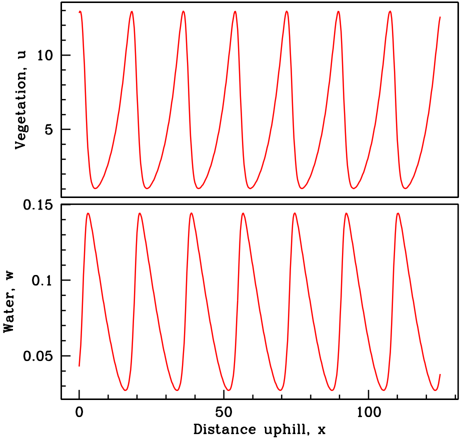

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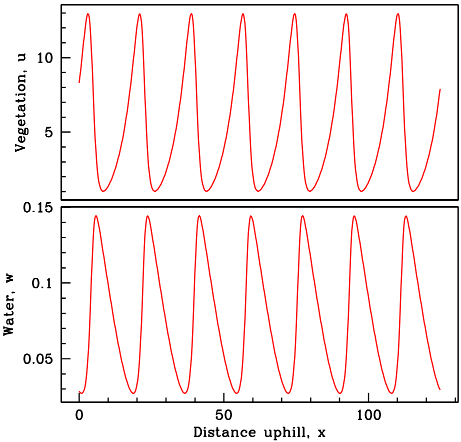

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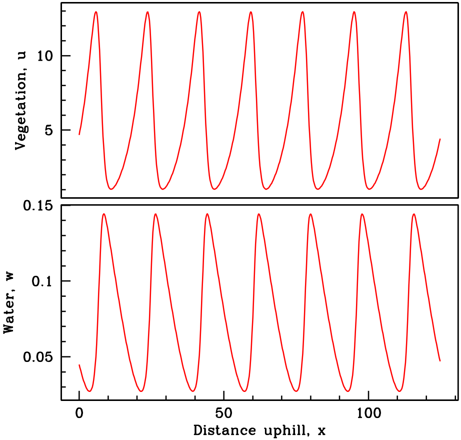

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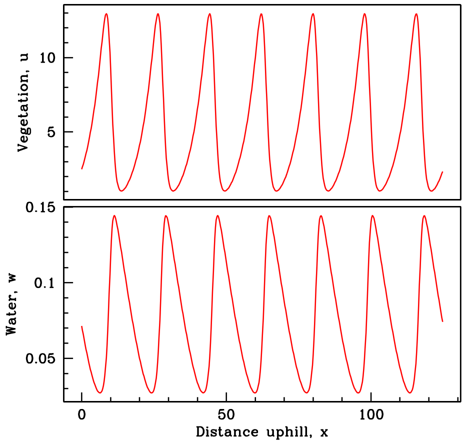

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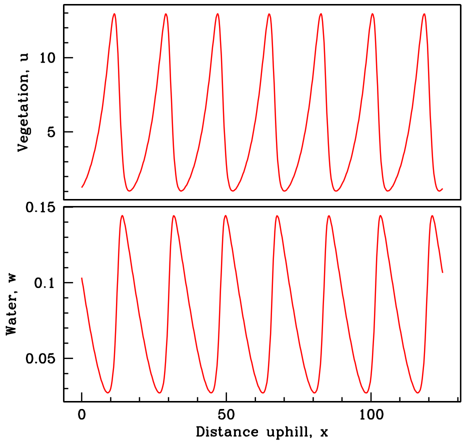

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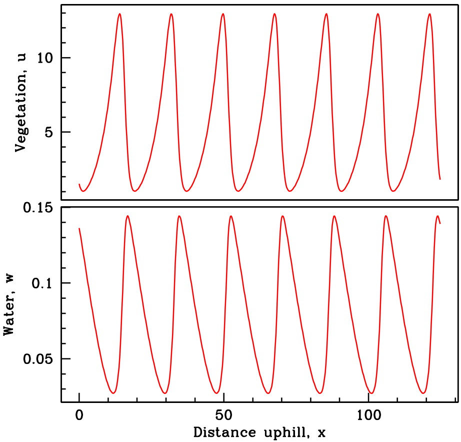

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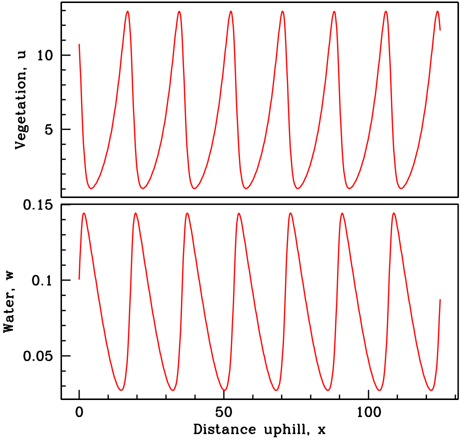

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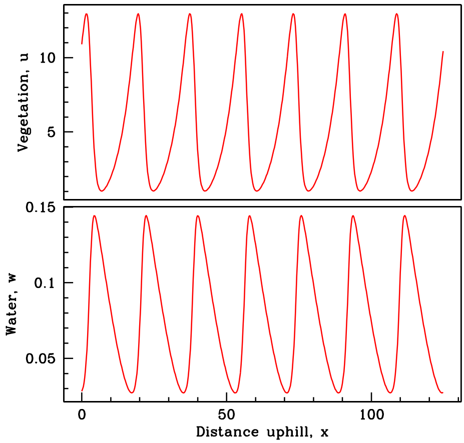

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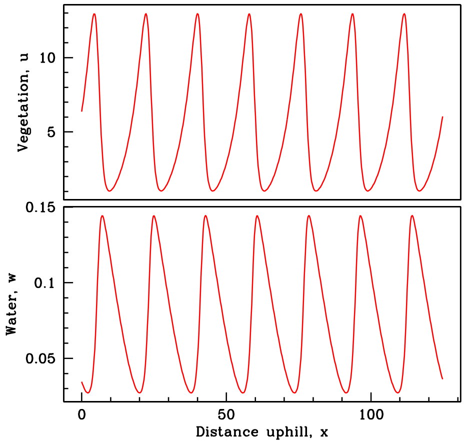

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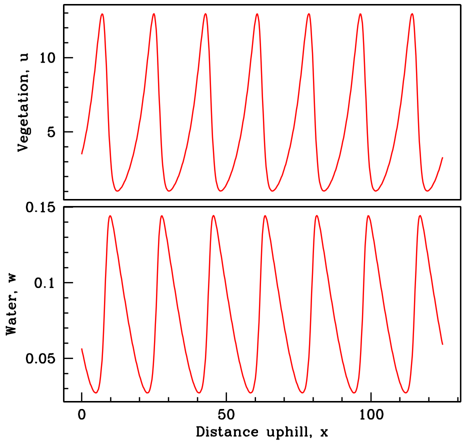

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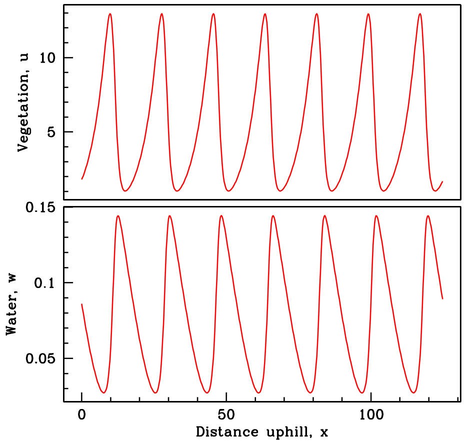

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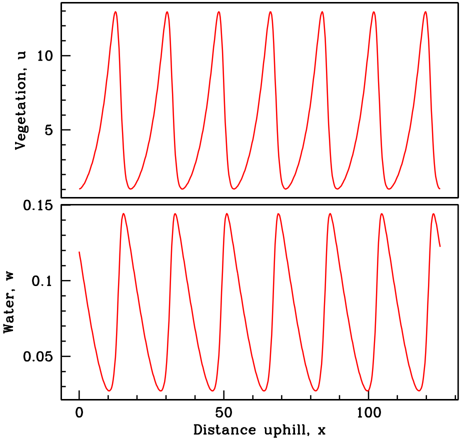

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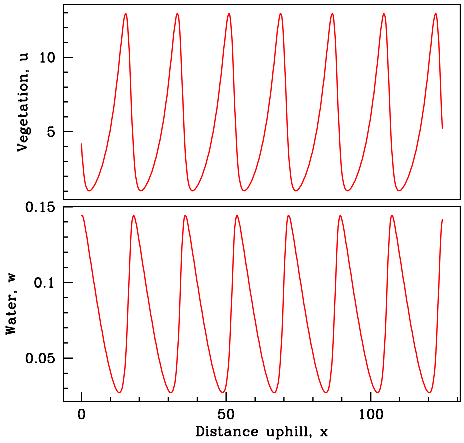

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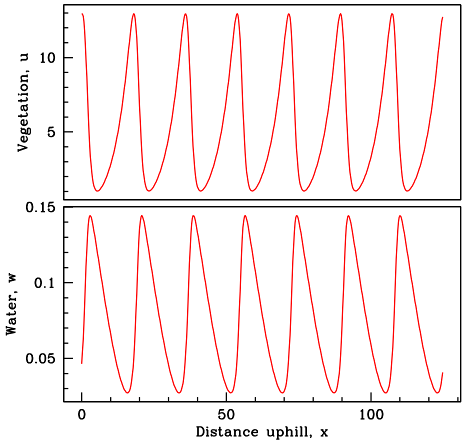

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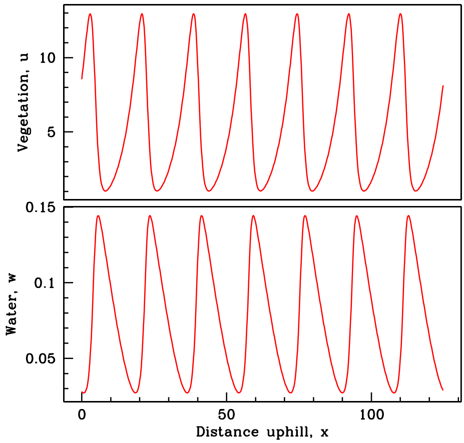

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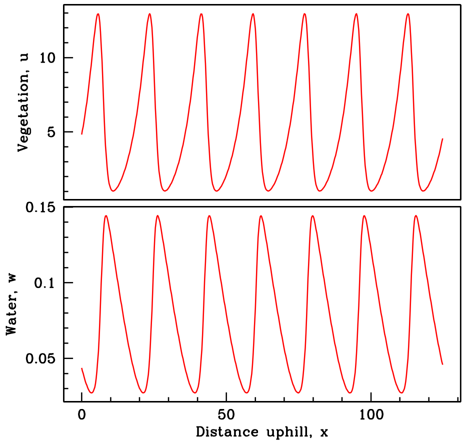

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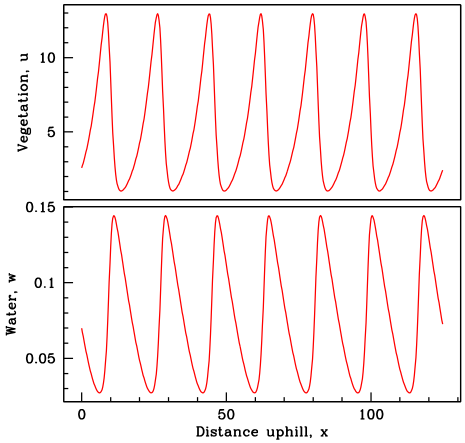

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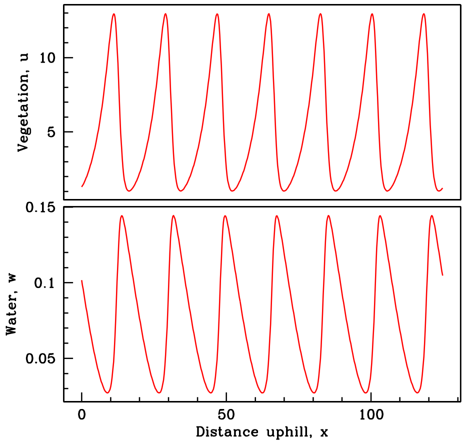

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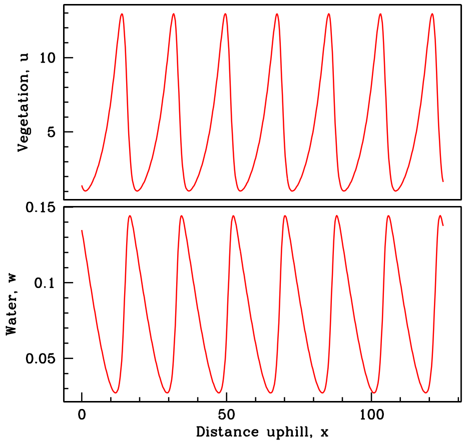

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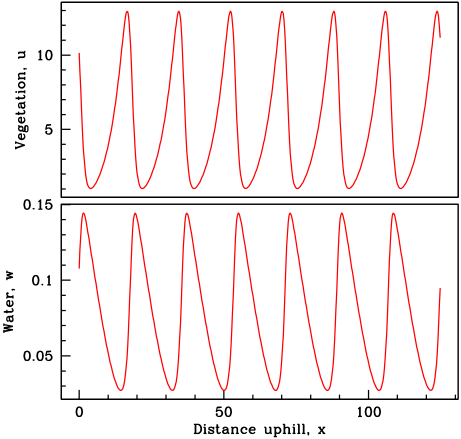

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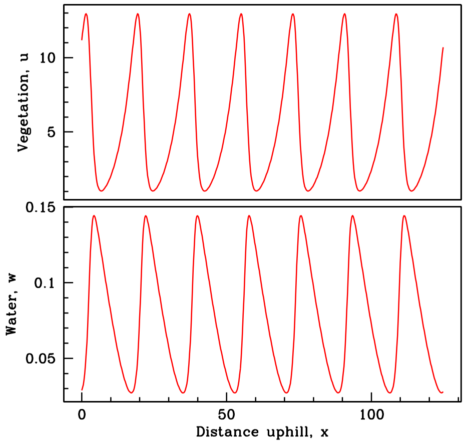

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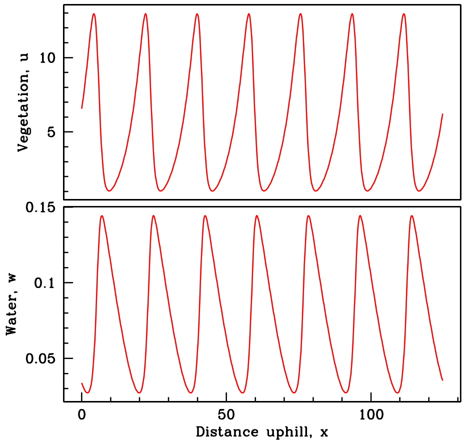

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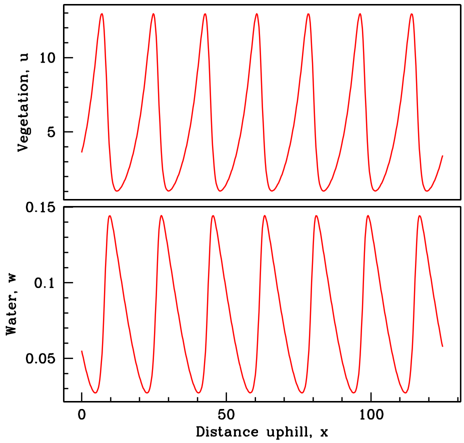

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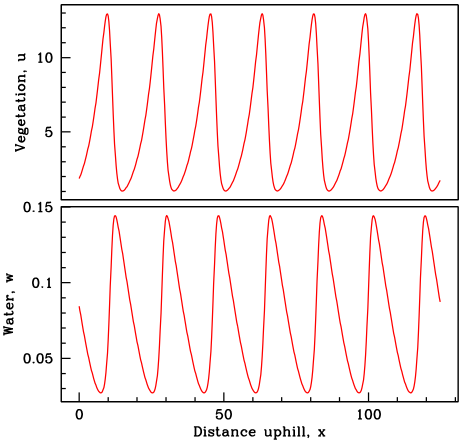

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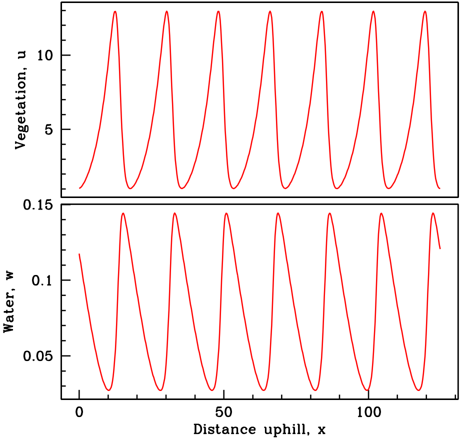

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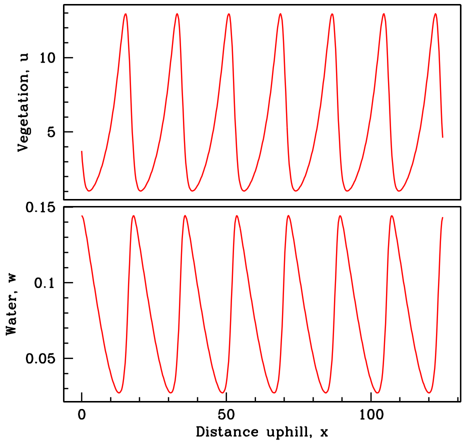

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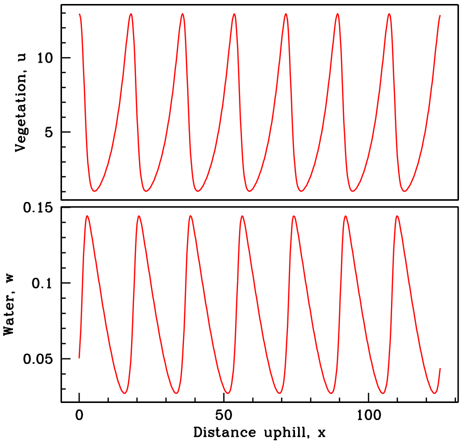
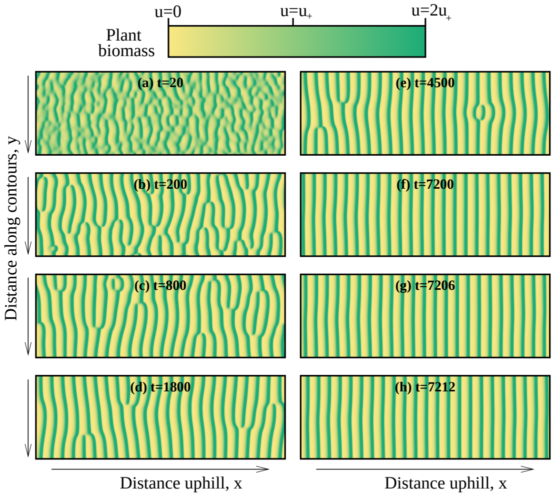


Typical Solution of the Model in Two Dimensions



Homogeneous Steady States

- The starting point for mathematical study of vegetation patterns is to determine homogeneous steady states
- Recall the model equations:

$$\begin{aligned}\partial w/\partial t &= A - w - wu^2 + \nu\partial w/\partial x \\ \partial u/\partial t &= wu^2 - Bu + \partial^2 u/\partial x^2\end{aligned}$$

For homogeneous steady states $A = w + wu^2, \quad Bu = wu^2$
$\Rightarrow u = 0, w = A \quad$ or $\quad uw = B, A - w - B^2/w = 0$

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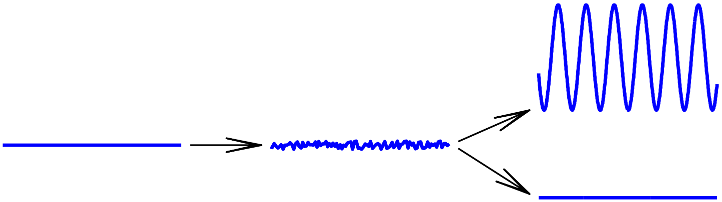
- For all parameter values, there is a stable "desert" steady state $u = 0$, $w = A$
- When $A \geq 2B$, there are also two non-trivial steady states satisfying $w^2 - Aw + B^2 = 0$, $u = B/w$

Stability of Homogeneous Steady States I

- Patterns can arise when a homogeneous steady state is unstable



- To determine stability:
  (i) Linearise the model about the steady state

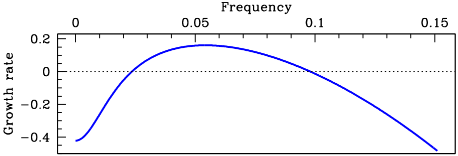
$$
\begin{aligned}
\partial \hat{w}/\partial t &= -(1+u_s^2)\hat{w} - 2B\hat{u} + \nu \partial \hat{w}/\partial x \\
\partial \hat{u}/\partial t &= u_s^2 \hat{w} + B\hat{u} + \partial^2 \hat{u}/\partial x^2 \\
& (\hat{w} = w - w_s, \quad \hat{u} = u - u_s)
\end{aligned}
$$

Stability of Homogeneous Steady States II

- To determine stability:
  (i) Linearise the model about the steady state
  (ii) Consider sinusoidal solutions:

$$(\hat{u}, \hat{w}) = \underbrace{(u_s, w_s)}_{\text{steady state}} + \underbrace{(\tilde{u}, \tilde{w})}_{\text{small constants}} \cdot \underbrace{e^{\lambda t}}_{\text{growth or decay}} \cdot \underbrace{\begin{matrix}\sin\\ \cos\end{matrix}(2\pi f x)}_{\text{Fourier mode frequency } f}$$

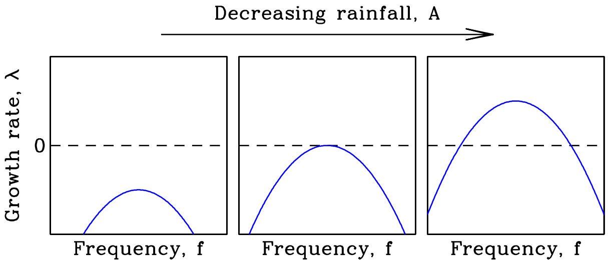
  Substituting into the model gives $\lambda(f)$ ("dispersion relation")



  (iii) The steady state is unstable if $\mathrm{Re}\,\lambda(f) > 0$ for some f

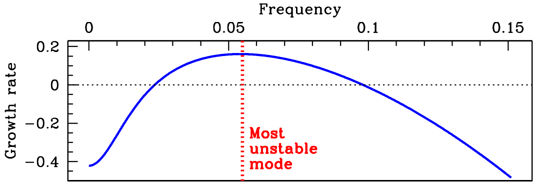
Stability of Homogeneous Steady States III

The steady state loses stability as rainfall A is decreased



Stability of Homogeneous Steady States IV

In many cases the pattern wavelength corresponds to the most unstable frequency (wavelength=1/frequency).



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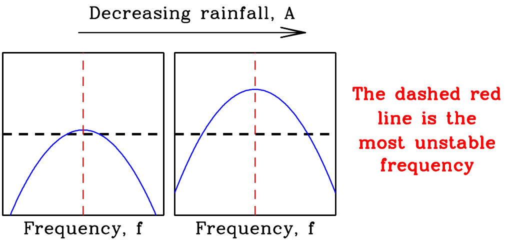

When A is just small enough for patterns, the most unstable frequency gives a reliable guide to wavelength. For smaller A, wavelength selection is more complicated.

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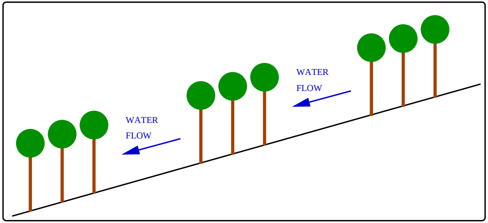

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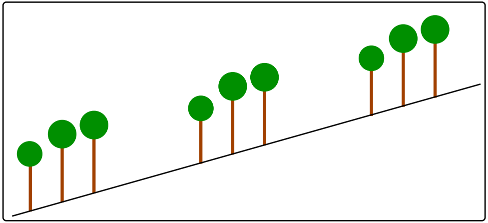

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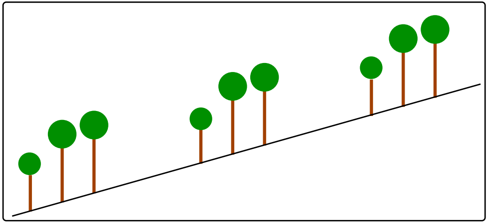

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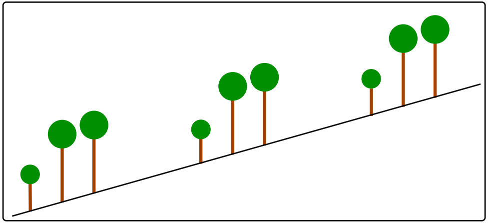

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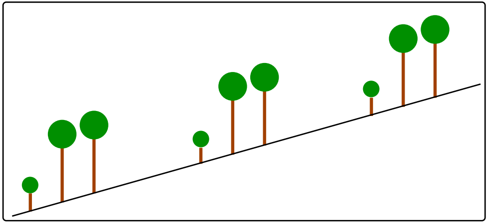

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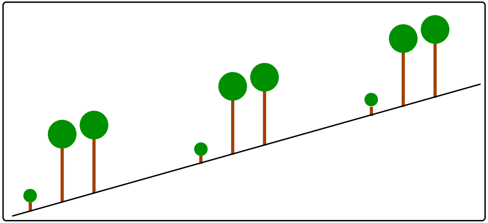

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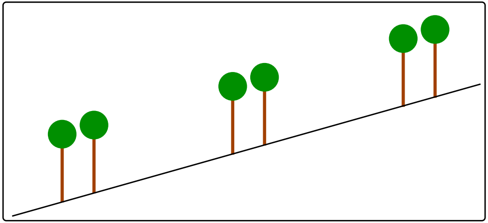

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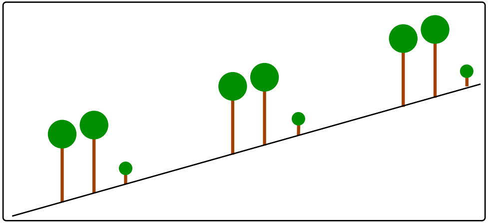

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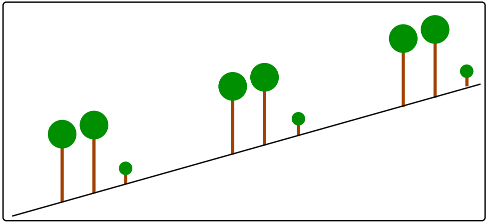

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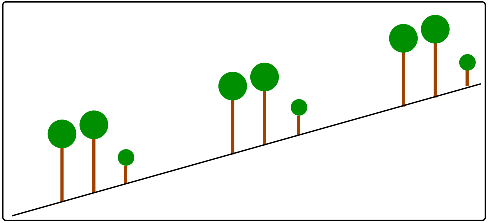

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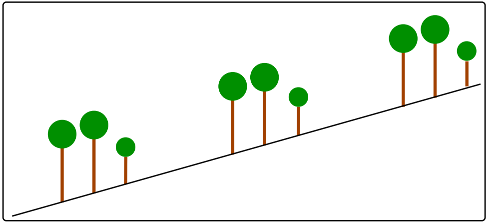

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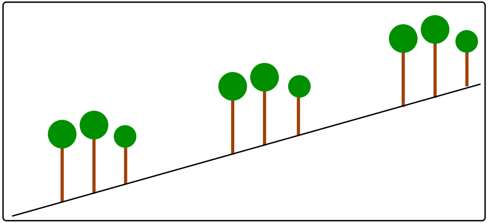

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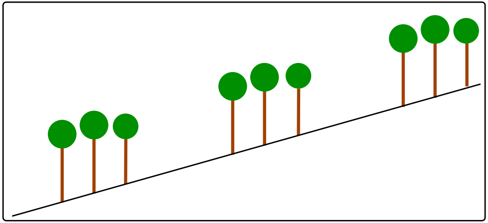

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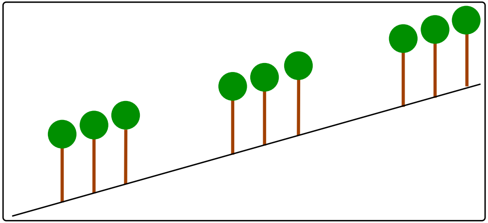

Travelling Wave Equations

The patterns move at constant shape and speed
$\Rightarrow u(x,t) = U(z),\ w(x,t) = W(z),\ z = x - ct$

$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

Patterns are periodic (limit cycle) solutions of these equations

Calculation of all possible patterns is done in three steps.

Step 1: Calculate the Locus of Hopf Bifurcations

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$$\begin{aligned} d^2U/dz^2 + c\,dU/dz + WU^2 - BU &= 0 \\ (\nu + c)dW/dz + A - W - WU^2 &= 0 \end{aligned}$$

Patterns lie on a solution branch that starts at a Hopf bifurcation point (in most cases)

Step 1: Calculate the locus of Hopf bifurcations in the *A*–*c* plane

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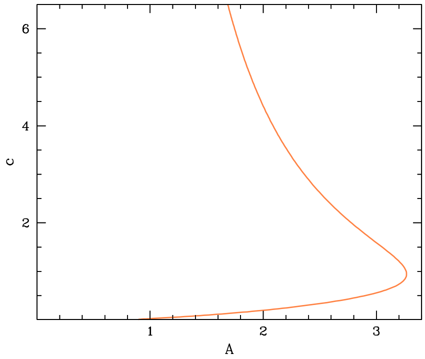
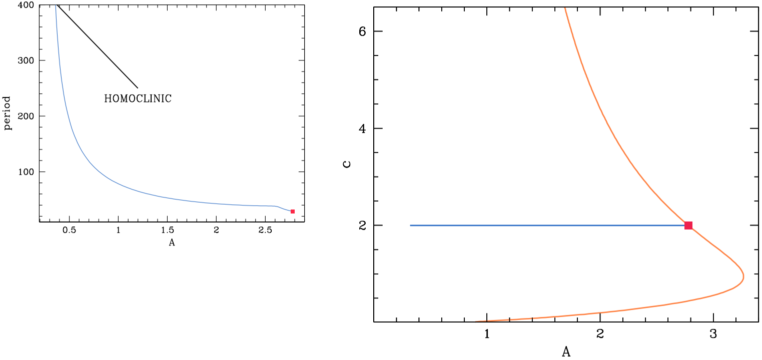
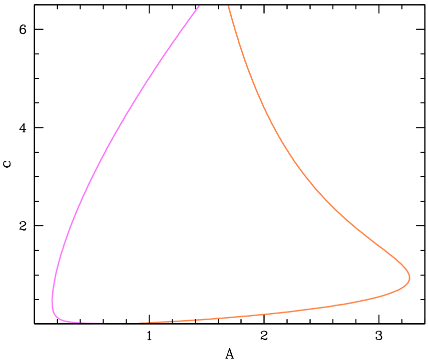


Step 2: Calculate Some Branches of Pattern Solutions



Step 2: Calculate some branches of pattern solutions. These end at a homoclinic solution

Step 3: Calculate the Locus of Homoclinic Solutions



Step 3: Calculate the homoclinic locus, approximated by the locus of patterns of a fixed, very long period

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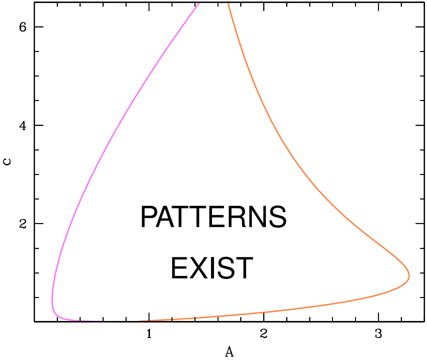

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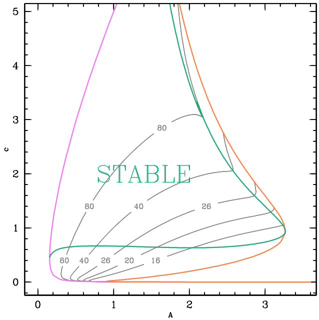
Pattern Stability

Not all of the possible patterns are stable as solutions of the model equations.



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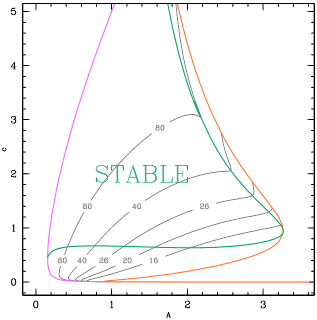

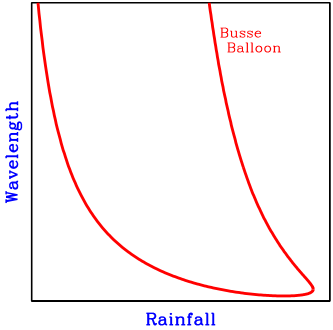

The parameter region with stable patterns is known as the “Busse balloon”.

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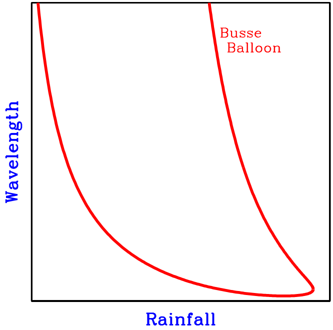
Key Result

For many rainfall levels, there are stable patterns with a range of wavelengths.



The Eigenvalue Problem

PDE model:
$$u_t = u_{zz} + cu_z + f(u,w)$$
$$w_t = \nu w_z + cv_z + g(u,w)$$

Periodic wave satisfies:
$$0 = U_{zz} + cU_z + f(U,W)$$
$$0 = \nu W_z + cW_z + g(U,W)$$

Consider
$$u(z,t) = U(z) + e^{\lambda t}\overline{u}(z) \quad \text{with } |\overline{u}| \ll |U|$$
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$\Rightarrow$ Eigenfunction eqn:
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Boundary conditions:
$$\overline{u}(0) = \overline{u}(L)e^{i\gamma} \quad (0 \le \gamma < 2\pi)$$
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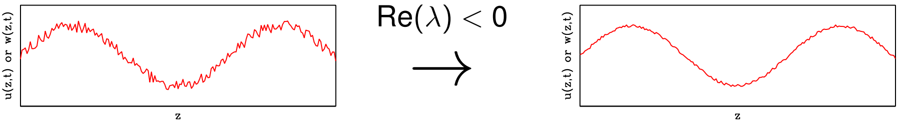

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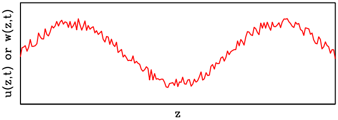
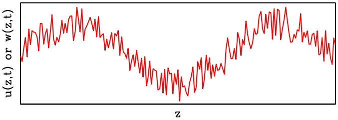


$\text{Re}(\lambda) > 0$

$\longrightarrow$



Numerical Calculation of Eigenvalue Spectrum

(based on Jens Rademacher, Bjorn Sandstede, Arnd Scheel Physica D 229 166-183, 2007)

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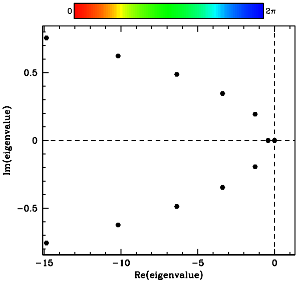

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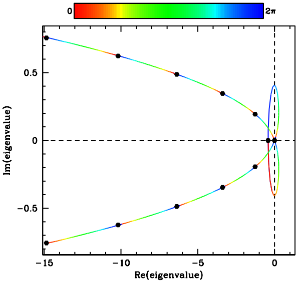

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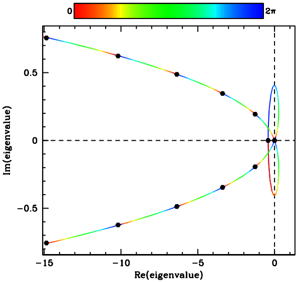

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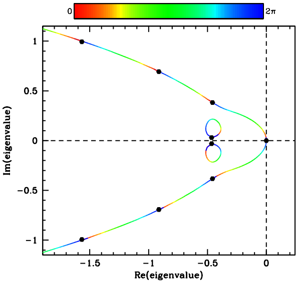
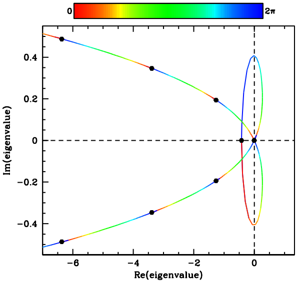


STABLE

Eckhaus instability



UNSTABLE

This gives the eigenvalue spectrum, and hence (in)stability

Stability in a Parameter Plane

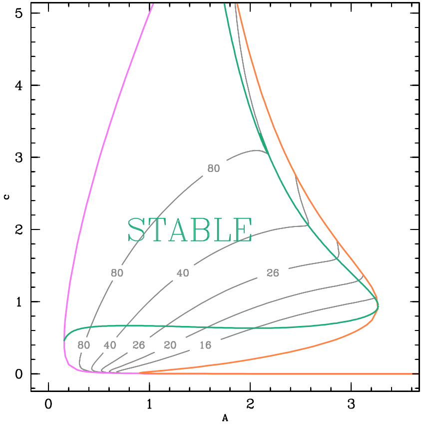
By following this procedure at each point on a grid in parameter space, regions of stability/instability can be determined.

In fact, stable/unstable boundaries can be computed accurately by numerical continuation of the point at which

$$\mathrm{Re}\lambda = \mathrm{Im}\lambda = \gamma = \partial^2 \mathrm{Re}\lambda / \partial \gamma^2 = 0$$

(Eckhaus instability point)

Stability in a Parameter Plane



Outline

1 Ecological Background

2 Detailed Calculation of Possible Wavelengths

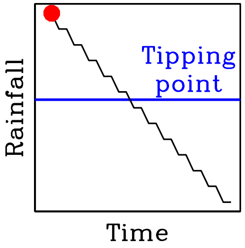
3 Effects of Changing Rainfall Levels

4 Wavelength Selection: Two Examples

5 Further Reading

The Onset of Patterning

At high rainfall levels, vegetation is uniform.
The transition to patterns is a “tipping point”



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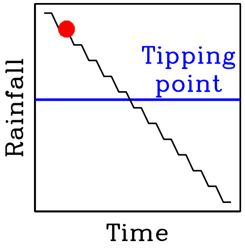

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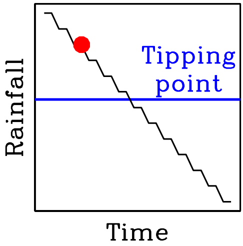

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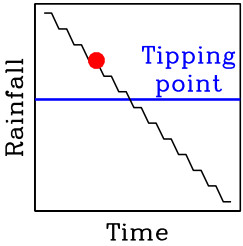

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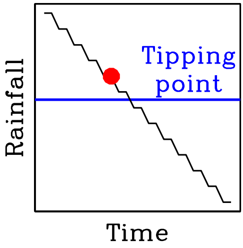

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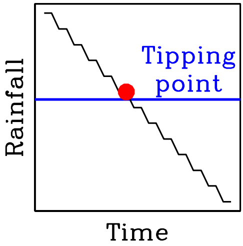

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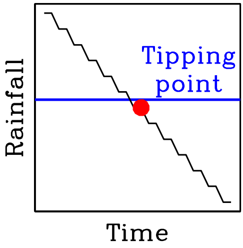

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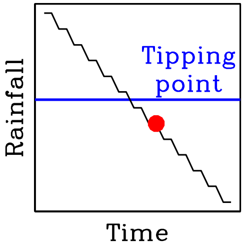

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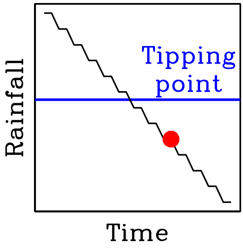

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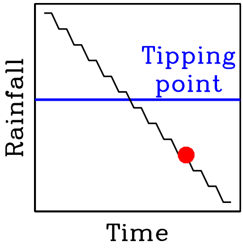

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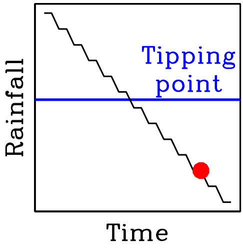

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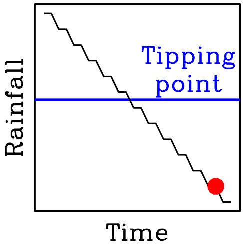

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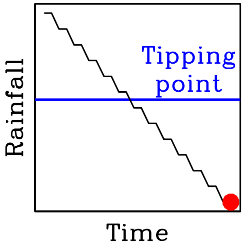

The tipping point occurs when the homogeneous steady state becomes unstable.

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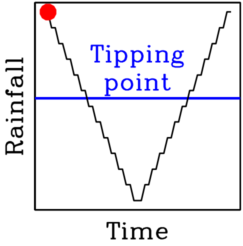

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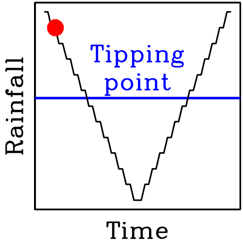

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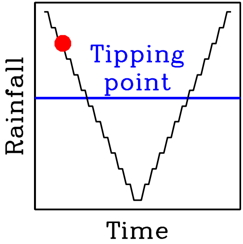

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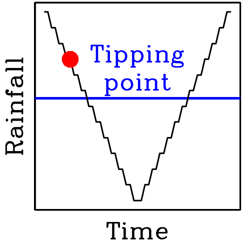

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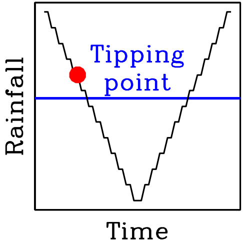

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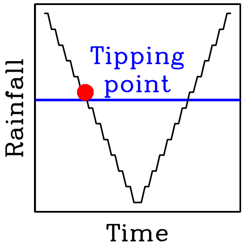

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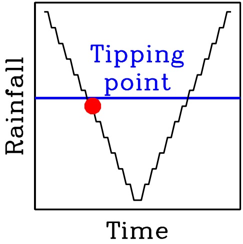

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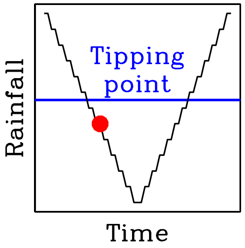

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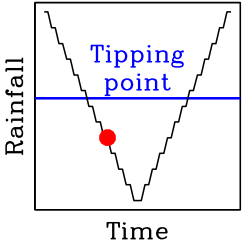

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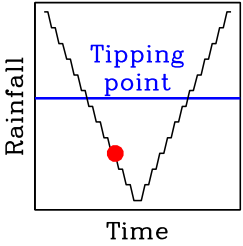

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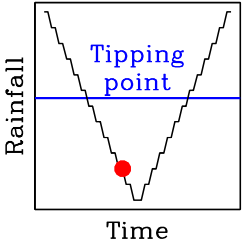

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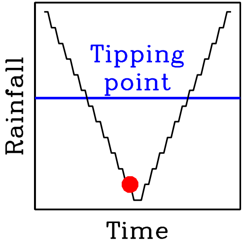

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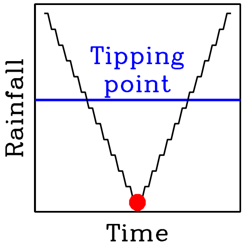

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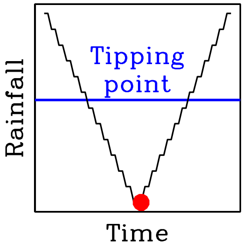

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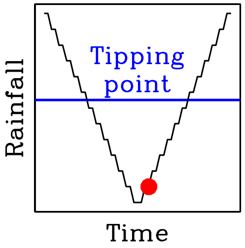

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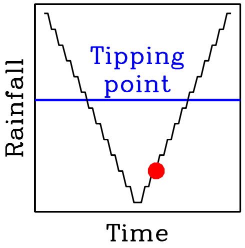

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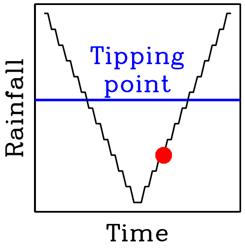

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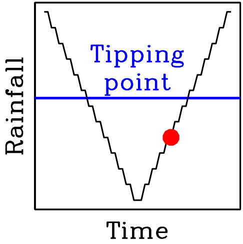

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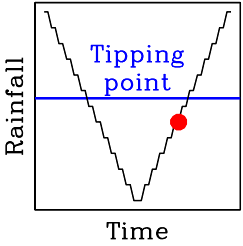

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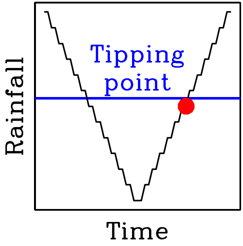

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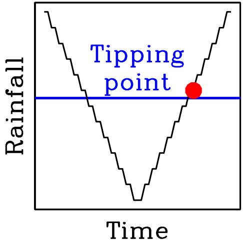

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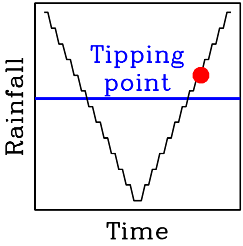

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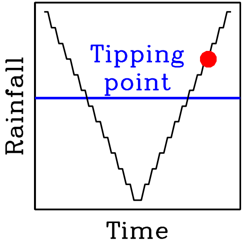

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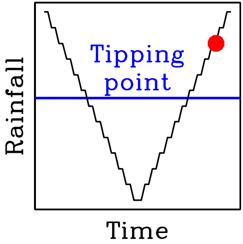

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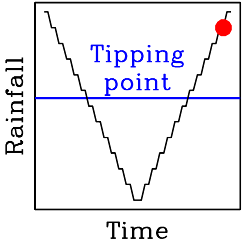

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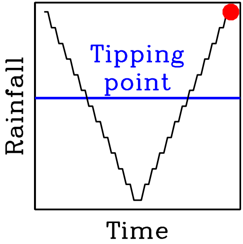

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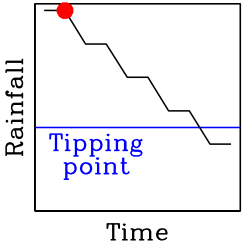

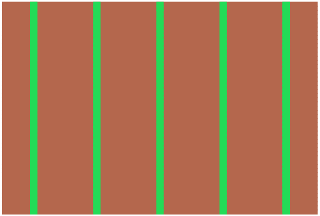

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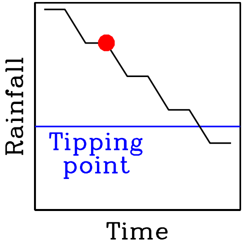

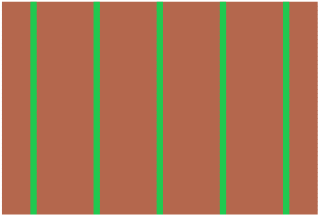

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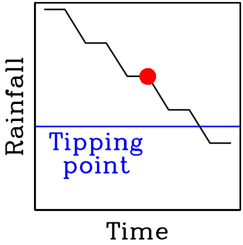

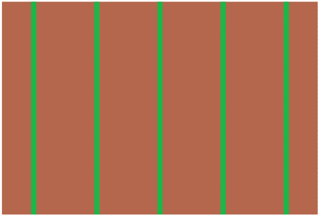

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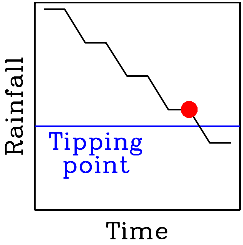

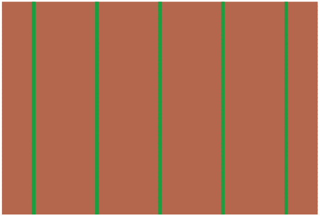

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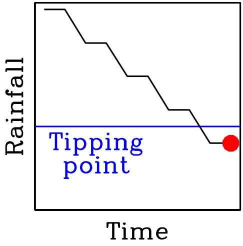

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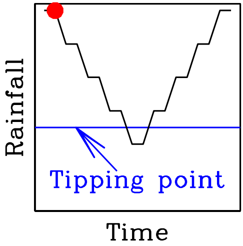

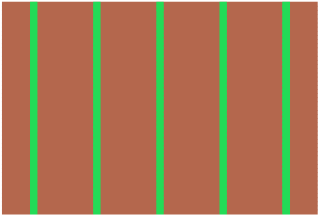

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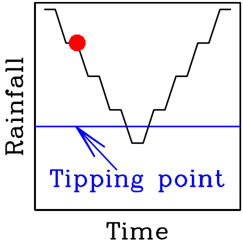

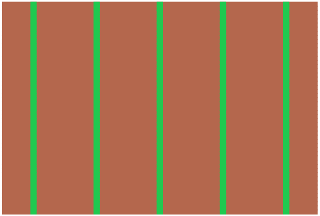

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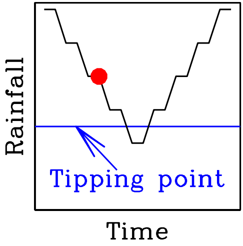

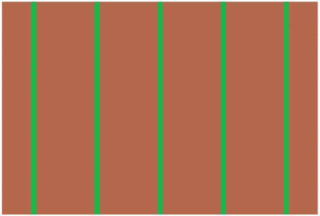

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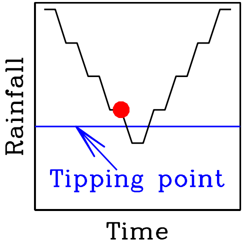

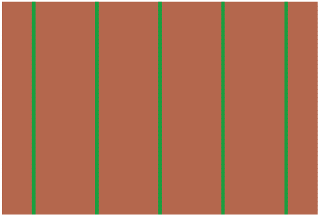

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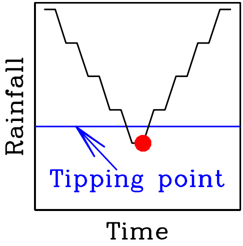

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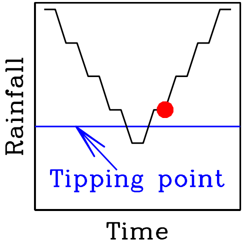

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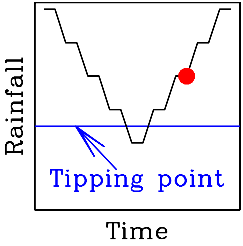

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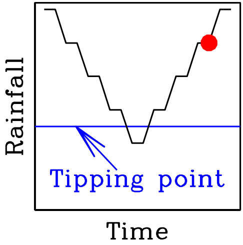

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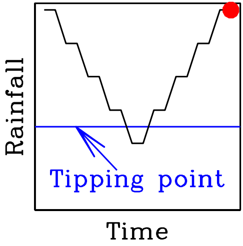

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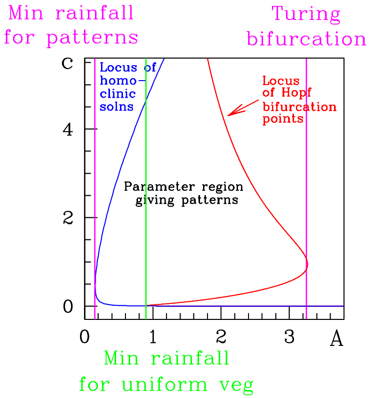
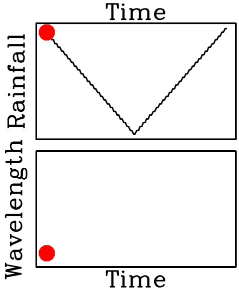
Stability in a Parameter Plane



History-Dependent Patterns

Model prediction: as rainfall is varied within the range giving patterns, abrupt changes in pattern wavelength occur.



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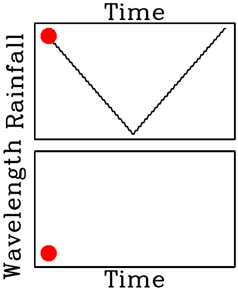

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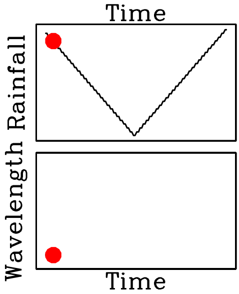

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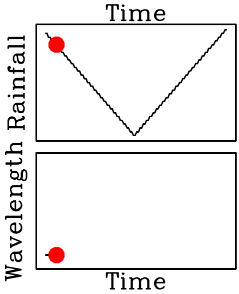

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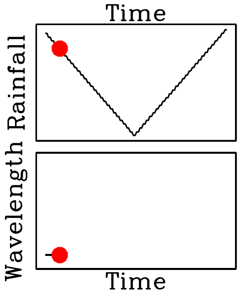
Model prediction: as rainfall is varied within the range giving patterns, abrupt changes in pattern wavelength occur.



History-Dependent Patterns

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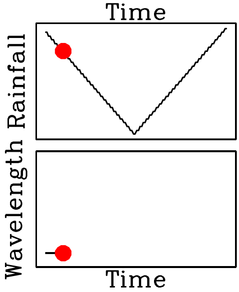

History-Dependent Patterns

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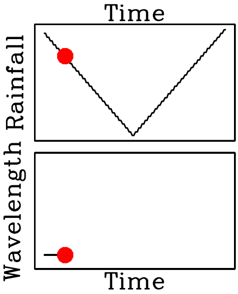

History-Dependent Patterns

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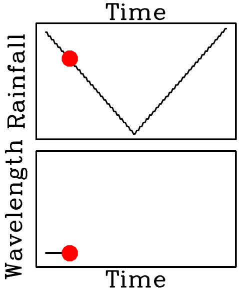

History-Dependent Patterns

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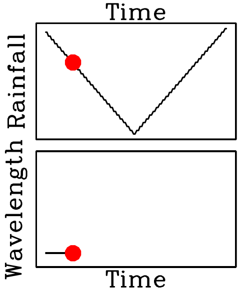

History-Dependent Patterns

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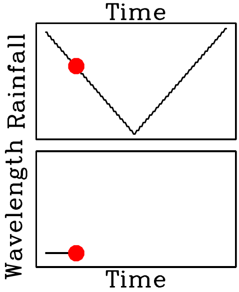

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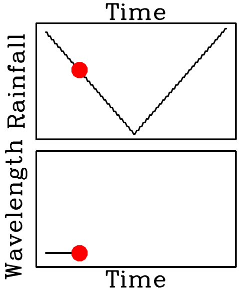

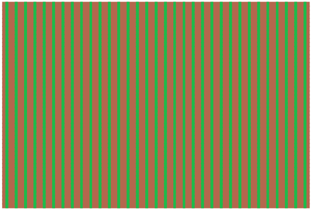

History-Dependent Patterns

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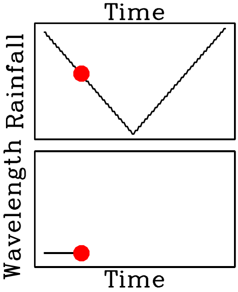

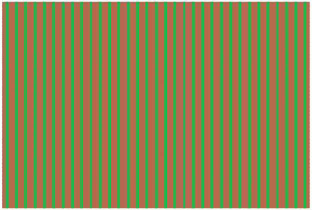

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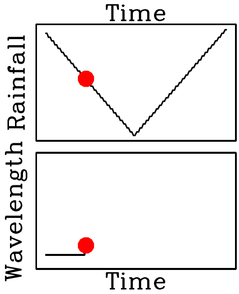

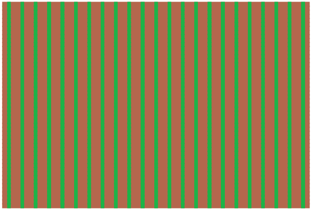

History-Dependent Patterns

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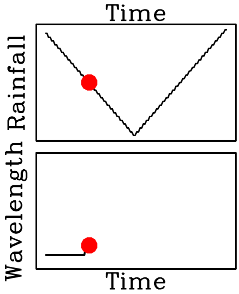

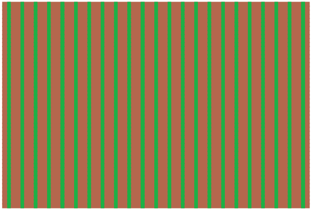

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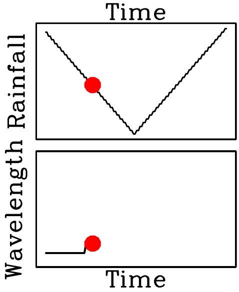

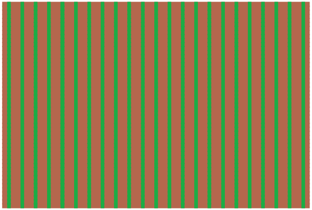

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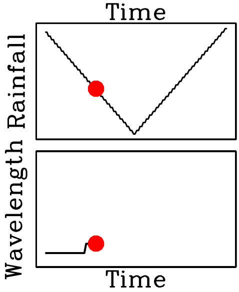

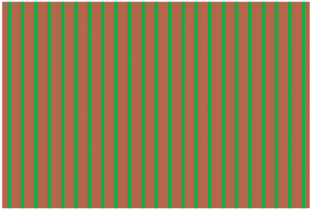

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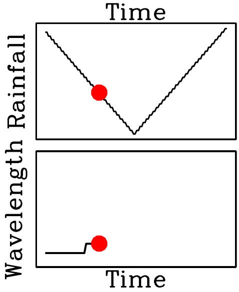

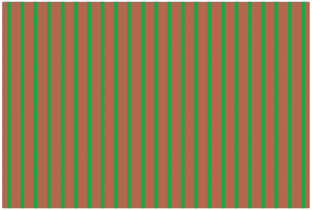

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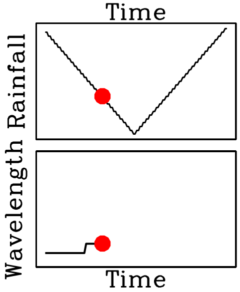

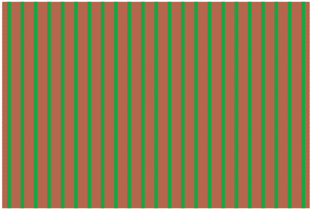

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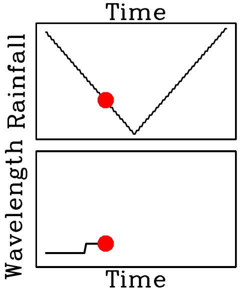

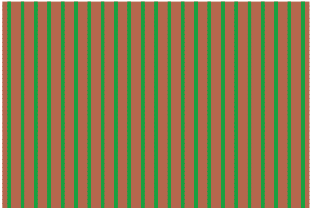

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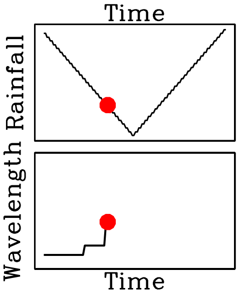

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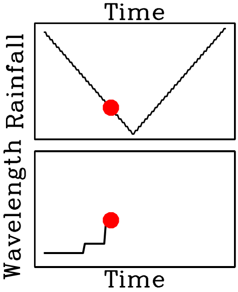

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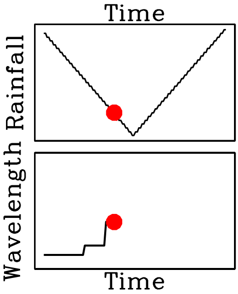

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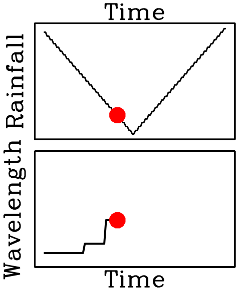

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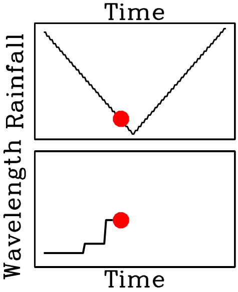

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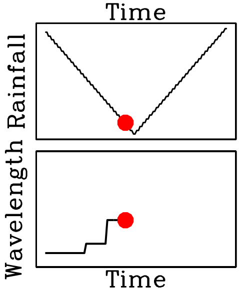

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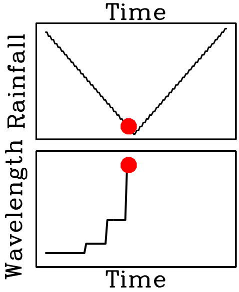

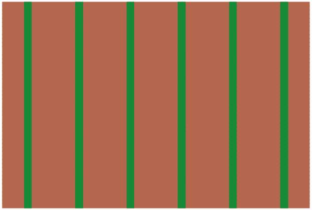

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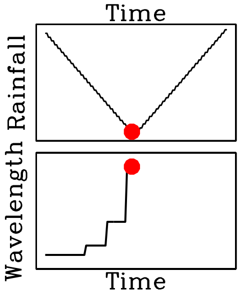

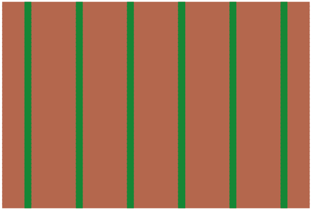

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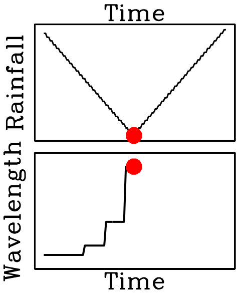

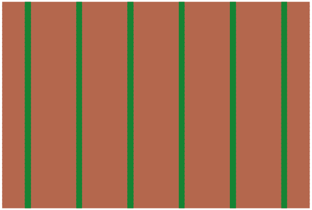

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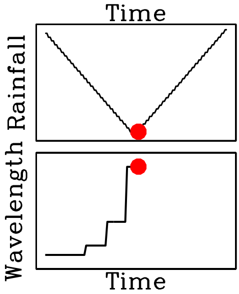

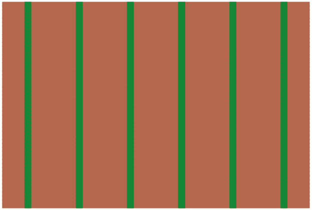

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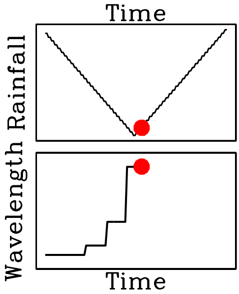

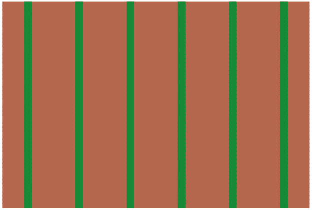

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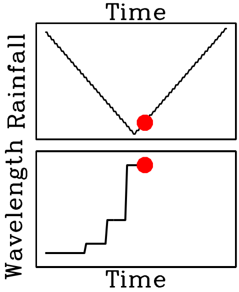

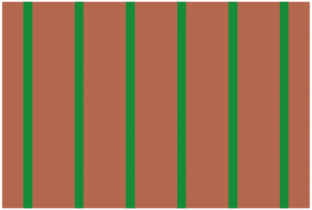

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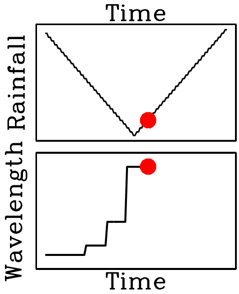

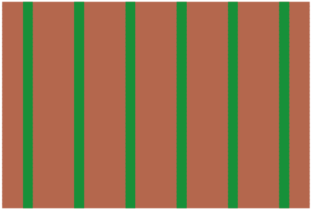

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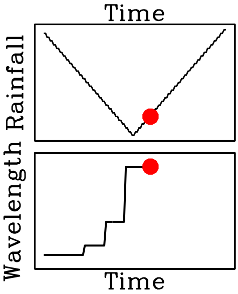

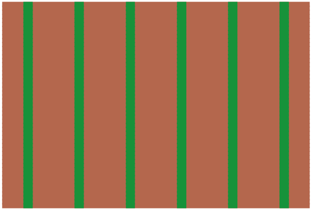

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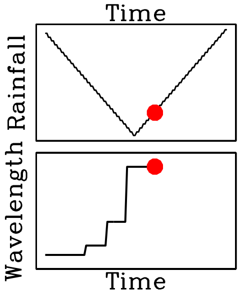

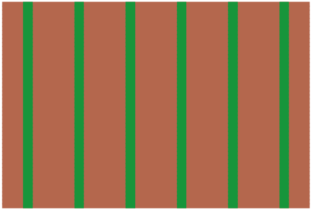

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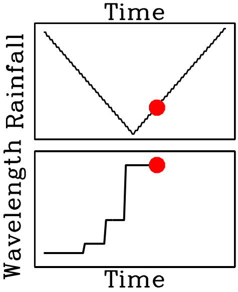

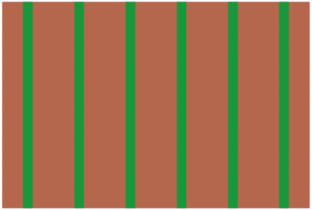

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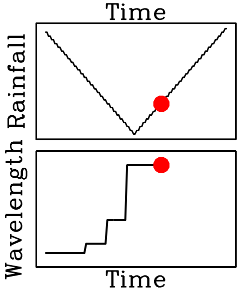

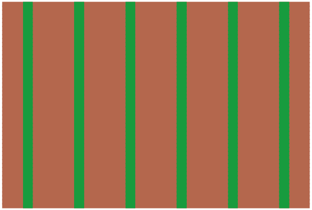

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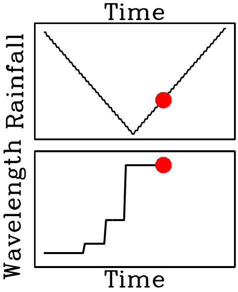

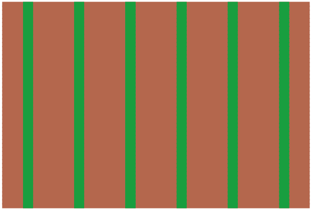

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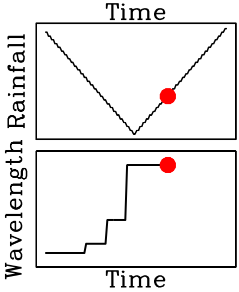

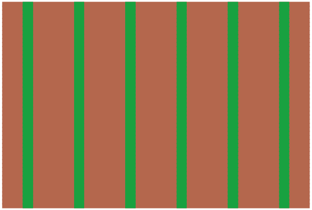

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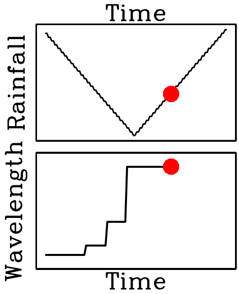

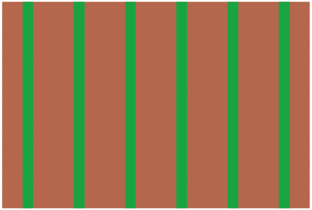

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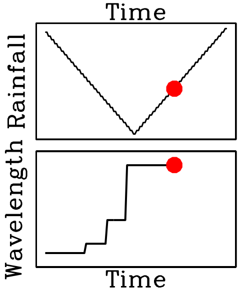

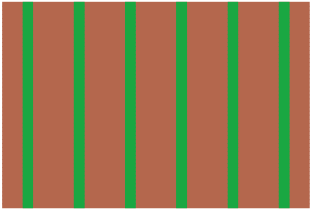

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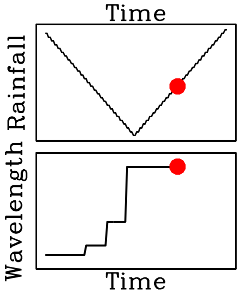

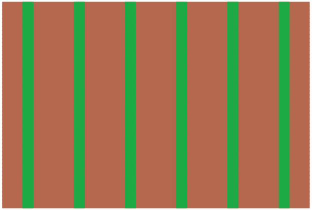

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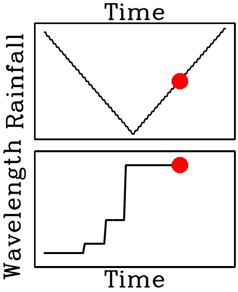

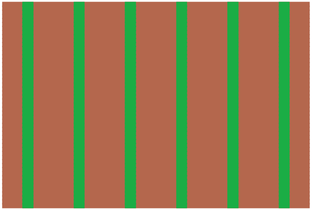

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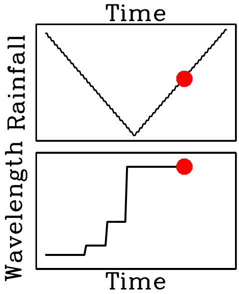

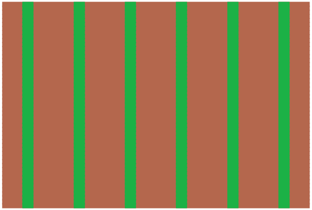

History-Dependent Patterns

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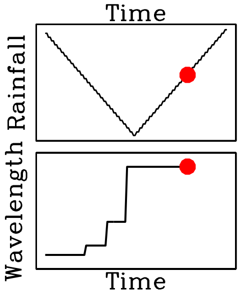

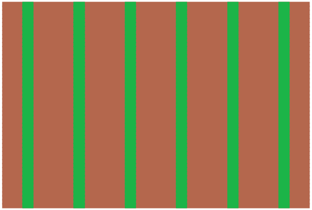

History-Dependent Patterns

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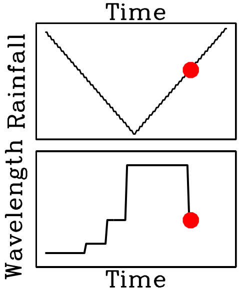

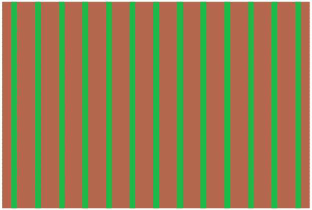

History-Dependent Patterns

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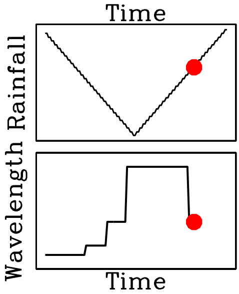

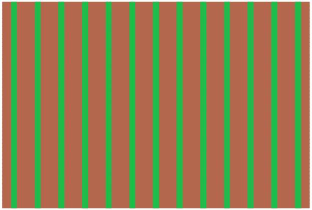

History-Dependent Patterns

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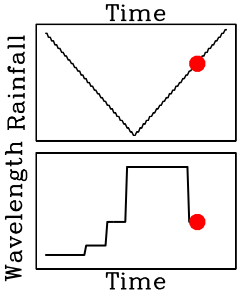

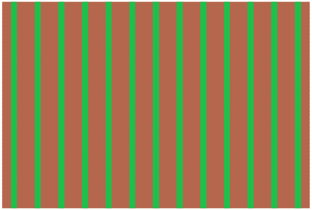

History-Dependent Patterns

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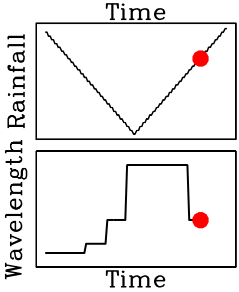

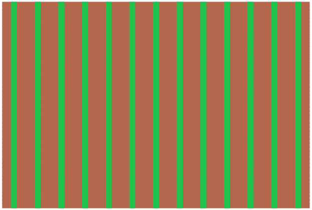

History-Dependent Patterns

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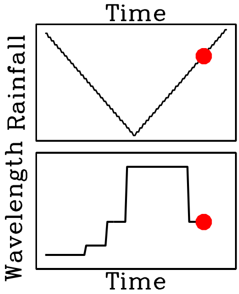

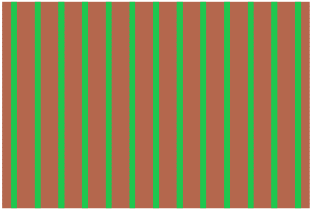

History-Dependent Patterns

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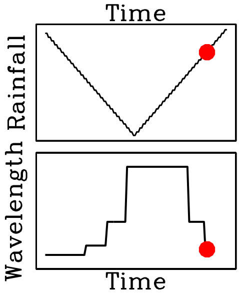

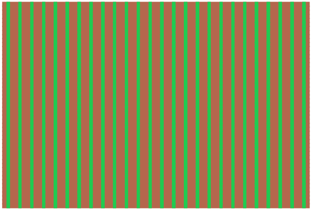

History-Dependent Patterns

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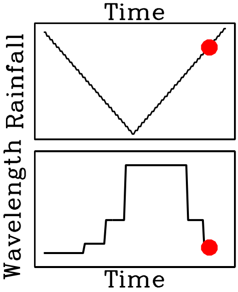

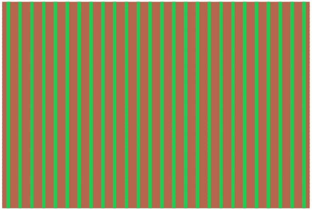

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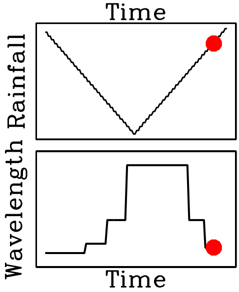

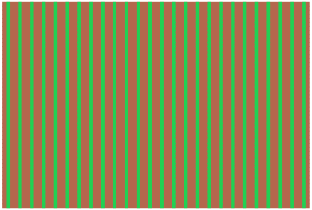

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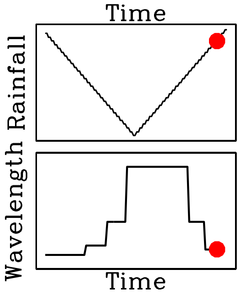

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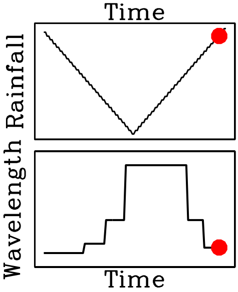

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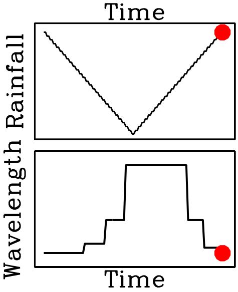

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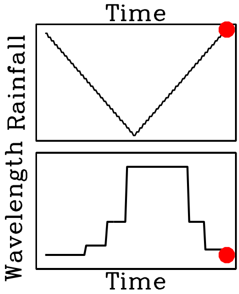

History-Dependent Patterns

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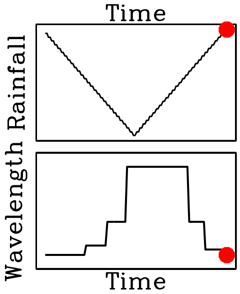
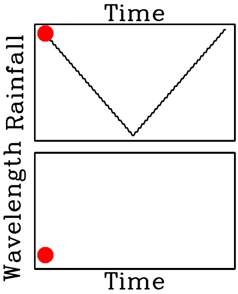


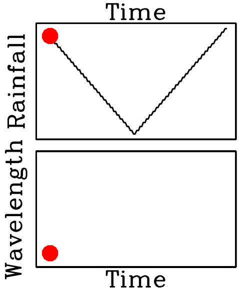
History-Dependent Patterns (Animation)

Model prediction: as rainfall is varied within the range giving patterns, abrupt changes in pattern wavelength occur.



History-Dependent Patterns (Animation)

Model prediction: as rainfall is varied within the range giving patterns, abrupt changes in pattern wavelength occur.



History-Dependent Patterns (Animation)

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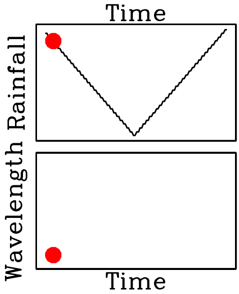

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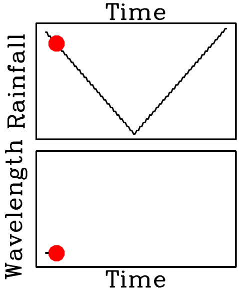

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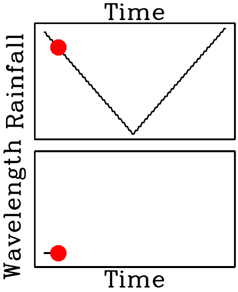

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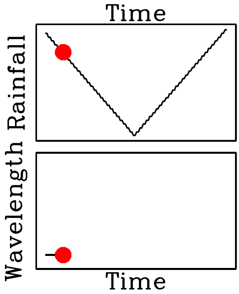

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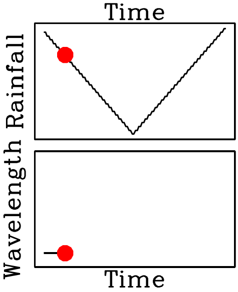

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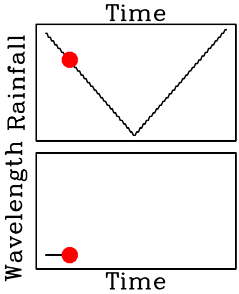

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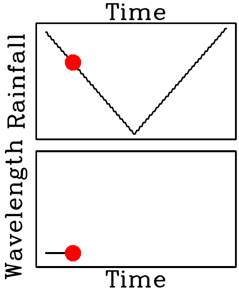

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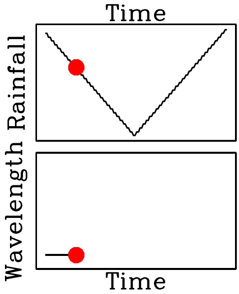

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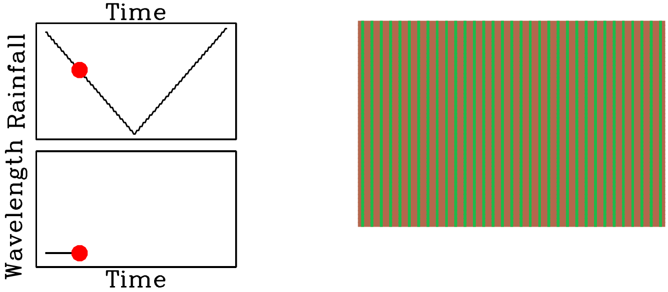

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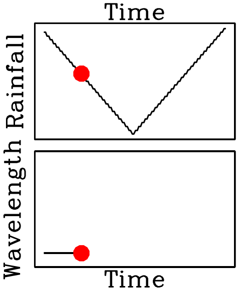

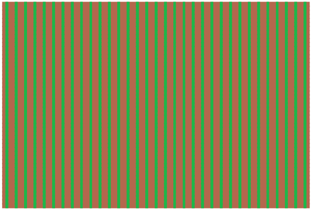

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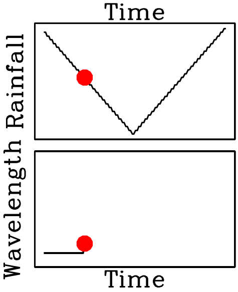

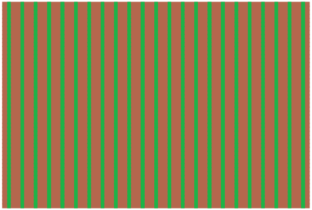

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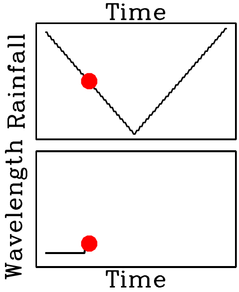

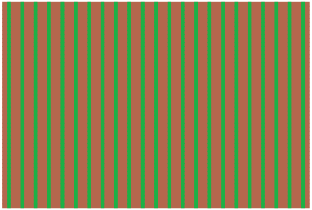

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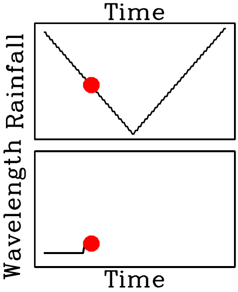

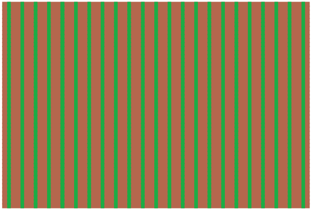

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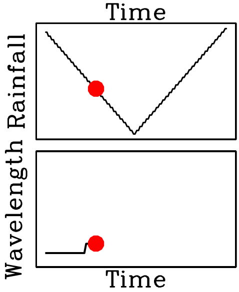

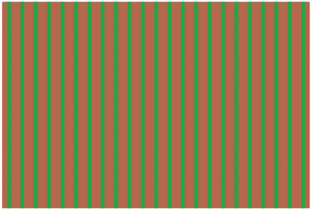

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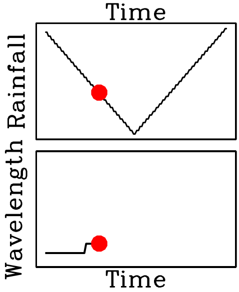

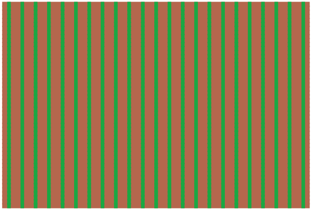

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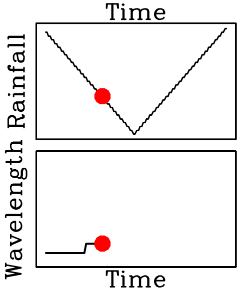

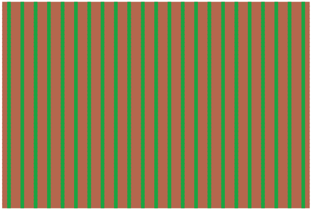

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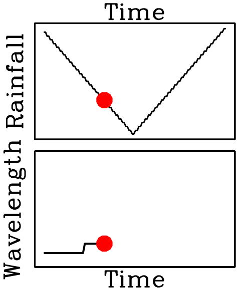

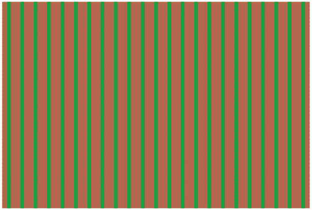

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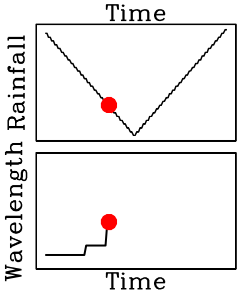

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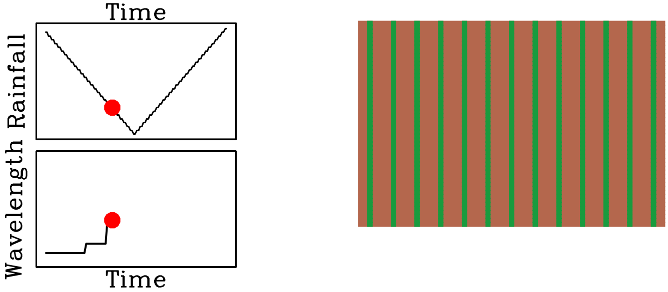

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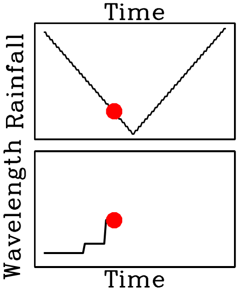

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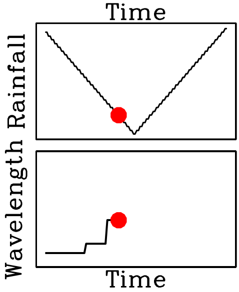

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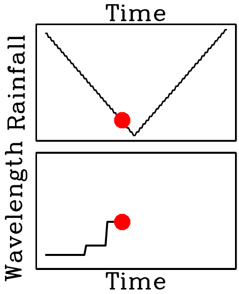

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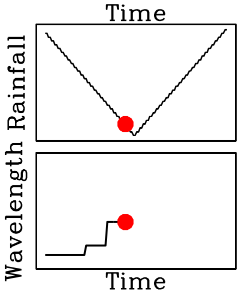

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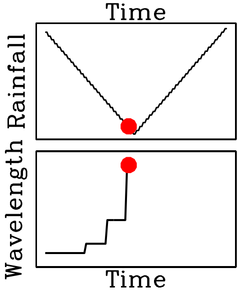

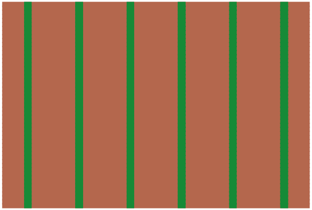

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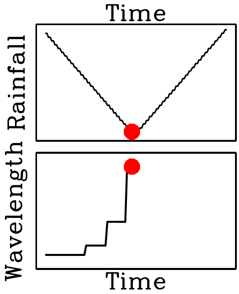

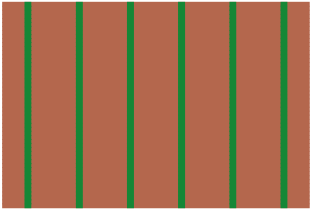

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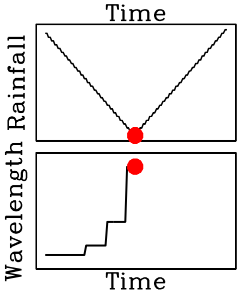

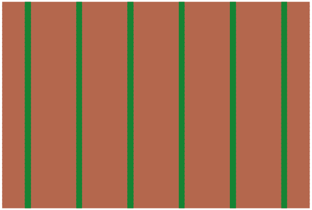

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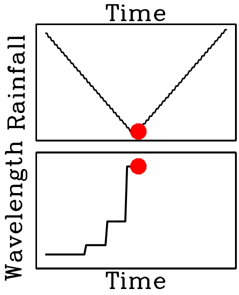

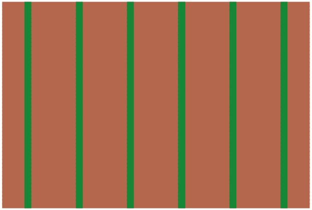

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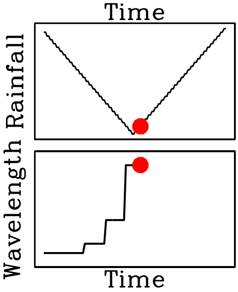

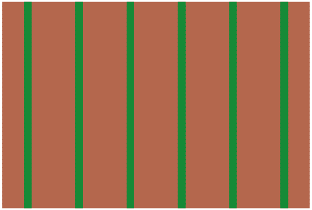

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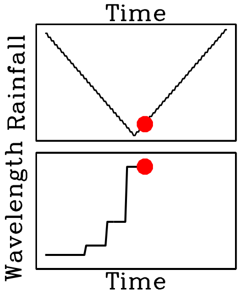

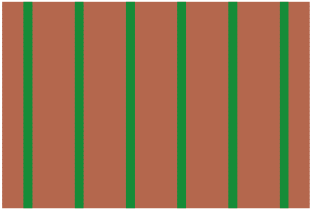

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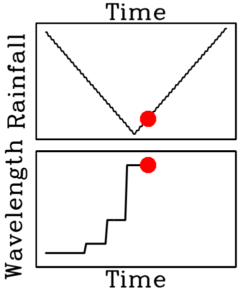

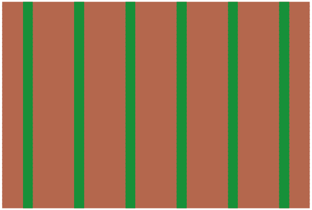

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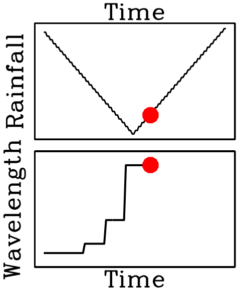

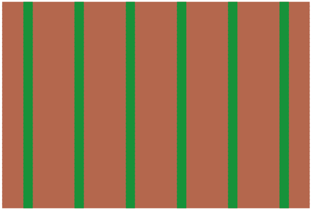

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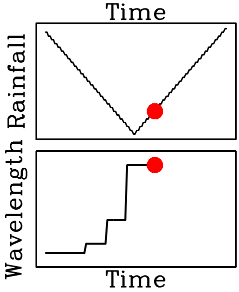

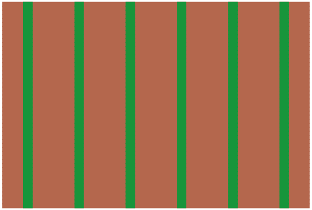

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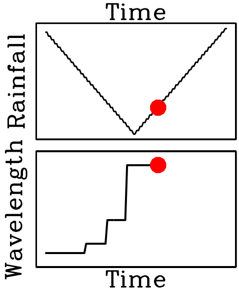

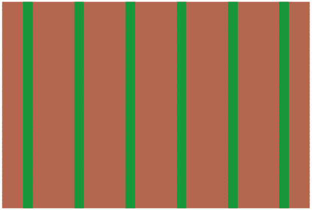

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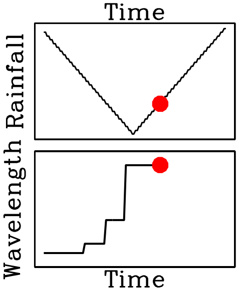

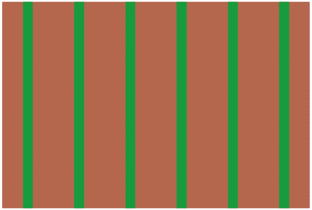

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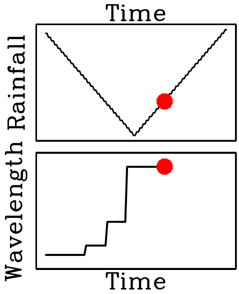

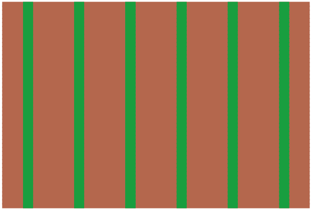

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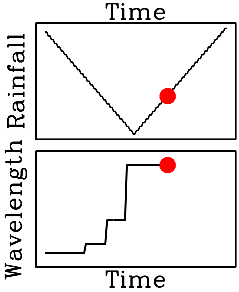

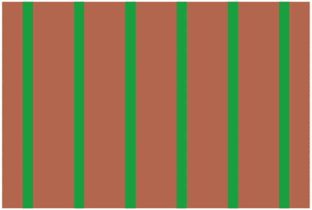

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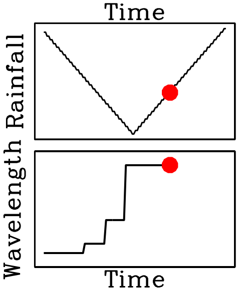

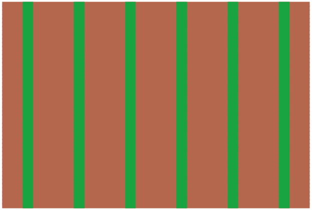

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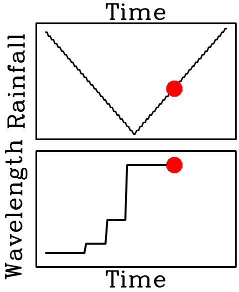

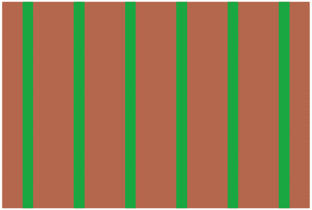

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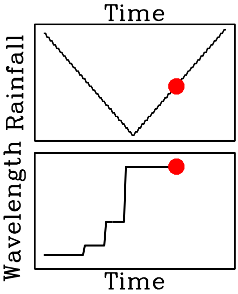

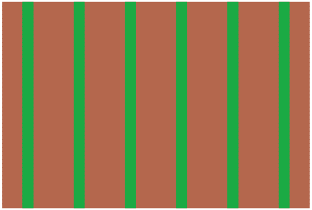

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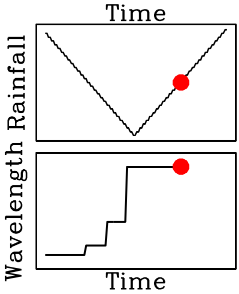

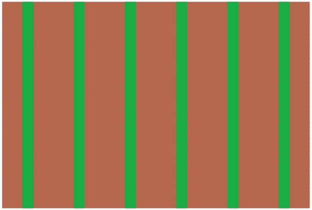

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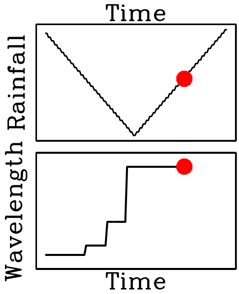

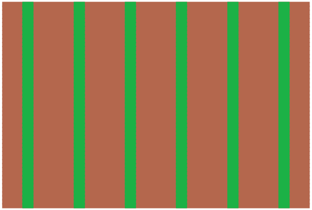

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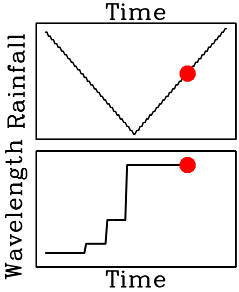

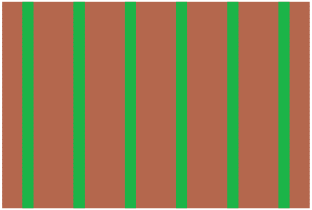

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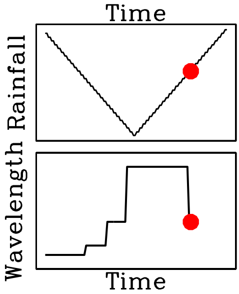

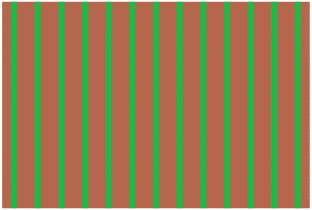

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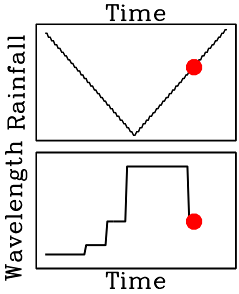

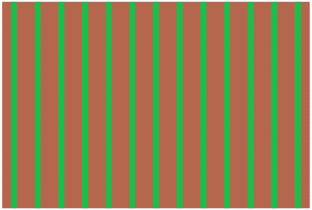

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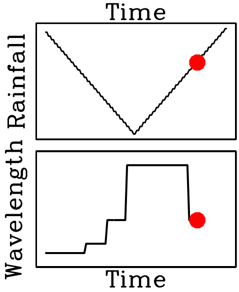

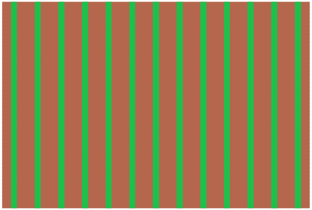

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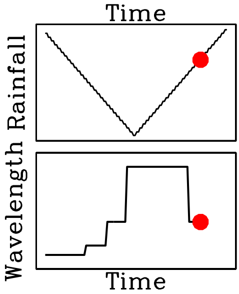

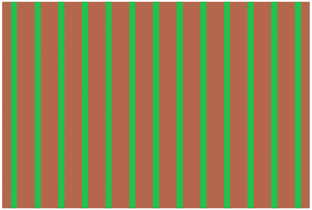

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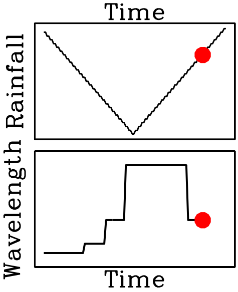

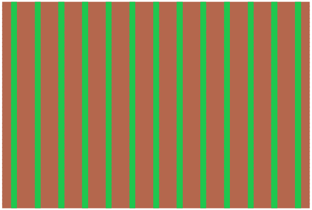

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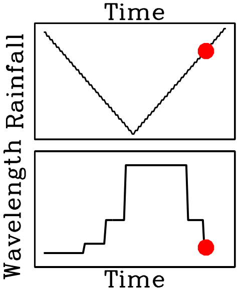

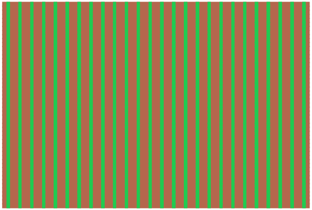

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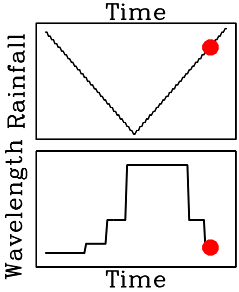

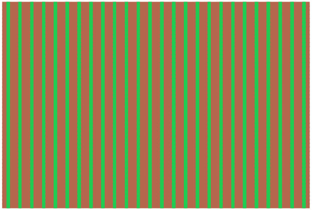

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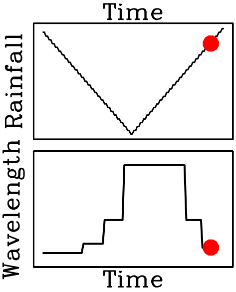

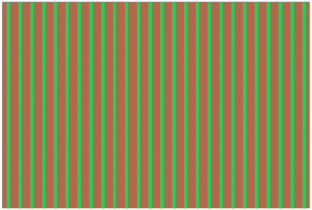

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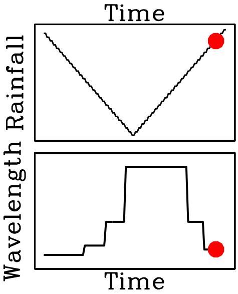

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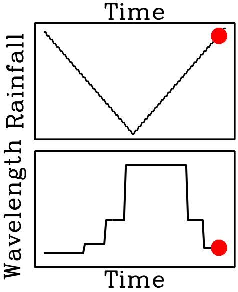

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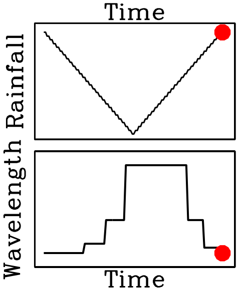

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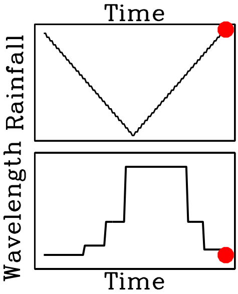

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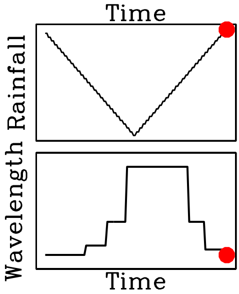
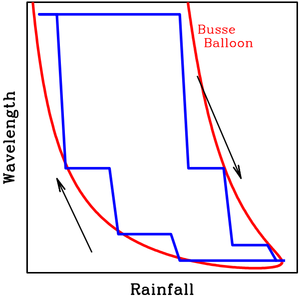


Mathematical Explanation of Hysteresis

Wavelength changes abruptly at the edge of the Busse Balloon.



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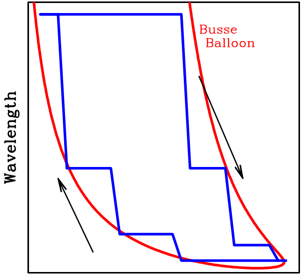
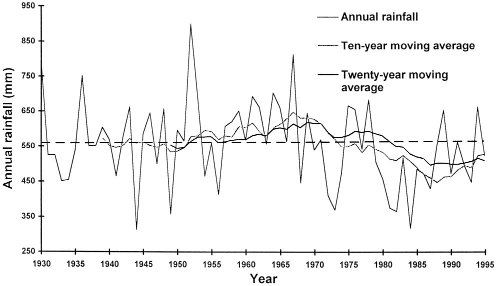
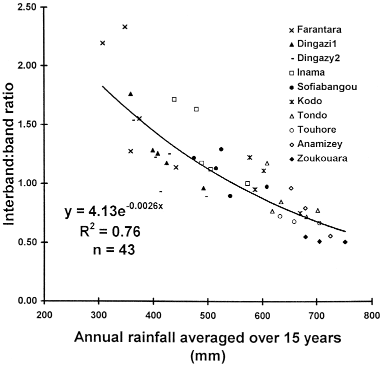


The Busse Balloon can be calculated using the software package WAVETRAIN (www.ma.hw.ac.uk/wavetrain)

Data on the Effects of Changing Rainfall





Data from 1950-1995 (C. Valentin & J.M. d'Herbès, Catena 37:231, 1999)

Ecological Conclusions

The mathematical model has predicted answers to the following key ecological questions:

- At what rainfall level is there a switch from uniform vegetation to patterns?
- At what rainfall level is there a transition to desert?
- What determines the wavelength of vegetation bands?

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  Wavelength depends on both parameters and patterning history.

Outline

1 Ecological Background

2 Detailed Calculation of Possible Wavelengths

3 Effects of Changing Rainfall Levels

4 Wavelength Selection: Two Examples

5 Further Reading

The Origin of Vegetation Patterns

Vegetation patterns develop via either degradation of uniform vegetation, or colonisation of bare ground.

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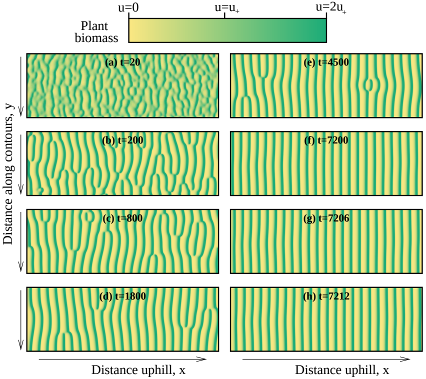

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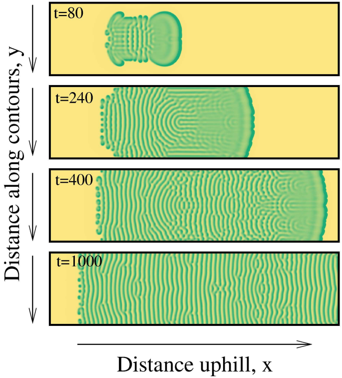

How to Predict Pattern Wavelength

Pattern wavelength
is history-dependent

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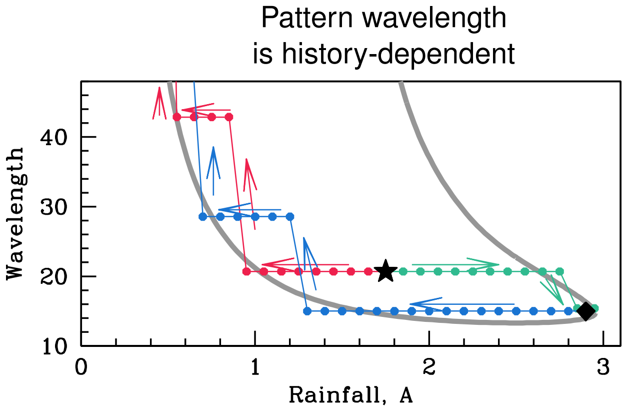


How to Predict Pattern Wavelength

Pattern wavelength
is history-dependent

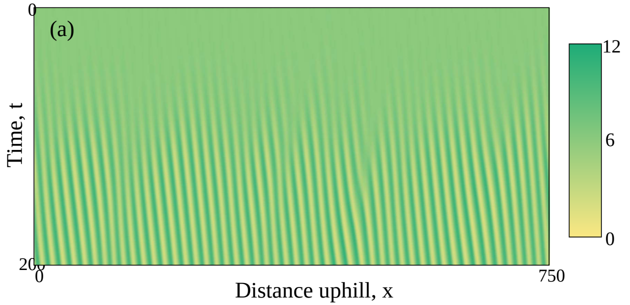
$\Downarrow$

We must focus on the
onset of patterning

$\swarrow$ $\searrow$

Degradation of uniform vegetation | Colonisation of bare ground

Wavelength for Degradation of Uniform Vegetation



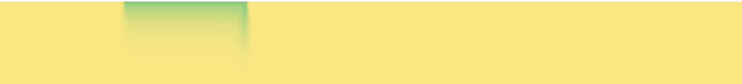
For degradation of uniform vegetation, pattern wavelength can be calculated via the stability of the homogeneous steady state: wavelength=1/(most unstable frequency).

When Does Vegetation Colonise Bare Ground?

Downhill ⟷ Uphill

Time ↓

Very low rainfall: an isolated vegetation patch dies out

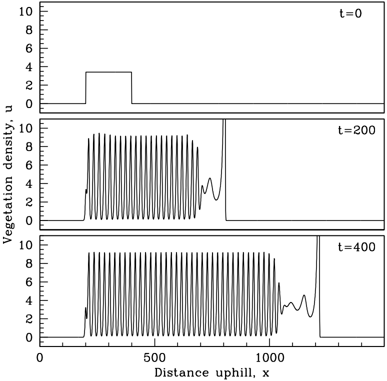


Slightly larger rainfall: both edges move uphill

Larger rainfall: the patch expands both uphill and downhill

When Does Vegetation Colonise Bare Ground?

The key critical case is when the downhill edge is stationary



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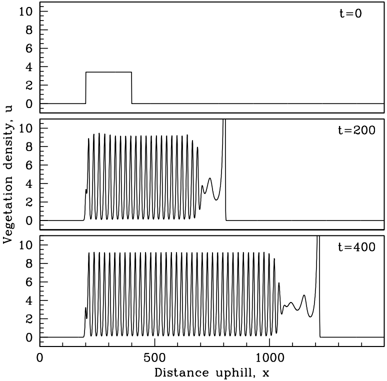
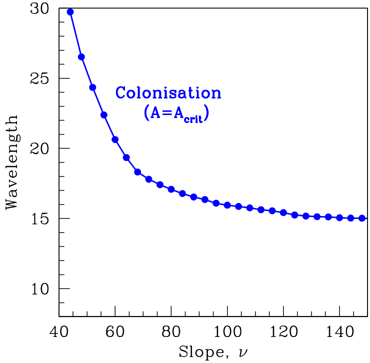
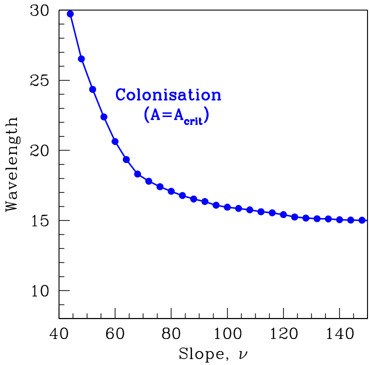


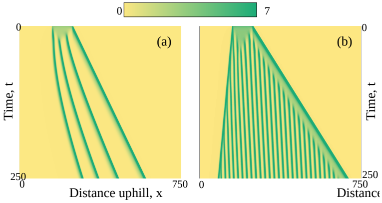
Wavelength can be calculated via numerical simulations in which rainfall A is chosen so that the downhill edge is stationary ($A = A_{crit}$, say).

Comparison of Wavelengths



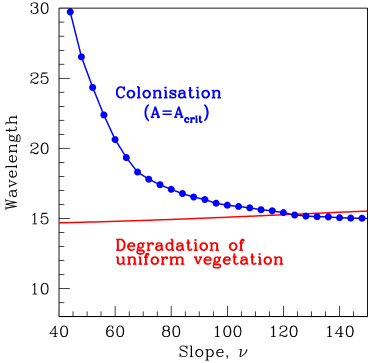
Comparison of Wavelengths





$A < A_{crit}$ $A > A_{crit}$

Comparison of Wavelengths



Degradation of uniform vegetation and colonisation of bare ground give patterns with different wavelengths.

Outline

1. Ecological Background
2. Detailed Calculation of Possible Wavelengths
3. Effects of Changing Rainfall Levels
4. Wavelength Selection: Two Examples
5. Further Reading

Further Reading

Dagbovie AS, Sherratt JA (2014) Pattern selection and hysteresis in the Rietkerk model for banded vegetation in semi-arid environments. J R Soc Interface 11:20140465

Klausmeier CA (1999) Regular and irregular patterns in semiarid vegetation. Science 284:1826-1828.

Sherratt JA (2005) An analysis of vegetation stripe formation in semi-arid landscapes. J. Math. Biol. 51:183-197.

Sherratt JA (2013) History-dependent patterns of whole ecosystems. Ecological Complexity 14:8-20.

Siero E, Doelman A, Eppinga MB, Rademacher J, Rietkerk M, Siteur K (2015) Stripe pattern selection by advective reaction-diffusion systems: resilience of banded vegetation on slopes. Chaos 25:036411

Siteur K, Siero E, Eppinga MB, Rademacher J, Doelman A, Rietkerk M (2014) Beyond Turing: the response of patterned ecosystems to environmental change. Ecological Complexity 20:81-96

List of Frames

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   - Vegetation Patterns
   - Pattern Wavelength: A Quantitative Statistic
   - Mathematical Model of Klausmeier
   - Typical Solution of the Model
   - Homogeneous Steady States
2. Detailed Calculation of Possible Wavelengths
   - Banded Patterns on Slopes Move Uphill
   - Travelling Wave Equations
   - Pattern Stability
3. Effects of Changing Rainfall Levels
   - The Onset of Patterning
   - Desertification
   - History-Dependent Patterns
   - Mathematical Explanation of Hysteresis
   - Ecological Conclusions
4. Wavelength Selection: Two Examples
   - The Origin of Vegetation Patterns
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5. Further Reading